THREE LAYERS SUPPLY CHAIN PRODUCTION INVENTORY MODEL UNDER PERMISSIBLE DELAY IN PAYMENTS IN IMPRECISE ENVIRONMENT

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Abstract:A three layer supply chain production inventory model(SCPIM) under conditionally permissible delay in payments formulated in fuzzy environment. Supplier produced the item at a finite rate and purchase the item to the manufacturer. Manufacturer has also purchased and produced the item in a certain rate which is the decision variable. Manufacturer sale his product to the retailer and also give the delay in payment to the retailer. Retailer purchase the item from manufacture and to sale the customers. Ideal costs of supplier, manufacturer and retailer have been taken into account. Integrated model has been developed and solved analytically in crisp and fuzzy environments and finally corresponding individual profits are calculated through numerically and graphically.
Keywords: Production inventory system, Fuzzy set, Three-layer supplier chain, Delay in payment.

1. Introduction

In 1965, Prof. Zadeh[17] first introduced a new concept "Fuzzy Set Theory" to accommodate the uncertainty in non-stochastic sense. Bellman and Zadeh[18], first introduced fuzzy set theory in decision making process. After that, Zimmermann[19] introduced fuzzy mathematical programming.

In a real-life problem, in addition crisp constraints, there may be other two types of uncertain constraints - stochastic and fuzzy constraints. Analogous to chance constraints, imprecise constraints are defined in possibility and / or necessity sense and

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defuzzied following fuzzy relations (cf. Zadeh [17], Dubios and Prade [2], Maity and Maiti[9]). These constraints are now-a-days applied in different areas including inventory control system. Inuiguchi, Sakawa and Kume[4], Maity[10], have been solved inventory production planning problems under possibility / Necessity/ Credibility constraints. Wang and Shu [16] developed a supply chain inventory model under possibility constraints. Mandal et. al.[11] developed a production-inventory problem with fuzzy expectation.

A supply chain model (SCM) is a network of supplier, producer, distributor and customer which synchronizes a series of inter-related business process in order to have: (i) optimal procurement of raw materials from nature; (ii) transportation of raw-materials into warehouse; (iii) production of the goods in the production center and (iv) distribution of these finished goods to retailer for sale to the customers. With a recent paradigm shift to the supply chain (SC), the ultimate success of a firm may depend on its ability to link supply chain members seamlessly.

One of the earliest efforts to create an integrated SCM dates back to Bookbinder, Cachon [5], Agarwal[1] and others. They developed a production, distribution and inventory (PDI) planning system that integrated three supply chain segments comprised of supply, storage / location and customer demand planning. The core of the PDI system was a network model and diagram that increased the decision maker's insights into supply chain connectivity. The model however was confined to a single-period and single-objective problem. Viswanathan and Piplani [15] concerned an integrated inventory model through common replenishment in the SC. All the above SCMs are considered with constant, known demand and production rates.

In the traditional economic order quantity (EOQ) model, it often assumed that the retailer must pay off as soon as the items

are received. In fact, the supplier offers the retailer a delay period, known as trade credit period, in paying for purchasing cost, which is a very common business practice. Suppliers often offer trade credit as a marketing strategy to increase sales and reduce on hand stock level. Once a trade credit has been offered, the amount of period that the retailers capital tied up in stock is reduced, and that leads to a reduction in the retailers holding cost of finance. In addition, during trade credit period, the retailer can accumulate revenues by selling items and by earning interests. As a matter of fact, retailers, especially small businesses which tend to have a limited number of financing opportunities, rally on trade credit as a source of short-term funds. In this research field, Goyal[6] was the first to establish an EOQ model with a constant demand rate under the condition of permissible delay in payments. Recently, Liang [7] was also established a two-warehouse inventory model for deteriorating items under conditionally permissible delay in payment. None can consider supply chain production inventory model under permissible delay in payments in imprecise environment. In this paper, we developed a three layers supply chain production inventory model under permissible delay in payments in fuzzy environment

2. **Possibility / Necessity / Credibility Measurements**

Any fuzzy subset $\tilde{a}$ of $\Re$ (where $\Re$ represents a set of real numbers) with membership function $\mu_{\tilde{a}}(x) : \Re \to [0,1]$ is called a fuzzy number. Let $\tilde{a}$ and $\tilde{b}$ be two fuzzy quantities with membership functions $\mu_{\tilde{a}}(x)$ and $\mu_{\tilde{b}}(x)$ respectively. Then according to Dubois and Prade [3], Liu and Liu[8] and Maity[9]

Pos $(\tilde{a} * \tilde{b}) = \{sup(min(\mu_{\tilde{a}}(x), \mu_{\tilde{b}}(y)),\ x, y \epsilon \Re, x * y\}$
Nes $(\tilde{a} * \tilde{b}) = \{inf(max(1 - \mu_{\tilde{a}}(x), \mu_{\tilde{b}}(y)),\ x, y \epsilon \Re, x * y\}$
where the abbreviation Pos represents possibility, Nes necessity and $*$ is any of the the relations $>, <, =, \leq, \geq$.

The dual relationship of possibility and necessity requires that
Nes$(\tilde{a} * \tilde{b}) = 1 - Pos(\overline{\tilde{a} * \tilde{b}})$
Also necessity measures satisfy the condition

$$\text{Min}(\text{Nes}(\tilde{a} * \tilde{b}\ ), \text{Nes}(\overline{\tilde{a} * \tilde{b}}))=0$$

The relationships between possibility and necessity measures satisfy also the following conditions (cf. Zimmermann[19], Dubois and Prade[2]:

Pos$(\tilde{a} * \tilde{b}) \geq$ Nes$(\tilde{a} * \tilde{b})$, Nes$(\tilde{a} * \tilde{b}) > 0 \Rightarrow$ Pos $(\tilde{a} * \tilde{b})$=1 and Pos$(\tilde{a} * \tilde{b}) < 1 \Rightarrow$ Nes$(\tilde{a} * \tilde{b})$=0

Credibility Measure:

Based on possibility and necessity measures, the third set function Cr, called credibility measure and fuzzy expectation were analyzed by Liu and Liu[8] is as follows:

$$Cr(A) = \frac{1}{2}[\,Pos(A) + Nec(A)] \text{ for any} A \in 2^R \tag{1}$$

It is easy to check that Cr satisfies the following conditions:
i) $Cr(\phi) = 0$ and $Cr(R) = 1$,
ii) $Cr(A) \leq Cr(B)$ when ever $A, B \in 2^R$ and $A \subset B$
Thus, Cr is also a fuzzy measure defined on $(R, 2^R)$. Besides, Cr is self dual, i.e., $Cr(A) = 1 - Cr(A^C)$ for any $A \in 2^R$.

In this paper, based on the credibility measure the following form can be defined as

$$Cr(A) = [\,\rho Pos(A) + (1 - \rho)Nec(A)\,] \tag{2}$$

for any $A \in 2^R$ and $0 < \rho < 1$. It also satisfies the above conditions.
Let $\tilde{A} = (\,a_1, a_2, a_3\,))$ is a triangular fuzzy number and r is a crisp number. Using Liu and Liu[8], we define possibility measure and necessity measure as following.

$$Pos(\tilde{A} \geq r) = \begin{cases} 1 & \text{if} \quad r \leq a_2 \\ \dfrac{a_3 - r}{a_3 - a_2} & \text{if} \quad a_2 \leq r \leq a_3 \\ 0 & \text{if} \quad r \geq a_3 \end{cases}$$

$$Nec(\tilde{A} \geq r) = \begin{cases} 1 & \text{if} \quad r \leq a_1 \\ \dfrac{a_2 - r}{a_2 - a_1} & \text{if} \quad a_1 \leq r \leq a_2 \\ 0 & \text{if} \quad r \geq a_2 \end{cases}$$

Using equ.(2), the credibility measure for TFN can be define as

$$Cr(\tilde{A} \geq r) = \begin{cases} 1 & \text{if} \quad r \leq a_1 \\ \dfrac{a_2 - \rho a_1}{a_2 - a_1} - \dfrac{(1-\rho)r}{a_2 - a_1} & \text{if} \quad a_1 \leq r \leq a_2 \\ \dfrac{\rho(a_3 - r)}{a_3 - a_2} & \text{if} \quad a_2 \leq r \leq a_3 \\ 0 & \text{if} \quad r \geq a_3 \end{cases} \tag{3}$$

$$Cr(\tilde{A} \leq r) = \begin{cases} 0 & \text{if} \quad r \leq a_1 \\ \rho\dfrac{r - a_1}{a_2 - a_1} & \text{if} \quad a_1 \leq r \leq a_2 \\ \dfrac{a_3 - \rho a_2 - (1-\rho)r}{a_3 - a_2} & \text{if} \quad a_2 \leq r \leq a_3 \\ 1 & \text{if} \quad r \geq a_3 \end{cases} \tag{4}$$

2.1. **Expected Value of a Fuzzy Variable.** Based on the credibility measure, Liu and Liu[8] presented the expected value operator of a fuzzy variable as follows.

Definition-1: Let X be a normalized fuzzy variable the expected value of the fuzzy variable X is defined by

$$E[X] = \int_0^\infty Cr(X \geq r)dr - \int_{-\infty}^0 Cr(X \leq r)dr \tag{5}$$

When the right hand side of (5) is of form $\infty - \infty$, the expected value is not defined. Also, the expected value operation has been proved to be linear for bounded fuzzy variables, i.e., for any two bounded fuzzy variables X and Y, we have $E[\,aX + bY\,] = aE[X] + bE[Y]$ for any real numbers a and b.

Lemma-1: The expected value of triangular fuzzy variable$\tilde{A} = (a_1, a_2, a_3)$ is defined as

$$E[\tilde{A}] = \frac{1}{2}[(1-\rho)a_1 + a_2 + \rho a_3] \tag{6}$$

Proof: Using (3) and (4), from the definition-1 of expected value of fuzzy variable $\tilde{A}$ is defined as follows,

$$E(\tilde{A}) = \int_0^\infty Cr(\tilde{A} \geq r)dr - \int_{-\infty}^0 Cr(\tilde{A} \leq r)dr$$

$$= \int_0^{a_1} Cr(\tilde{A} \geq r)dr + \int_{a_1}^{a_2} Cr(\tilde{A} \geq r)dr + \int_{a_2}^{a_3} Cr(\tilde{A} \geq r)dr + 0$$

$$= \int_0^{a_1} dr + \int_{a_1}^{a_2} [\tfrac{a_2 - \rho a_1}{a_2 - a_1} - \tfrac{(1-\rho)r}{a_2 - a_1}]dr + \int_{a_2}^{a_3} [\tfrac{\rho a_3}{a_3 - a_2} - \tfrac{\rho r}{a_3 - a_2}]dr$$

$= \frac{1}{2}[\, a_1(1-\rho) + a_2 + \rho a_3 \,]$

3. **Assumptions and notations**

The following assumption and notation are consider to develop the model:

Assumptions:

(i) Model is developed for single item product.
(ii) Lead time is negligible.
(iii) Joint effect of supplier, manufacturer, retailer is consider in a supply chain management.
(iv) Supplier produced the item with constant rate p_s unit per unit time.
(v) Total production rate of manufacturer is equal to the demand rate of manufacturer which is decision variable .
(vi) The manufacturer give the opportunity to the retailer conditionally permissible delay in payment.
(vii) Idle cost of suppliers, manufacturer and retailer are taken into account.

Notations:

p_s = constant production rate for the suppliers.
p_m = demand rate or production rate for the manufacturer(decision variable).
D_R = constant demand rate for the retailer.
D_c = constant demand rate of customer.
C_s = purchase cost of unit item for suppliers.
C_m = selling price of unit item for suppliers which is also purchase cost for manufacturer.
C_r = selling price of unit item for manufacturer which is also purchase cost for retailers.
C_{r_1} = selling price for retailers.
t_s = production time for supplers.
T_s = cycle length for the suppliers.
T_R = length of each time period of retailer.
T' = last cycle length of the retailer.
T = total time for the integrated model.
h_s = holding cost per unit per unit time for suppliers.
h_m = holding cost per unit per unit time for manufacturer.
h_r = holding cost per unit per unit time for retailers.
A_s = ordering cost for suppliers.
A_m = ordering cost for manufacturer.
h_r = ordering cost for retailers.
id_s = idle cost per unit time for suppliers.
id_m = idle cost per unit time for manufacturer.
id_r = idle cost per unit time for retailers.
n = number of cycle for retailers.
r = number of cycle where manufacturer stop the production.

M = retailers trade credit period offered by the manufacturer to the retailers in years.
I_p = interest payable to the manufacturer by the retailers.
$I_{re}, \tilde{I_{re}}$ = interest earned by the retailers in crisp and fuzzy environment respectively.
$q_s(t)$ = inventory level of suppliers in time [0,T].
$q_m(t)$ = inventory level of manufacturer in time [0,T].
$q_r(t)$ = inventory level of retailers in time [0,T].
ATP = average total profit for the integrated models.
p_{m^*} = optimum value of p_m for integrated models.
ATP^* = optimum value of average total profit for the integrated models.

4. Model Description and Diagrammatic Representation.

The integrated inventory model in Fig.-1 starts with stock zero at $t = 0$. At that time the supplier starts his production with constant rate p_s unit per unit time and sales to the manufacturer at the rate p_m unit per unit time, when $t = t_s$ suppliers stop his production and at $t = T_s$ the inventory level of suppliers become zero. The total time of the integrated model is T, so the idle time for supplier is $T - T_s$. Similarly manufacturer starts his production at the same time $t = 0$ with production rate p_m unit per unit time and sales this production at the rate D_R unit per unit time to the retailer in the time gap T_R, which is the bulk pattern. At time $t = T_s$ manufacturer stop the production and at $t = (n+1)T_R$ ($n = [\frac{p_m T_s}{D_R}]$) stock of manufacturer is zero. So idle period for manufacturer is $T - (n+1)T_R$. Retailer start his business of this production to the customer at time $t = T_R$ and end at $T = (n+1)T_R + \frac{p_m T_s - nD_R}{D_c}$. The idle period for retailer is T_R.

5. Mathematical Formulation of the Model

5.1. **Formulation of suppliers individual average profit.** Differential equation for the supplier in Fig.-2 in [0,T] is given by

$$\frac{dq_s}{dt} = \begin{cases} p_s - p_m, & 0 \le t \le t_s \\ -p_m, & t_s \le t \le T_s \end{cases}$$

with boundary condition $q_s(t) = 0, t = 0, T_s$. Solving the differential equation with boundary condition, we have

$$q_s(t) = \begin{cases} (p_s - p_m)t, & 0 \le t \le t_s \\ p_m(T_s - t) & t_s < t \le T_s \end{cases} \tag{7}$$

By continuity at $t = t_s$, we get $p_m T_s = p_s t_s$ and total unit produced by the supplier in $[0, t_s]$
$Q_s = p_s t_s = p_m T_s$(= Total demand during $[0, T_s]$)

$$\begin{aligned} H_s &= \text{Holding cost of supplier} \\ &= h_s\left[\int_0^{t_s}(p_s - p_m)t\,dt + \int_{t_s}^{T_s} p_m(T_s - t)dt\right] \\ &= h_s\left[\frac{p_s t_s^2}{p_m} - p_s t_s^2\right] \end{aligned}$$

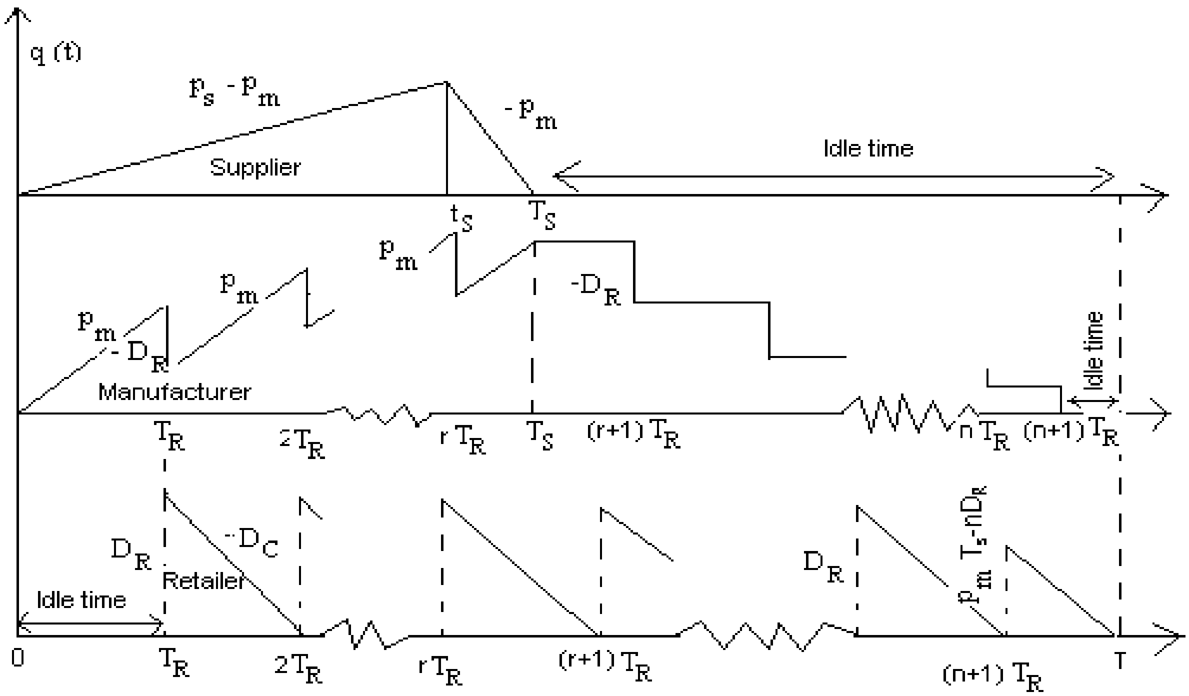

FIGURE 1. Inventory level for integrated model

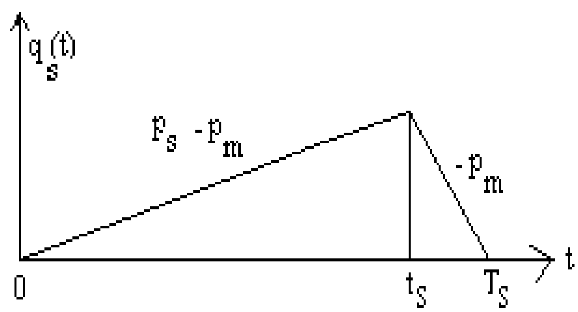

FIGURE 2. Inventory level of supplier

The idle cost of supplier= $id_s\big[T_R + p_s t_s\big(\frac{1}{D_c} - \frac{1}{p_m}\big)\big]$
Total purchase cost= $c_s p_m T_s$
Total selling price = $c_m p_m T_s$
and ordering cost is = A_s

$$
\begin{aligned}
\text{APS} &= \text{Average profit of supplier.} \\
&= \frac{1}{T}\ [\text{revenue from sale -(purchase + holding +idle +ordering) cost.}] \\
&= \frac{1}{T}[(c_m - c_s)p_s t_s - h_s(\frac{p_s^2 t_s^2}{p_m} - p_s t_s^2) - id_s(T_R + p_s t_s(\frac{1}{D_c} - \frac{1}{p_m})) - A_s] \qquad (8)
\end{aligned}
$$

5.2. **Formulation of manufacturer individual average profit.** Inventory level of manufacturer in Fig.-3 in [0,T] is given by

$$q_m(t) = \begin{cases} p_m t, & 0 \le t \le T_R \\ p_m t - iD_R, & iT_R < t \le (i+1)T_R, \text{ i=1,2,....,(r-1)} \\ p_m t - rD_R, & rT_R < t \le T_s \\ p_m T_s - rD_R, & T_s < t \le (r+1)T_R \\ p_m T_s - iD_R, & iT_R < t \le (i+1)T_R, \text{ i=r+1,r+2,....,n-1} \\ p_m T_s - nD_R, & nT_R < t \le (n+1)T_R \end{cases} \tag{9}$$

with boundary condition $q_m(0) = 0$, and $q_m(iT_R + 0) = q_m(iT_R) - D_R$

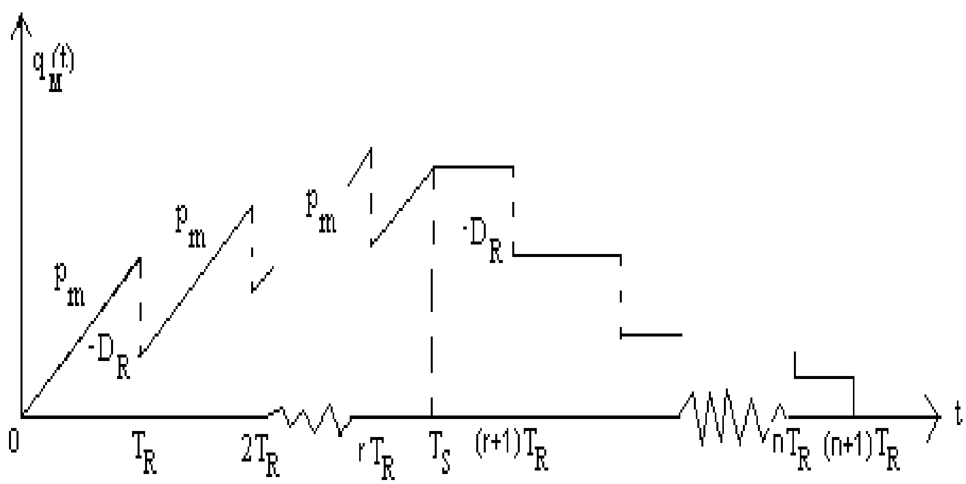

FIGURE 3. Inventory level of manufacturer

$$\begin{aligned} H_m &= \text{Holding cost for manufacturer.} \\ &= h_m\Big[\int_0^{T_R} p_m t dt + \sum_1^{r-1}\int_{iT_R}^{(i+1)T_R}(p_m t - iD_R)dt + \int_{rT_R}^{(T_s}(p_m t - rD_R)dt \\ &+ \int_{T_s}^{(r+1)T_R}(p_m T_s - rD_R)dt + \sum_{r+1}^{n-1}\int_{iT_R}^{(i+1)T_R}(p_m T_s - iD_R)dt + \int_{nT_R}^{(n+1)T_R}(p_m T_s - nD_R)dt\Big] \\ &= h_m\Big[np_m T_s T_R - \frac{n^2+n-2r-2}{2}T_R D_R - \frac{p_s^2 t_s^2}{2p_m}\Big] \end{aligned}$$

The idle cost of manufacturer= $id_m\left[\frac{p_m T_m - nD_R}{D_c}\right]$
Total purchase cost $= c_m p_m T_s$
Total selling price= $c_r p_m T_s$
and ordering cost $= A_m$

5.2.1. **Case-I (When** $M \leq T' \leq T_R$**).**

$$\begin{aligned}
I_{em} = I_{pr} &= \text{Amount of interest earned by the manufacturer in [0,T] from retailer.}\\
&= \text{Amount of interest paid by the retailer to the manufacturer in [0,T].}\\
&= c_r i_p\Big[n\int_M^{T_R}(D_R - D_c t)dt + \int_M^{T'}(p_m T_s - nD_R - D_c t)dt\Big]\\
&= \frac{nc_r i_p}{2}\big[T_R D_R + D_c M^2 - 2MD_R\big] + c_r i_p\Big[\big(\frac{(p_m T_s - nD_R)^2}{2D_c}\\
&+ (p_m T_s - nD_R)M + \frac{D_c M^2}{2}\Big]
\end{aligned}$$

$$\begin{aligned}
APM_1 &= \text{Average profit of Manufacturer.}\\
&= \frac{1}{T}\ \text{[revenue from sale -purchase cost-holding cost-idle cost}\\
&+ \text{earned interest-ordering cost.]}\\
&= \frac{1}{T}\Big[(c_r - c_m)p_m T_s - h_m\Big(np_m T_s T_R - \frac{n^2+n-2r-2}{2}T_R D_R - \frac{p_s^2 t_s^2}{2p_m}\Big)\\
&- id_m\Big(\frac{p_m T_m - nD_R}{D_c}\Big) + \frac{nc_r i_p}{2}\big[T_R D_R + D_c M^2 - 2MD_R\big]\\
&+ c_r i_p\Big(\frac{(p_m T_s - nD_R)^2}{2D_c} + (p_m T_s - nD_R)M + \frac{D_c M^2}{2}\Big) - A_m\Big] \qquad (10)
\end{aligned}$$

5.2.2. **Case-II (When** $T' \leq M \leq T_R$**).**

$$\begin{aligned}
I_{em} = I_{pr} &= \text{Amount of interest earned by the manufacturer in [0,T] from retailer.}\\
&= \text{Amount of interest paid by the retailer to the manufacturer in [0,T].}\\
&= c_r i_p\Big[n\int_M^{T_R}(D_R - D_c t)dt\Big]\\
&= \frac{nc_r i_p}{2}\big[T_R D_R + D_c M^2 - 2MD_R\big]
\end{aligned}$$

$$\begin{aligned}
APM_2 &= \text{Average profit of Manufacturer.}\\
&= \frac{1}{T}\ \text{[revenue from sale -purchase cost-holding cost-idle cost}\\
&+ \text{earned interest-ordering cost.]}\\
&= \frac{1}{T}\Big[(c_r - c_m)p_m T_s - h_m\Big(np_m T_s T_R - \frac{n^2+n-2r-2}{2}T_R D_R - \frac{p_s^2 t_s^2}{2p_m}\Big)\\
&- id_m\Big(\frac{p_m T_m - nD_R}{D_c}\Big) + \frac{nc_r i_p}{2}\big[T_R D_R + D_c M^2 - 2MD_R\big] - A_m\Big] \qquad (11)
\end{aligned}$$

5.3. **Formulation of Retailer individual average profit.** Inventory level of retailer in Fig.-4 in [0,T] is given by

$$q_r(t) = \begin{cases} D_c t, & iT_R \leq t \leq (i+1)T_R \\ p_m T_s - nD_R - D_c t, & (n+1)T_R \leq t \leq T \end{cases} \qquad (12)$$

with boundary condition $q_r((n+1)T_R) = 0$, and $q_r(T) = 0$

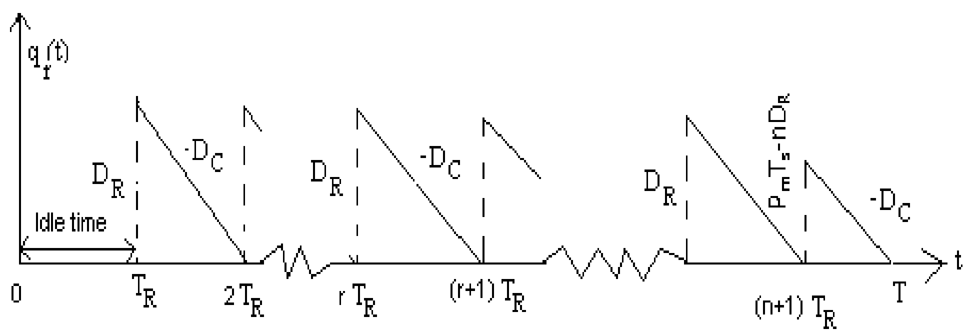

FIGURE 4. Inventory level of retailer

$$\begin{aligned} H_r &= \text{Holding cost of retailer.} \\ &= nh_r\left[\int_0^{T_R}(D_R - D_c t)dt + \int_0^{T'}(p_m T_s - nD_R - D_c t)dt\right] \\ &= \frac{h_r}{2}\left[\frac{p_m^2 T_s^2}{D_c} - 2np_m T_s T_R - (2n+1)T_R D_R\right] \end{aligned}$$

The idle cost of retailer$= id_r T_R$
Total purchase cost $= c_r p_m T_s$
Total selling price $= c_{r1} p_m T_s$
and ordering cost $= A_r$

5.3.1. **Case-I (When $M \leq T' \leq T_R$).** Interest earned by the retailers for $(n+1)$ cycle is given by

$$\begin{aligned} I_{er} &= \text{Amount of interest earned by the retailer from Bank in (n+1) cycle.} \\ &= (n+1)c_{r_1} i_e\left[\int_0^M (M-t)D_c dt\right] \\ &= \frac{(n+1)c_{r_1} i_e D_c M^2}{2}, \end{aligned}$$

$$
\begin{aligned}
I_{pr} &= \text{Amount of interest paid by the retailer to the manufacturer in [0,T].}\\
&= c_r i_p\left[n\int_M^{T_R}(D_R - D_c t)dt + \int_M^{T'}(p_m T_s - nD_R - D_c t)dt\right]\\
&= nc_r i_p\left[\frac{T_R D_R + D_c M^2 - 2MD_R}{2}\right] + c_r i_p\left[\frac{(p_m T_s - nD_R)^2}{2D_c}\right.\\
&+ \left.(p_m T_s - nD_R)M + \frac{D_c M^2}{2}\right],\\
APR_1 &= \text{Average profit of retailer.}\\
&= \frac{1}{T}[\text{revenue from sale -purchase cost-holding cost}\\
&+ \text{earned interest-payable interest-idle cost-ordering cost}]\\
&= \frac{1}{T}\left[c_{r_1}p_m T_s - c_r p_m T_s - \frac{h_r}{2}\left(\frac{p_m^2 T_s^2}{D_c} - 2np_m T_s T_R - (2n+1)T_R D_R\right)\right.\\
&+ \frac{(n+1)c_{r_1} i_e D_c M^2}{2} - \frac{nc_r i_p}{2}\left[T_R D_R + D_c M^2 - 2MD_R\right] - id_r T_R - A_r\\
&+ \left.(p_m T_s - nD_R)M + \frac{D_c M^2}{2}\right) - c_r i_p\left(\frac{(p_m T_s - nD_R)^2}{2D_c}\right]
\end{aligned} \tag{13}
$$

5.3.2. **Case-II (When $T^{'} \le M \le T_R$).** Interest earned by the retailer for $(n+1)$ cycle

$$
\begin{aligned}
I_{er} &= \text{Amount of interest earned by the retailer from Bank in n cycle.}\\
&= c_{r_1} i_e\left[n\int_0^{M}(M-t)D_c dt + \int_0^{T'}(T^{'}-t)D_c dt + (M - T^{'})(p_m T_s - nD_R)\right]\\
&= \frac{nc_{r_1} i_e D_c M^2}{2} + \frac{c_{r_1} i_e}{2}(p_m T_s - nD_R)(2M - T^{'})
\end{aligned}
$$

Interest payable by the retailers for 1st n cycle is given by

$$
\begin{aligned}
I_{pr} &= \text{Amount of interest paid by the retailer to the manufacturer in [0,T].}\\
&= c_r i_p\left[n\int_M^{T_R}(D_R - D_c t)dt\right]\\
&= \frac{nc_r i_p}{2}\left[T_R D_R + D_c M^2 - 2MD_R\right]
\end{aligned}
$$

$$
\begin{aligned}
APR_2 &= \text{Average profit for retailer.}\\
&= \frac{1}{T}\ [\text{revenue from sale -purchase cost-holding cost}\\
&+ \text{earned interest-payable interest-idle cost-ordering cost}].\\
&= \frac{1}{T}\Bigg[(c_{r_1}-c_r)p_mT_s - \frac{h_r}{2}\bigg(\frac{p_m^2T_s^2}{D_c} - 2np_mT_sT_R - (2n+1)T_RD_R\bigg)\\
&+ \frac{nc_{r_1}i_eD_cM^2}{2} + \frac{c_{r_1}i_e}{2}(p_mT_s - nD_R)(2M - T')\\
&- \frac{nc_ri_p}{2}\big[T_RD_R + D_cM^2 - 2MD_R\big] - id_rT_R - A_r\Bigg] \qquad (14)
\end{aligned}
$$

6. Integrated model

6.1. In Crisp environment:

6.1.1. Case-I (when $M \le T' \le T_R$).

$$
\begin{aligned}
ATP_1 &= \text{Total average profit for integrated model}\\
&= APS + APM_1 + APR_1\\
&= \frac{D_c}{p_mT_s + D_R}\Bigg[(c_m - c_s)p_mT_s - h_s\bigg(\frac{p_st_s^2}{p_m} - p_st_s^2\bigg) - id_s\bigg(T_R + p_st_s(\frac{1}{D_c} - \frac{1}{p_m})\bigg)\\
&- A_s + (c_r - c_m)p_mT_s - h_m\bigg(np_mT_sT_R - \frac{n^2+n-2r-2}{2}T_RD_R - \frac{p_s^2t_s^2}{2p_m}\bigg)\\
&- id_m - \frac{p_mT_m - nD_R}{D_c} - A_m + (c_{r_1} - c_r)p_mT_s - \frac{h_r}{2}\bigg(\frac{p_m^2T_s^2}{D_c}\bigg)\\
&- 2np_mT_sT_R - (2n+1)T_RD_R + \frac{(n+1)c_{r_1}i_eD_cM^2}{2} - id_rT_R - A_r\Bigg]\\
&= \frac{D_c}{p_mT_s + D_R}\Bigg[\frac{h_m - h_s}{2}p_mT_s^2 + (\frac{h_s}{2p_s} - \frac{h_r}{2D_c})p_m^2T_s^2 + Ap_mT_s + B\Bigg] \qquad (15)
\end{aligned}
$$

where $p_mT_s = p_st_s$,

$$
\begin{aligned}
A &= (c_{r_1} - c_s) + n(h_r - h_m)T_R - \frac{id_m + id_s}{D_c} \qquad (16)\\
\text{and}\quad B &= \big[h_m\frac{n^2+n-2r-2}{2} + h_r\frac{2n+1}{2}\big]T_RD_R + \frac{(n+1)c_{r_1}i_eD_cM^2}{2}\\
&+ (nid_m - id_s - id_r)T_R + id_sT_s - (A_s + A_m + A_r)
\end{aligned}
$$

$$
\begin{aligned}
&\frac{d(ATP_1)}{dp_m} = 0\\
&\Rightarrow \bigg(\frac{h_s}{2p_s} - \frac{h_r}{2D_c}\bigg)p_m^2T_s^2 + 2\bigg(\frac{h_s}{2p_s} - \frac{h_r}{2D_c}\bigg)p_mT_sD_R + \frac{(h_m - h_s)}{2}T_sD_R + AD_R - B = 0\\
&\Rightarrow Xp_m^2T_s^2 + 2Xp_mT_sD_R + Y = 0
\end{aligned}
$$

where X and Y are given by

$$\begin{aligned} X &= \frac{h_s}{2p_s} - \frac{h_r}{2D_c}, \\ Y &= \frac{(h_m - h_s)}{2} T_s D_R + AD_R - B = 0 \end{aligned}$$

solving we get

$$P_m = \frac{1}{T_s}\left[\sqrt{D_R^2 - \frac{Y}{X}} - 1\right] \tag{17}$$

$$\frac{d^2(ATP_1)}{dp_m^2} < 0$$
$$\text{if } \; X < 0 \qquad \text{and} \quad XD_R^2 < Y$$

Therefore ATP_1 is concave if

$$X < 0 \qquad \text{and} \quad XD_R^2 < Y \tag{18}$$

6.1.2. **Case-II (When $T' \leq M \leq T_R$).**

$$\begin{aligned} ATP_2 &= \text{Total average profit for integrated model} \\ &= APS + APM_1 + APR_1 \\ &= \frac{D_c}{p_m T_s + D_R}\Bigg[(c_m - c_s)p_m T_s - h_s\left(\frac{p_s t_s^2}{p_m} - p_s t_s^2\right) - id_s\left(T_R + p_s t_s\left(\frac{1}{D_c} - \frac{1}{p_m}\right)\right) \\ &- h_m\left(np_m T_s T_R - \frac{n^2+n-2r-2}{2} T_R D_R - \frac{p_s^2 t_s^2}{2p_m}\right) - id_m\left(\frac{p_m T_m - nD_R}{D_c}\right) \\ &- A_m - A_s + (c_{r1} - c_r)p_m T_s - \frac{h_r}{2}\left(\frac{p_m^2 T_s^2}{D_c} - 2np_m T_s T_R - (2n+1)T_R D_R\right) \\ &+ (c_r - c_m)p_m T_s + \frac{nc_{r_1} i_e D_c M^2}{2} + \frac{c_{r_1} i_e}{2}(p_m T_s - nD_R)(2M - T) - id_r T_R - A_r\Bigg] \\ &= \frac{D_c}{p_m T_s + D_R}\left[\frac{h_m - h_s}{2} p_m T_s^2 + \left(\frac{h_s}{2p_s} - \frac{h_r}{2D_c}\right)p_m^2 T_s^2 + Ap_m T_s + C\right] \end{aligned} \tag{19}$$

where $p_m T_s = p_s t_s$, A is given in (16) and

$$C = \left(h_m \frac{n^2+n-2r-2}{2} + h_r \frac{2n+1}{2}\right)T_R D_R + \big(nid_m - id_s - id_r\big)T_R + id_s T_s \tag{20}$$

nonumber (21)

$$+ \quad \frac{nc_{r1} i_e D_c M^2}{2} + \frac{c_{r_1} i_e}{2}(p_m T_s - nD_R)(2M - T') - (A_s + A_m + A_r)$$

$$\begin{aligned} &\frac{d(ATP_2)}{dp_m} = 0 \\ &\Rightarrow \left(\frac{h_s}{2p_s} - \frac{h_r}{2D_c}\right)p_m^2 T_s^2 + 2\left(\frac{h_s}{2p_s} - \frac{h_r}{2D_c}\right)p_m T_s D_R + \frac{(h_m - h_s)}{2} T_s D_R + AD_R - C = 0 \\ &\Rightarrow Xp_m^2 T_s^2 + 2Xp_m T_s D_R + Y = 0 \end{aligned}$$

where X and Y are given by

$$\begin{aligned} X &= \frac{h_s}{2p_s} - \frac{h_r}{2D_c}, \\ Y &= \frac{(h_m - h_s)}{2} T_s D_R + AD_R - C = 0 \end{aligned}$$

solving we get

$$P_m = \frac{1}{T_s}\left[\sqrt{D_R^2 - \frac{Y}{X}} - 1\right] \tag{22}$$

$$\frac{d^2(ATP_2)}{dp_m^2} < 0$$
$$\text{if} \quad X < 0 \quad \text{and} \quad XD_R^2 < Y$$

$$\text{Therefore } ATP_2 \text{ is concave if } \quad X < 0 \quad \text{and} \quad XD_R^2 < Y \tag{23}$$

6.2. **In fuzzy Environment:** We consider $\tilde{I_{re}} = (I_{re_1}, I_{re_2}, I_{re_3})$ be a triangular fuzzy number. Then the objective reduce to
6.2.1 For case-I $M \le T' \le T_R$

$$\begin{aligned} A\tilde{T}P_1 &= \frac{D_c}{p_m T_s + D_R}\Bigg[(c_m - c_s)p_m T_s - h_s\left(\frac{p_s t_s^2}{p_m} - p_s t_s^2\right) - id_s\left(T_R + p_s t_s\left(\frac{1}{D_c} - \frac{1}{p_m}\right)\right) \\ &- A_s + (c_r - c_m)p_m T_s - h_m\left(np_m T_s T_R - \frac{n^2+n-2r-2}{2}T_R D_R - \frac{p_s^2 t_s^2}{2p_m}\right) \\ &- A_m + (c_{r_1} - c_r)p_m T_s - \frac{h_r}{2}\left(\frac{p_m^2 T_s^2}{D_c} - 2np_m T_s T_R - (2n+1)T_R D_R\right) \\ &- id_m\left(\frac{p_m T_m - nD_R}{D_c}\right) + \frac{(n+1)c_{r_1}\tilde{I_{re}}]D_c M^2}{2} - id_r T_R - A_r\Bigg] \\ &= \frac{D_c}{p_m T_s + D_R}\left[\frac{h_m - h_s}{2}p_m T_s^2 + \left(\frac{h_s}{2p_s} - \frac{h_r}{2D_c}\right)p_m^2 T_s^2 + Ap_m T_s + B_1\right] \end{aligned} \tag{24}$$

where $p_m T_s = p_s t_s$, A is given by (16)

$$\begin{aligned} B_1 &= \left[h_m\frac{n^2+n-2r-2}{2} + h_r\frac{2n+1}{2}\right]T_R D_R + \frac{(n+1)c_{r_1}E[\tilde{I_{re}}]D_c M^2}{2} \\ &+ (nid_m - id_s - id_r)T_R + id_s T_s - (A_s + A_m + A_r) \end{aligned} \tag{25}$$

6.2.1. **For case-II ($T^{'} \leq M \leq T_R$).**

$$
\begin{aligned}
A\tilde{T}P_2 &= \frac{D_c}{p_mT_s+D_R}\Bigg[(c_m-c_s)p_mT_s-h_s\Big(\frac{p_st_s^2}{p_m}-p_st_s^2\Big)-id_s\Big(T_R+p_st_s(\frac{1}{D_c}-\frac{1}{p_m})\Big)\\
&- (A_m+A_s)+(c_{r1}-c_m)p_mT_s-h_m\Big(np_mT_sT_R-\frac{n^2+n-2r-2}{2}T_RD_R-\frac{p_s^2t_s^2}{2p_m}\Big)\\
&- id_m\Big(\frac{p_mT_m-nD_R}{D_c}\Big)-\frac{h_r}{2}\Big(\frac{p_m^2T_s^2}{D_c}-2np_mT_sT_R-(2n+1)T_RD_R\Big)\\
&+ \frac{nc_{r_1}\tilde{I_{re}}D_cM^2}{2}+\frac{c_{r_1}\tilde{I_{re}}]}{2}(p_mT_s-nD_R)(2M-T^{'})-id_rT_R-A_r\Bigg]\\
&= \frac{D_c}{p_mT_s+D_R}\Bigg[\frac{h_m-h_s}{2}p_mT_s^2+(\frac{h_s}{2p_s}-\frac{h_r}{2D_c})p_m^2T_s^2+Ap_mT_s+C_1\Bigg] \qquad (26)
\end{aligned}
$$

where $p_mT_s = p_st_s$, A is given by (16)

$$
\begin{aligned}
A &= (c_{r_1}-c_s)+n(h_r-h_m)T_R-\frac{id_m+id_s}{D_c},\\
C_1 &= \big(h_m\frac{n^2+n-2r-2}{2}+h_r\frac{2n+1}{2}\big)T_RD_R+\big(nid_m-id_s-id_r\big)T_R+id_sT_s\\
&+ \frac{nc_{r1}\tilde{I_{re}}]D_cM^2}{2}+\frac{c_{r_1}\tilde{I_{re}}}{2}(p_mT_s-nD_R)(2M-T^{'})-(A_s+A_m+A_r)
\end{aligned}
$$

6.3. **Equivalent crisp Model:** Using lemma-1, We put $E[\tilde{\tilde{I_e}}] = \frac{1}{2}\big[(1-\rho)I_{re_1}+I_{re_2}+\rho I_{re_3}\big]$ where $0 < \rho < 1$. The expected total average profit

6.3.1. **For case-I ($M \leq T^{'} \leq T_R$).**

$$
\begin{aligned}
E[ATP_1] &= \frac{D_c}{p_mT_s+D_R}\Bigg[(c_m-c_s)p_mT_s-h_s\Big(\frac{p_st_s^2}{p_m}-p_st_s^2\Big)-id_s\Big(T_R+p_st_s(\frac{1}{D_c}-\frac{1}{p_m})\Big)\\
&- h_m\Big(np_mT_sT_R-\frac{n^2+n-2r-2}{2}T_RD_R-\frac{p_s^2t_s^2}{2p_m}\Big)-id_m\Big(\frac{p_mT_m-nD_R}{D_c}\Big)\\
&- A_m-A_s+(c_{r_1}-c_r)p_mT_s-\frac{h_r}{2}\Big(\frac{p_m^2T_s^2}{D_c}-2np_mT_sT_R-(2n+1)T_RD_R\Big)\\
&+ (c_r-c_m)p_mT_s+\frac{(n+1)c_{r_1}\big[(1-\rho)I_{re_1}+I_{re_2}+\rho I_{re_3}\big]D_cM^2}{4}-id_rT_R-A_r\Bigg]
\end{aligned}
$$

6.3.2. For case-II ($T^{'} \leq M \leq T_R$).

$$\begin{aligned}
E[ATP_2] &= \frac{D_c}{p_mT_s+D_R}\Bigg[(c_m-c_s)p_mT_s - h_s\Big(\frac{p_st_s^2}{p_m}-p_st_s^2\Big) - id_s\Big(T_R+p_st_s(\frac{1}{D_c}-\frac{1}{p_m})\Big)\\
&- h_m\Bigg[(np_mT_sT_R-\frac{n^2+n-2r-2}{2}T_RD_R-\frac{p_s^2t_s^2}{2p_m}\Big)-id_m\Bigg[(\frac{p_mT_m-nD_R}{D_c}\Big)\\
&- A_m+(c_{r1}-c_r)p_mT_s-\frac{h_r}{2}\Big(\frac{p_m^2T_s^2}{D_c}-2np_mT_sT_R-(2n+1)T_RD_R\Big)\\
&- A_s+(c_r-c_m)p_mT_s+\frac{nc_{r_1}\big[(1-\rho)I_{re_1}+I_{re_2}+\rho I_{re_3}\big]D_cM^2}{4}\\
&+ \frac{c_{r_1}\big[(1-\rho)I_{re_1}+I_{re_2}+\rho I_{re_3}\big]}{4}(p_mT_s-nD_R)(2M-T^{'})-id_rT_R-A_r\Bigg] \qquad (27)
\end{aligned}$$

7. Numerical Example

7.1. Crisp Environment.

Table-1: Input data of different parameter in crisp environment for case-I and case-II

$Parameter$	$case-I$	$case-II$		$parameter$	$case-I$	$case-II$
c_s	8	8		D_R	120	120
c_m	14	14		D_C	50	50
c_r	25	25		id_s	1	1
c_{r_1}	30	30		id_m	2	2
h_s	.05	.05		id_r	3	3
h_m	.1	.1		A_s	20	20
h_r	.2	.2		A_m	30	30
T_s	10	10		A_r	40	40
p_s	150	150		i_e	.09	.09
n	5	6		M	1.6	2
r	4	4		i_p	.1	.1

Table-2:Optimal values of objective and decision variable in crisp environment for case-I and case-II

$Parameter$	$case-I$	$case-II$		$parameter$	$case-I$	$case-II$
ATP^*	1039.68	1107.91		APS^*	249.28	253.92
p_m^*	70.81	79.34		APM^*	468.96	463.62
$T^{'}$	2.16	1.46		APR^*	321.45	390.36

7.2. Equivalent Crisp Environment.

Table-3: Input data of different parameter in equivalent crisp environment for case-I and case-II

$Parameter$	$case-I$	$case-II$	$parameter$	$case-I$	$case-II$
c_s	8	8	D_R	120	120
c_m	14	14	D_C	50	50
c_r	25	25	id_s	1	1
c_{r1}	30	30	id_m	2	2
h_s	.05	.05	id_r	3	3
h_m	.1	.1	A_s	20	20
h_r	.2	.2	A_m	30	30
T_s	10	10	A_r	40	40
p_s	150	150	$\tilde{\tilde{I}}_e$	(.08, .09, .11)	(.08, .09, .11)
n	5	6	M	1.6	2
r	4	4	i_p	.1	.1
ρ	.5	.5			

Table-4: Optimal values of objective and decision variable in equivalent crisp environment for case-I & case-II

$Parameter$	$case-I$	$case-II$	$parameter$	$case-I$	$case-II$
ATP^*	1041.43	1110.75	APS^*	248.71	253.91
p_m^*	69.86	79.31	APM^*	467.76	463.61
$T^{'}$	1.97	1.46	APR^*	324.96	393.23

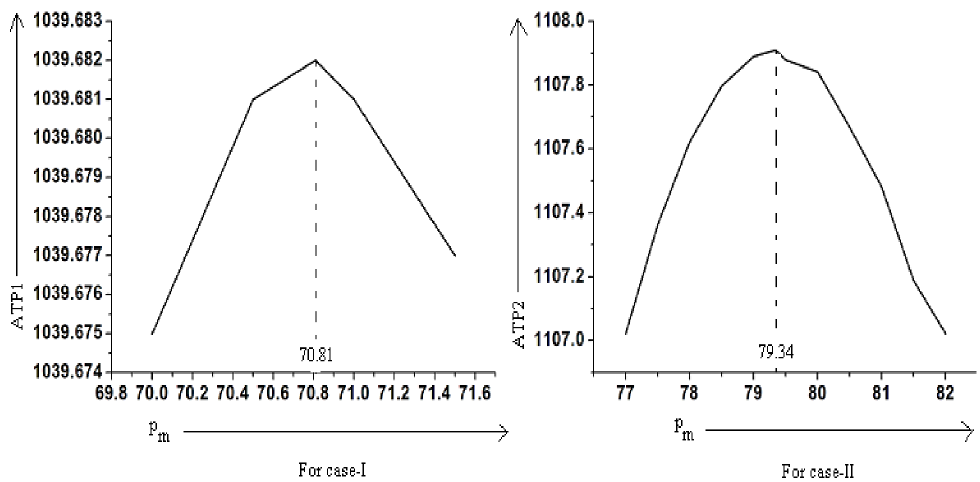

Figure 5. Average profit versus supply rate of supplier's in case-I and case-II

8. **Discussion:**

From table-2 and table-4, we see that in both environment optimal individual profits for the supplier and retailers are better in case-II(i.e. when $T^{'} \leq M \leq T_R$) then case-I (i.e. when $T^{'} \leq M \leq T_R$).But manufacturer individual profit is less in case-II then case-I.This is due to large delay in payment,retailer earned interest is greater for greater value of delay in payment. Concave nature of the integrated model is shown analytically and graphically in Fig.-5 for the both cases.

9. Conclusion:

In this model, we developed production inventory three layers supply chain model under fuzzy environment. Here suppliers is also manufacturer, collect the row material(ore) and produce the row material of actual manufacture. For example, In petrochem industries suppliers collect the ore and produced the naphthalene , which is the row material of manufacturer. Then manufacturer produce the usable product to sale the retailer.

In this paper, we have developed a production inventory supply chain model under fuzzy environment. The paper can be extended to imperfect production inventory system. Deterioration can be allowed for produced items of retailer, manufacturer and also in case of retailer.

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References

[1] V. Agarwal, X. Chao, S. Seshadri, Dynamic balancing of inventory in supply chains, European Journal of operation Research. 159(2004)296-317.

[2] D. Dubois, H. Prade, The mean value of a fuzzy number, Fuzzy Sets and Systems. 24(1987)279-300.

[3] D. Dubois, H. Prade, Possibility Theory, New York, London, 1988.

[4] M. Inuiguchi, M. Sakawa, Y. Kume, The usefulness of possibility programming in production planning problems, International journal of Production Economics. 33(1994)45-52.

[5] G.P.Cachon, P.H.Zipkin, Competitive and cooperative inventory policies in a two-stage supply chain, Management Science. 45(1999) 936-953.

[6] S.K. Goyal, Economic order quantity under conditions of permissible delay in payments, Journal of the Operational research Society. 36 (1985)35-38.

[7] Y. Liang, F. Zhou, A two-warehouse inventory model for deteriorating items under conditionally permissible delay in payment, Applied Mathematical Modelling. 35 (2011) 2221-2231.

[8] B. Liu, Y.K. Liu, Expected value of Fuzzy variable and Fuzzy expected value Models, IEEE Transactions of Fuzzy Systems. 10(4)(2002)45-450.

[9] K. Maity, M. Maiti, A numerical approach to a multi-objective optimal inventory control problem for deteriorating multi-items under fuzzy inflation and discounting, Computers & Mathematics with Applications. 55(2008) 1794-1807.

[10] K. Maity, Possibility and necessity representations of fuzzy inequality and its application to two warehouse production-inventory problem, Applied Mathematical Modelling. 35(2011)1252-1263

[11] S. Mandal, A.K. Maity, K. Maity, S. Mondal, M. Maiti, Multi-item multi-period optimal production problem with variable preparation time in fuzzy stochastic environment, Applied Mathematical Modelling. 35(2011) 4341-4353

[12] A.M.Radzikowska, E.E.Kerre, A comparative study of rough sets, Fuzzy Sets and Systems. 126(2002) 137-155.

[13] Y. Wang, Mining stock price using fuzzy rough set system, Expert Systems with Applications. 24 (2003) 13-23.

[14] R. B. Bhatt, M. Gopal, On fuzzy-rough sets approach to feature selection, Pattern Recognition Letters. 26(2005) 965-975.

[15] S.Viswanathan, R. Piplani, Coordinating supply chain inventories through common replenishment epochs, European Journal of Operational Research, 129(2001) 277-286.

[16] J. Wang, Y.F. Shu, Fuzzy decision modelling for supply chain management, Fuzzy sets and systems. 150(2005) 107-127.

[17] L.A. Zadeh, Fuzzy sets as a basis for a theory of possibility, Fuzzy Sets and Systems. 1(1978)3-28.

[18] L.A. Zadeh, Fuzzy Sets, Information and Control. 8(1965)338-356.

[19] H.J. Zimmermann, Description and Optimization of Fuzzy System, International Journal of General Systems. 2(1985)209-215.