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Reconstruction method for conformal mapping in two dimensions

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Abstract:

In this paper, we proposed a conformal mapping tool for reconstruction a part of boundary curve of a two -dimensional bounded domain from Cauchy data of a holomorphic function. The approach is based on determined a holomorphic function that maps a disk unit onto the unknown domain.The boundary values of this holomorphic function on the part of the circle are obtained from solving a non local differential Bessel's equation. Then the unknown boundary is found as image of the part of the circle by solving an ill-posed Cauchy problem for holomorphic functions via a regularized power expansion. In this paper, the case of a homogeneous Dirichlet condition on one part of the boundary was considered. And in the either part of the boundary, the Cauchy data were restricted to a non vanishing function and to a normal derivatives without zeros. Our analysis includes convergence result by minimizing a functional to obtain a numerical approximation to the unknown boundary. Numerical examples show that the proposed method is feasible.

Mathematics Subject classification: 34K29, 30C30,31A25,45Q05, 65E05

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1. Introduction

A conformal (or angle-preserving) map between two domains is a function which preserves oriented angles between curves as well as their direction. Such function preserves both angles and the shapes of infinitesimally small figures, but not necessarily their size. Conformal mapping has for more then a century, been powerful tool in mathematics, engineering, physics and a lot of other subjects of the science, especially in solving various partial differential equations (PDEs).
Classical applications of conformal mapping to many stationary problems of mathematical physics go back over a century and continue to the present. As was indicated, these applications usually deal with solutions of Laplace's equation which remains invariant if the original plane is subject to a conformal transformation. Applications of conformal mapping techniques in certain areas, such as the mathematical theory of elasticity, are considerably more complex. In other cases the physics may not be the problem, but the shape of the boundary is.

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There are various classes of conformal mappings that frequently arise in applications. Some of these are: Moebius Transformations see [6], Schwarz-Christoffel Mapping see [7] and Riemann Map see [21].
General methods of approximate conformal map building can be found in survey works by [15], [16] and [25].

One major approach in developing methods for numerical conformal mapping is based on the following interpretation of the Riemann mapping theorem see [21] page 173, there exists a conformal mapping $\phi : D \longmapsto \Omega$ with $\phi(z_0) = 0$ and $\phi^{'}(z_0) \neq 0$, where $z_0 \in D$, and this function has a power series expansion $\phi(z) = a_0(z - z_0) + \sum_{n=1}^{\infty} a_n(z - z_0)^n$, with $a_0 \neq 0$ and $z_0 \in D$, which converges uniformly in very closed disk which center z_0 and contained in D.

The inverse problems we are concerned with is to determine the unknown part of boundary curve $\Gamma = \partial\Omega\backslash\Gamma_0$ from the Cauchy data

$$u = g \quad \text{on} \quad \Gamma_0 \quad \text{and} \quad \frac{\partial u}{\partial \nu} = h \quad \text{on} \quad \Gamma_0 \tag{1}$$

on Γ_0 of a solution $u \in H^1(\Omega)$ of the Laplace equation

$$-\triangle u = \lambda u \quad \text{in} \quad \Omega \tag{2}$$

satisfying the condition

$$u = g \quad \text{on} \quad \Gamma \tag{3}$$

where g is a given positive continuous function prescribed on Γ_{D_1}, h a given definite negative bounded continuous function, the corresponding Neumann data measured on Γ_0 and ν is outer normal vector unit.
The main purpose of this paper is to present the iterative methods for solving the inverse problem (1)-(2)-(3) via using ideas from conformal mapping as develop over the last decade. For references therein see [9], [15], [16]. In this approach, the reconstruction of the boundary curve $\Gamma = \partial\Omega\backslash\Gamma_0$ from over determined Cauchy data on the accessible boundary Γ_0 is based on a conformal map $\Phi : D_1 \mapsto \Omega$ that takes the disk D_1 with radius one centered at the origin which boundary $\Gamma_{D_1} = \Gamma_1 \cup \Gamma_2$ onto Ω. The cauchy-Riemann equations provide a nonlinear ordinary differential equations via Bessel equations for the boundary values $\psi|_{\Gamma_2}$ on this part of circle that can be solved by successive approximations. This is then followed by the solution of a several ill-posed Cauchy problem for the holomorphic function Φ by regularized a power series expansion to retrieve the unknown part of the boundary curve via $\Phi(\Gamma_1) = \Gamma$. We present the foundations of this algorithm including a convergence analysis result.

2. Presentation of the conformal mapping method

In the sequel we identify the Euclidian space $\mathbb{R}^2$ and the complex plane $\mathbb{C}$ in the usual way. We introduce the unit disk D_1 centered at zero which boundary Γ_{D_1} is divided by two arcs

$\Gamma_{D_1} = \Gamma_1 \cup \Gamma_2$. Introduce the notations:

$$\begin{aligned} \Gamma_1 &= \{e^{i\theta},\ \theta \in [\alpha\pi, 2\pi],\ 0 < \alpha < 2\} \\ \Gamma_2 &= \{e^{i\theta},\ \theta \in [0, \alpha\pi],\ 0 < \alpha < 2\}. \end{aligned}$$

We denote by $\Phi : D_1 \mapsto \Omega$ a bijective holomorphic function that conformally maps D_1 onto Ω such that Γ_2 and Γ_1 are mapped onto Γ_0 and Γ respectively.
We also parameterize

$$\begin{aligned} \Gamma_0 &= \{\gamma(\theta),\ \theta \in [0, \alpha\pi]\} \\ \Gamma &= \{\gamma(\theta),\ \theta \in [\alpha\pi, 2\pi]\}. \end{aligned}$$

by a continuous differentiable 2π periodic function $\gamma : \mathbb{R} \mapsto \mathbb{C}$ with the property that $\gamma|_{[0,2\pi]}$ is injective. The latter, in particular, implies that $|\gamma(\theta)|^{'} \neq 0$ for all $\theta \in [0, 2\pi]$. We can normalize the mapping Φ by prescribing $\Phi(1) = \gamma(0)$, i.e by relating two points on the outer boundaries Γ_2 and Γ_0. Assuming the counter-clockwise orientation, there exists a strictly monotonically bijective boundary correspondence function $\phi : [0, 2\pi] \mapsto [0, \alpha\pi]$ by setting

$$\Phi(e^{it}) := \gamma o \phi(t),\ \ t \in [0, \alpha\pi]$$

Note that the function ϕ maps arc length Γ_2 onto arc length on the data boundary Γ_0 of Ω and ϕ is define by

$$\phi(t) = \frac{\alpha\pi}{2\pi} t + \Phi(t)$$

where Φ is a 2π periodic function with $\Phi(0) = 0$. Clearly, since Γ_0 , that is γ is given, knowing the function ϕ is equivalent to knowing of the holomorphic function $\Phi|_{\Gamma_1}$.
Define the operator

$$A : H^{\frac{1}{2}}([0, \alpha\pi]) \mapsto H^{-\frac{1}{2}}([0, \alpha\pi])$$

in the usual trace spaces see [14] that maps the Dirichlet data $\hat{u}_{|\Gamma_2}$ *on* Γ_2 in the sense of traces theorem of a harmonic function $\hat{u} \in H_0^1(D_1)$ with Dirichlet data $\hat{u}_{|\Gamma_1} = g$ *on* Γ_1 onto its normal derivative $\frac{\partial \hat{u}}{\partial n}$ *on* Γ_2.
For the solution u to the problem (1)-(2)-(3), then $\hat{u} := u o \phi,$ *in* D_1 is a harmonic function i.e For $z \in D_1, z = re^{i\theta}$, $0 < r < 1$ we have $\hat{u}(z) = u o \Phi(z)$. We have also $\Delta \hat{u}(z) = |\Phi^{'}(z)|^2 \Delta u(\Phi(z)) = -\lambda |\Phi^{'}(z)|^2 \hat{u}(z)$ and satisfying the boundary conditions:

1. if $z \in \Gamma_2 \cup \Gamma_1$ then $z = e^{i\theta}$, $\hat{u}(e^{i\theta}) = u o \Phi(e^{i\theta}) = g o \Phi(e^{i\theta}) = \hat{g}$, as we know $u = g$ *on* $\Gamma_0 \cup \Gamma$.
2. Using the Cauchy Riemann equations for u and $\hat{u}$ and their harmonic conjugates to derive the nonlocal differential equation. If $z \in \Gamma_2$, then $z = (e^{i\theta})$, $\frac{\partial \hat{u}}{\partial \nu} = \frac{\partial u}{\partial \nu}(\Phi(e^{i\theta}))\Phi^{'}((e^{i\theta}))$

we deduce the following relation

$$|\Phi^{'}((e^{i\theta}))| = \frac{|\frac{\partial \hat{u}}{\partial \nu}|}{|\frac{\partial u}{\partial \nu}(\Phi(e^{i\theta}))|}$$

We obtain $\hat{u} \in H_0^1(D_1)$ solution of the following problem

$$\begin{cases} -\triangle \hat{u} &= \lambda|\Phi^{'}(z)|^2\hat{u}(z) \quad \text{in} \quad D_1 \\ \hat{u} &= \hat{g} \quad \text{on} \quad \Gamma_2 \cup \Gamma_1 \\ \dfrac{\partial \hat{u}}{\partial \nu} &= \frac{\partial u}{\partial \nu}(\Phi(e^{i\theta}))\Phi^{'}((e^{i\theta})) \quad \text{on} \quad \Gamma_2, \end{cases} \tag{4}$$

Then, in polar coordinates, the solution $\hat{u}$ of the Dirichlet problem in the disk D_1 with boundary $\hat{u} = \hat{g}$ *on* $\Gamma_2 \cup \Gamma_1$ is given by the expansion

$$\hat{u}(r,\theta) = \sum_{k=0}^{+\infty} \hat{u_k}(r)e^{ik\theta}$$

The expression of $-\triangle \hat{u}(r,\theta)$ is given by

$$\begin{aligned} -\triangle \hat{u}(r,\theta) &= -\frac{1}{r}\frac{\partial \widehat{u}}{\partial r} - \frac{\partial^2 \widehat{u}}{\partial r^2} - \frac{1}{r^2}\frac{\partial^2 \widehat{u}}{\partial \theta}^2 \\ &= -\widehat{u}_k^{''} - \frac{\widehat{u}_k^{'}}{r} + \frac{k^2\widehat{u}_k}{r^2} = \widehat{\lambda}\widehat{u}_k \end{aligned}$$

Boundaries condition in polar coordinates of the problem (4) are expressed by

$$\widehat{u}(1,\theta) = \sum_{k=0}^{+\infty} \widehat{g}_k(1)e^{ik\theta} = \widehat{g}(1,\theta) \text{ on } \Gamma_2 \cup \Gamma_1$$

In the sequel, we assume that $g = 0$ this implies that $\widehat{u_k}(1) = 0, \ \forall\, k \in \mathbb{N}$ on the boundary $\Gamma_2 \cup \Gamma_1$.
We denote by $\widehat{u}_k = \widehat{u}(r)$ the solution of the following problem

$$\begin{aligned} -\widehat{u}_k^{''} - \tfrac{\widehat{u}_k^{'}}{r} + \tfrac{k^2\widehat{u}_k}{r^2} &= \widehat{\lambda}\widehat{u}_k \\ \widehat{u}_k(1) &= 0 \qquad \forall k, \text{ on } \quad \Gamma_2 \cup \Gamma_1 \end{aligned} \tag{5}$$

The first line of the equation (5) is not properly Bessel equation. To obtain the standard form of Bessel equation, we put $w = lr$ by taking $\widehat{\lambda} = l^2$, then we obtain Bessel equation of order k

$$w^2\widehat{u}_k^{''} + w\widehat{u}_k^{'} + (w^2 - k^2)\widehat{u}_k = 0 \tag{6}$$

The equation (6) admits a solution see [23] denoted by

$$u_k(w) = AJ_k(w) + BY_k(w).$$

where $J_k = \sum\limits_{p=0}^{+\infty} \frac{(-1)^p}{p!(k+p)!}(\frac{x}{2})^{2p+k}$ is the Bessel function of the first kind of order k and $Y_k = \lim\limits_{m\to k} \frac{J_m(x)\cos(m\pi) - J_{-m}(x)}{\sin(m\pi)}$ is the Bessel function of the second kind of order k.
The asymptotic behavior of J_k as w goes to zero is $J_k \simeq 1, if\ k = 0$ and $J_k \simeq \frac{w^k}{2^k n!},\ if,\ k > 0$. And $Y_k(w) \simeq \frac{2}{\pi}\ln w,\ if\ k = 0$ and $\frac{-2^k(k-1)!}{\pi}w^{-k},\ if,\ k > 0$

Note that the behavior of J_k and Y_k appear to be similar to $\sin w$ and $\cos w$ for large w except that oscillations of J_k and Y_k decay to zero, see [23].
Note that J_k goes to zero as w goes to zero while Y_k has a logarithmic singularity at zero, and we have necessarily $B = 0$, see [23]. The condition $\widehat{u}_k(l) = 0$ on $\Gamma_2 \cup \Gamma_1$ means that $J_k(l) = 0,\ \forall k,\ on\ \Gamma_2 \cup \Gamma_1$. According the eigenvalue l_{k_n}, we can take $l_{k_n} = w_{k_n}$.
The limit condition $\widehat{u}_k(l) = 0,\ \forall k\, on\ \Gamma_2 \cup \Gamma_1$ can be rewritten by $J_k(l_{kn}) = 0,\ \forall k,\ on\ \Gamma_2 \cup \Gamma_1$.
We obtain the following problem

$$\begin{array}{lcll} \widehat{u}_{k_n}(r) & = & AJ_k(l_{k_n} r) & \\ \widehat{u}_{k_n} & = & 0 & \forall k, \text{ on } \quad \Gamma_2 \cup \Gamma_1 \end{array} \tag{7}$$

So, we can rewrite the solution to the problem (4), we have

$$\widehat{u}_{k_n}(r,\theta) = A_{kn} J_k(r l_{k_n}) e^{ik\theta}.$$

On Γ_2,

$$\begin{array}{rcl} \dfrac{\partial \hat{u}}{\partial \nu} & = & h(\Phi(e^{i\theta}))\Phi^{'}(e^{i\theta}) = \dfrac{\partial \widehat{u}_{k_n}}{\partial r}(1,\theta) \\ \text{Or } \dfrac{\partial \widehat{u}_{k_n}}{\partial r} & = & A_{kn} l_{k_n} J_k^{'}(r l_{k_n}) e^{ik\theta}, \\ \text{we deduce } \Phi^{'}(e^{i\theta}) & = & \dfrac{A_{kn} l_{kn} J_k^{'}(l_{k_n}) e^{ik\theta}}{h(\Phi(e^{i\theta}))} \end{array}$$

According the above expression, with

$$\begin{array}{rcl} J_k(x) & = & \dfrac{1}{\pi}\displaystyle\int_0^{\pi} \cos(k\tau - x\sin\tau) d\tau \\ A_{k,n}^2 & = & \dfrac{1}{2\pi \int_0^1 J_k^2(l_{k,n} r) r dr}, \quad \forall k \geq 0, \ \ n \geq 1 \end{array}$$

we use the following result to give an approximate value of $|\Phi^{'}|$.

Proposition 2.1 *let' s take $f : [a,b] \to \mathbb{C}$ such that $\int_a^b f(x)dx < \infty$. Then*

$$\int_a^b f(x)dx \simeq (b-a) f(\frac{a+b}{2})$$

□

Then we have

$$\begin{array}{rcl} J_k^{'}(x) & \simeq & \sin(k\dfrac{\pi}{2} - x) \ \text{ and} \\ A_{k,n}^2 & \simeq & \dfrac{2\pi(2n-1+k)^2}{\pi^2(2n-1+k)^2 - 2(1+(-1)^k)}, \quad \forall k \geq 0, \ \ n \geq 1 \end{array}$$

We deduce

$$\Phi^{'}(e^{i\theta}) \simeq \frac{\frac{\frac{9\pi}{2}\sqrt{2\pi}}{\sqrt{9\pi^2-4}}}{h(\Phi(e^{i\theta}))} \text{ on } \Gamma_2$$

Let

$$C = \frac{\frac{9\pi}{2}\sqrt{2\pi}}{\sqrt{9\pi^2-4}} \tag{8}$$

$$\Leftrightarrow \Phi^{'}(e^{i\theta})h(\Phi(e^{i\theta})) \simeq C \tag{9}$$

C is a number real et h is a real function then according above expression $\Phi^{'}(e^{i\theta})$ is a real function.
Therefore $\Im(\Phi^{'}(e^{i\theta})) = 0$,
So

$$\Phi(e^{i\theta}) = \Re(\Phi(e^{i\theta})) + i\rho \text{ with } \rho \text{ a constant.} \tag{10}$$

When Φ is known then Γ and u are determined by

$$\begin{aligned} u &= \widehat{u}o\Phi^{-1} \text{ and} \\ \Phi(\Gamma_1) &= \Gamma \\ \Phi(\Gamma_2) &= \Gamma_0 \end{aligned}$$

And inversely, if $\Omega,\ \Gamma_0$ and u are known then Φ will be determined by

$$\begin{aligned} \widehat{u} &= uo\Phi \text{ and} \\ \Phi^{'}(e^{i\theta})h(\Phi(e^{i\theta})) &= \frac{\partial\hat{u}}{\partial\nu} \quad \forall\ \theta \in \mathbb{R} \\ \Phi(\Gamma_2) &= \Gamma_0. \end{aligned}$$

Finally, determine Γ and u returns to find the injective conformal mapping Φ defined on D_1 such that for all $\alpha,\ 0 < \alpha < 2$, the following problem

$$\begin{cases} -\Delta\widehat{u} = \widehat{\lambda}\widehat{u} & \text{in } D_1, \text{ with } \widehat{\lambda} = \lambda|\Phi^{'}(z)|^2 \\ \widehat{u} = \widehat{g} & \text{on } \Gamma_2 \cup \Gamma_1 \\ \Phi(\Gamma_2) = \Gamma_0 & \\ \Phi^{'}(e^{i\theta})h(\Phi(e^{i\theta})) = \frac{\partial\hat{u}}{\partial\nu} & \text{on } \Gamma_2 \end{cases} \tag{11}$$

has solution.

Apparently, solving the problem (11) may suffer from difficulties arising from the determination of the holomorphic function Φ. To try to avoid these difficulties, we can proceed by the minimization of a functional J defined on a set S by the least squares method when h given, see([26] and [18]).

The problem consist to find Φ such that the problem (11) admits a solution. Let' s take the set

$$\mathbf{S} := \{\Phi \text{ holomorphic on } \overline{D}_1 \text{ bijective such that the problem (11) admits a solution.}\}$$

Let' s take $\Phi(z) = P(x, y) + iQ(x, y)$, in polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$, we rewrite $\Phi(z) = P(r, \theta) + iQ(r, \theta)$.
But we have

$$\Phi(z) = \Re(\Phi(z)) + i\rho \text{ on } \Gamma_2, \rho \text{ constant.}$$

Ask

$$F(z) = \Re(\Phi(z))$$

So we have

$$\Phi(z) = F(z) + i\rho \text{ on } \Gamma_2 \text{ with } \rho \text{ constant.} \tag{12}$$

So we have

$$F^{'}(e^{i\theta})ho\Phi(e^{i\theta}) \simeq C \text{ on } \Gamma_2$$

We can write the functional J in the following form

$$J(\Phi) = \int_0^{\alpha\pi} L(|h(\Phi(e^{i\theta}))|^2|F^{'}(e^{i\theta})|^2 - C^2)d\theta \text{ with } 0 < \alpha < 2$$

where the function L is define such that the functional J is coercive

$$\begin{array}{rcl} L : \mathbb{R} & \rightarrow & \mathbb{R}^+ \\ t & \mapsto & t^2 \\ L(t) = 0 & \Leftrightarrow & t = 0 \end{array}$$

L is even function and there exists $\beta > 0 : L(t) \geq \beta t^2$ for t big enough.
We have

$$J(\Phi) = \int_0^{\alpha\pi} (|h(\Phi(e^{i\theta}))|^2|F^{'}(e^{i\theta})|^2 - C^2)^2 d\theta \quad \text{with } 0 < \alpha < 2.$$

Let ' s take

$$\mathbf{H} := \{\Phi \text{ holomorphic on } \overline{D}_1\}.$$

As we know the functional J is bounded below by zero, therefore reaches its minimum in a Φ^* of the set $\mathbf{H}$, where $J(\Phi^*) = 0$. The existence of such conformal mapping Φ^* is due to the origin of the problem. Let ' s take Φ belonging to $\mathbf{H}$, then this function has a power series expansion on D_1,

$$\Phi(z) = \sum_0^{+\infty} a_n z^n$$

where $a_n = \alpha_n + i\beta_n$ with $\alpha_n, \beta_n \in \mathbb{R}$ and $z = re^{i\theta}$ with $0 < r \leq 1$.
In polar coordinates

$$F(re^{i\theta}) := \Re(\Phi(z)) = \alpha_0 + \sum_{n=1}^{+\infty} r^n(\alpha_n \cos n\theta - \beta_n \sin n\theta)$$

and Cauchy conditions imply

$$\rho := \Im(\Phi(z)) = \beta_0 + \sum_{n=0}^{+\infty} r^n(\alpha_n \sin n\theta + \beta_n \cos n\theta)$$

3. Minimization of the approximate functional

In this section we used a numerical method for building a conformal mapping to solve the problem (11). To do this we are working in finite dimension by introducing the space of trigonometric polynomials. Let' s take

$$\mathbf{H}_N := \{\Phi_N \text{ define on } \overline{D}_1 \text{ such that } \Phi_N(z) = \sum_{0}^{N} a_n^N z^n, \text{ where } a_n^N \in \mathbb{C},\ N \in \mathbb{N}\}$$

As we see H_N being a subset of H, it is a subspace of $\mathbb{R}^{2N+2}$ Let' s take $\Phi_N(z) = F_N(re^{i\theta}) + i\rho$ where

$$\begin{aligned} F_N(re^{i\theta}) &= \sum_{n=0}^{N} r^n(\alpha_n \cos n\theta - \beta_n \sin n\theta) \text{ and} \\ \rho &= \sum_{n=0}^{N} r^n(\alpha_n \sin n\theta + \beta_n \cos n\theta) \end{aligned}$$

In polar coordinates, we obtain

$$\begin{aligned} \Phi'_N(re^{i\theta}) &= \frac{\partial}{\partial x}P_N - i\frac{\partial}{\partial y}P_N \\ &= (\cos\theta\frac{\partial}{\partial r}P_N - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}P_N) - i(\sin\theta\frac{\partial}{\partial r}P_N + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}P_N) \end{aligned}$$

By developing and considering $\Phi_N(re^{i\theta}) = F_N(re^{i\theta}) + i\rho$, we have the following expression

$$F'_N(re^{i\theta}) = \sum_{n=0}^{N} nr^{n-1}(\alpha_n \cos(n-1)\theta - \beta_n \sin(n-1)\theta)$$

We deduce that minimize J on $\mathbf{H}_N$ is equivalent to minimize the functional J on $\mathbb{R}^{2N+2}$. Then the functional J is defined by

$$\begin{aligned} J : \mathbb{R}^{2N+2} &\to \mathbb{R} \\ \overrightarrow{X} &\mapsto J(\Phi_N) \end{aligned}$$

where $\overrightarrow{X} = (\alpha_0, \alpha_1, ..., \alpha_N, \beta_0, \beta_1, ..., \beta_N)$ and

$$J(\overrightarrow{X}) = \int_0^{\alpha\pi} (|h(\Phi_N(e^{i\theta}))|^2 |F_N'(e^{i\theta})|^2 - C^2)^2 d\theta.$$

The function Φ_N is determined if only if the coordinates of the vector $\overrightarrow{X}$ are known. We have the following result.

Proposition 3.1 *The functional $J(\overrightarrow{X})$ admits at least a local minimum in $\mathbb{R}^{2N+2}$, moreover $J'(\overrightarrow{X}) = 0$*

Preuve de la proposition 3.1 *we remark that if $|\overrightarrow{X}|_2$ goes to $+\infty$ then $(F_N')^2(e^{i\theta})$ goes to $+\infty$ then $(\Phi_N')^2(e^{i\theta})$ goes to $+\infty$.*
h definite bounded continuous function and C constant then $|h(\Phi_N(e^{i\theta})|^2 |F_N'(e^{i\theta})|^2 - C^2$ goes to $+\infty$ when $|\overrightarrow{X}|_2$ goes to $+\infty$. This implies that $\lim_{||\overrightarrow{X}||_2 \to +\infty} J(\overrightarrow{X}) = +\infty$.
According the definition of the function L, the functional J is continue on $\mathbb{R}^{2N+2}$, positive then bounded below by zero. Let's take $m = \inf_{\overrightarrow{X} \in \mathcal{R}^{2N+2}} J(\overrightarrow{X})$ then there exists a minimizing sequence $(\overrightarrow{X}_k)$ of $\mathbb{R}^{2N+2}$ such that $\lim_{k \to +\infty} J(\overrightarrow{X}_k) = m$.The sequence $(\overrightarrow{X}_k)_{k \in \mathbb{N}}$ is bounded if not for all $A > 0$ there exists $k' \in \mathbb{N}$ such that for all $k \geq k'$ we get $||\overrightarrow{X}_k|| > A$ either $\lim_{||\overrightarrow{X}_k||_2 \to +\infty} J(\overrightarrow{X}_k) = +\infty$ then $m = +\infty$ contradiction. Then there exists a subsequence $(\overrightarrow{X}_{k_i})_{k_i}$ converges weakly on $\overrightarrow{X} \in \mathbb{R}^{2N+2}$. In finite dimension this convergence is strong.
Let ' take $k = k_i$ then $J(\overrightarrow{X}_k)$ converges on $J(\overrightarrow{X})$, we deduce that $J(\overrightarrow{X}) = m$.
As the functional J is differentiable on the open $\mathbb{R}^{2N+2}$, then $J'(\overrightarrow{X}) = 0$. □

We assume that $\Phi_N(z)$ be a solution to the problem $\min_{\overrightarrow{X} \in \mathbb{R}^{2N+2}} J(\overrightarrow{X})$, on $\mathbf{H}_N$. Our difficulty is to show that the sequence $(\Phi_N)_N$ converges to Φ^*, where Φ^* is solution to the problem $\min_{\Phi \in \mathbf{H}} J(\Phi)$, or converges at least to Φ_0 such that $J'(\Phi_0) = 0$, that is a solution to the minimization problem on $\mathbf{H}$.

4. Convergence results

We assume that $\Phi(e^{i\theta})$ et $F'(e^{i\theta})$ exist such that the functional J is well posed. We have the following proposition.

Proposition 4.1 *Let's take $\Phi \in$ **H**. We assume that:*

1. *Φ is a diffeomorphism and $\mathcal{C}^2$ class, define on $D_1 \to \Omega$.*
2. *Γ_{D_1} is **Jordan** curve*
3. *For $n > 0$ $a_n = o(\frac{1}{n^3})$ where a_n is the nth coefficient of power expansion of Φ*

Then Φ^N converges uniformly to Φ^* on $\overline{D}_1$ where Φ^N is the regularization by truncating Φ^* that is $\Phi^N(z) = \sum_0^N a_n^* z^n$ and $\Phi^*(z) = \sum_0^{+\infty} a_n^* z^n$ for all $z \in D_1$

For the two first hypothesis see [17].

Preuve de la proposition 4.1 *We have*

$$\begin{aligned}
|\Phi^N(z) - \Phi^*(z)| &= |\sum_0^N a_n^* z^n - \sum_0^{+\infty} a_n^* z^n| \\
&\leq \sum_{n\geq N+1} |a_n^*||z^n| \\
&\leq \sum_{n\geq N+1} |a_n^*|
\end{aligned}$$

By hypothesis $a_n^* = o(\frac{1}{n^3})$, *then the term* $\sum_{n\geq N+1} |a_n^*|$ *is bounded above by* $c \sum_{n\geq N+1} |\frac{1}{n^3}|$, *where* c, *a constant not depends to* N.
The series $\sum_{n\in\mathbb{N}} \frac{1}{n^3}$ *converges, then* $\lim_{n\to+\infty} \sum_{n\geq N+1} \frac{1}{n^3} = 0$
Then the series $(\Phi^N(z), a_n^*)$ *converge normally* Φ^* *then uniformly to* Φ^* *on* $\overline{D}_1$. $\square$

We have the following result:

Proposition 4.2 $J(\Phi^N)$ *converges to* $J(\Phi^*)$ *when* N *goes to* $+\infty$ *and* $\lim_{N\to+\infty} J(\Phi_N) = 0$.

Preuve de la proposition 4.2 *let' s take*

$$J(\Phi^N) = \int_0^{\alpha\pi} L(|h(\Phi^N(e^{i\theta}))|^2 |(F^N)'(e^{i\theta})|^2 - C^2) d\theta.$$

On Γ_2, (Φ^N) *converges uniformly to* Φ^* *when* N *goes to* $+\infty$.
The real part $\Re\Phi$ *and the imaginary part* $\Im\Phi$ *are continuous functions then* $\Re\Phi^N$, $\Im\Phi^N$ *converges respectively uniformly to* $\Re\Phi^* = F^*$, $\Im\Phi^*$ *when* N *goes to infinity. We also have, in one part*

$$\begin{aligned}
|(F^N)'(e^{i\theta}) - (F^*)'(e^{i\theta})| &= |(\sum_0^N a_n^* e^{in\theta})' - (\sum_1^{+\infty} a_n^* e^{in\theta})'| \\
&\leq \sum_{n\geq N+1} n|a_n^*||e^{i(n-1)\theta}| \;\leq\; \sum_{n\geq N+1} n|a_n^*| \\
&\leq \sum_{n\geq N+1} n\frac{1}{n^3}
\end{aligned}$$

Then $(F^N)'$ *converges uniformly to* $(F^*)'$.
In either part, the function L *being continuous on the compact set* $[0;\alpha\pi]$ *of* $\mathbb{R}$, *then*

$$\lim_{N\to+\infty} \int_0^{\alpha\pi} L(|h(\Phi^N(e^{i\theta}))|^2 |(F^N)'(e^{i\theta})|^2 - C^2) d\theta = \int_0^{\alpha\pi} L(|h(\Phi^*(e^{i\theta}))|^2 |(F^*)'(e^{i\theta})|^2 - C^2) d\theta \quad (13)$$

The relation (13) implies that $J(\Phi^N)$ *goes to* $J(\Phi^*)$. *As* $\Phi^N \in \boldsymbol{H}_N$, *then* $J(\Phi^N) \geq \min_{\overrightarrow{X}\in\mathbb{R}^{2N}} J(\overrightarrow{X}) = J(\Phi_N)$, *or* $J(\Phi_N) \geq 0$ *et* $\lim_{N\to+\infty} J(\Phi^N) = J(\Phi^*) = 0$, *then we get* $\lim_{N\to+\infty} J(\Phi_N) = 0$ $\square$.

We have the following proposition.

Proposition 4.3 *1. there exists a subsequence $(\Phi_{N_k})_{N_k}$ de $(\Phi_N)_N$ and Φ_0 such tha Φ_{N_k} uniformly converges to Φ_0 on all compact set of D_1.*

2. there exists $\Phi_2 \in L^2(\Gamma_2)$ such that Φ_{N_k} weakly converges to Φ_2 in $H^1(\Gamma_2)$

3. there exists $\Phi_1 \in L^2(\Gamma_1)$ such that Φ_{N_k} weakly converges to Φ_1 in $L^2(\Gamma_1)$

4.

$$\Phi_0(z) = \frac{1}{2i\pi}\int_{\Gamma_2}\frac{\Phi_2(\xi)}{(\xi - z)}d\xi + \frac{1}{2i\pi}\int_{\Gamma_1}\frac{\Phi_1(\xi)}{(\xi - z)}d\xi,\ \forall z \in D_1.$$

Φ_0 has a power series expansion on D_1

$$\Phi_0(z) = \sum_0^{+\infty} a_n z^n,\ \ \textit{where}\ a_n = \frac{1}{2i\pi}(\int_{\Gamma_2}\frac{\Phi_2(\xi)}{\xi^{n+1}}d\xi + \int_{\Gamma_1}\frac{\Phi_1(\xi)}{\xi^{n+1}}d\xi),\ \ n \in \mathbb{N}$$

Preuve de la proposition 4.3 *1. Show that the sequence $(\Phi_N)_N$ is bounded in $L^2(\Gamma_1)$ and $H^1(\Gamma_2)$.*
We know that $J(\Phi^N) \geq J(\Phi_N) \geq 0$, $\lim\limits_{N\to+\infty} J(\Phi^N) = 0$ and $\lim\limits_{N\to+\infty} J(\Phi_N) = 0$.
Then we have $\lim\limits_{N\to+\infty} \int_0^{\alpha\pi} L(|h(\Phi_N(e^{i\theta}))|^2|(F_N)'(e^{i\theta})|^2 - C^2)d\theta = 0$.
by hypothesis on the function L, $L(t) \geq \beta t^2$, we get :
$(|h(\Phi_N(e^{i\theta}))|^2|F_N'(e^{i\theta})|^2 - C^2)^2$ is bounded for all N.
Then there exists $M > 0 :$ $\int_0^{\alpha\pi}|(F_N)'(e^{i\theta})|^2 d\theta \leq M$ because h is continuous bounded and, C^2 is a constant value and the function to $(e^{i\theta})$ associates C^2 is continuous on Γ_2.
*By the **Parseval** formula, we have*

$$\int_0^{\alpha\pi}|F_N'(e^{i\theta})|^2 d\theta = \int_0^{\alpha\pi} F_N'(e^{i\theta})\overline{F_N'(e^{i\theta})}d\theta \tag{14}$$

*We deduce by the **Parseval** formula*

$$\begin{aligned}\int_0^{\alpha\pi}|(F_N)'(e^{i\theta})|^2 d\theta &= \alpha\pi(\sum_0^N n^2|a_n^N|^2) \leq M,\ \forall N\\ \int_0^{\alpha\pi}|\Phi_N(e^{i\theta})|^2 d\theta &= \int_0^{\alpha\pi}\Phi_N e^{i\theta})\overline{\Phi_N(e^{i\theta})}d\theta\end{aligned}$$

We also have by the same formula

$$\int_0^{\alpha\pi}|(\Phi_N)(e^{i\theta})|^2 d\theta = \alpha\pi(\sum_0^N |a_n^N|^2) \leq M,\ \forall N\ .$$

As $\Phi_N' = F_N'$ and Φ_N are bounded in $L^2(\Gamma_2)$, this implies that $(\Phi_N)_{N\in\mathbb{N}}$ is also bounded in $H^1(\Gamma_2)$. We also can prove that $(\Phi_N)_{N\in\mathbb{N}}$ is bounded in $L^2(\Gamma_1)$.

$\dim(\Gamma_2) = 1$ *then the injection of* $H^1(\Gamma_2)$ *is compact in* $C(\Gamma_2)$. *Then* $(\Phi_N)_{N\in\mathbb{N}}$ *is uniformly bounded for all* $\theta \in [0; 2\pi]$.
$(\Phi_N)_{N\in\mathbb{N}}$ *is bounded in* $H^1(\Gamma_2)$ *and in* $L^2(\Gamma_1)$*; the Hilbert's spaces are reflexive then there exist* $\Phi_2 \in H^1(\Gamma_2)$, $\Phi_1 \in L^2(\Gamma_1)$ *and* $N_k \in \mathbb{N}$ *such that* Φ_{N_k} *weakly converges to* Φ_2 *in* $H^1(\Gamma_2)$ *and* Φ_{N_k} *weakly converges to* Φ_1 *in* $L^2(\Gamma_1)$.

2. *Now show that the uniform convergence of a subsequence* $(F_{N_{k'}})_{N_{k'}}$ *on all compact set on* D_1.
Let' s take $z \in D_1$, $\Phi_N(z)$ *is holomorphic function on* D_1. *According to* ***Cauchy*** *formula, we have*

$$\begin{aligned}\Phi_N(z) &= \frac{1}{2i\pi}\int_{\Gamma_{D_1}} \frac{\Phi_N(\xi)}{\xi - z}d\xi \\ &= \frac{1}{2i\pi}(\int_{\Gamma_2} \frac{\Phi_N(\xi)}{\xi - z}d\xi + \int_{\Gamma_1} \frac{\Phi_N(\xi)}{\xi - z}d\xi),\end{aligned}$$

By ***Cauchy-Schwarz*** *inequality :*

$$|\Phi_N(z)| = |\frac{1}{2i\pi}\int_{\Gamma_{D_1}} \frac{\Phi_N(\xi)}{\xi - z}d\xi| = |\frac{1}{2i\pi}(\int_{\Gamma_2} \frac{\Phi_N(\xi)}{\xi - z}d\xi + \int_{\Gamma_1} \frac{\Phi_N(\xi)}{\xi - z}d\xi)|$$
$$\leq \frac{1}{2\pi}(\int_{\Gamma_2} \frac{1}{|\xi - z|^2}d|\xi|)^{\frac{1}{2}}|\Phi_N|_{L^2(\Gamma_2)} + \frac{1}{2\pi}(\int_{\Gamma_1} \frac{1}{|\xi - z|^2}d|\xi|)^{\frac{1}{2}}|\Phi_N|_{L^2(\Gamma_1)}$$

where

$$(\int_{\Gamma_2} \frac{1}{|\xi - z|^2}d|\xi|) \text{ and } (\int_{\Gamma_1} \frac{1}{|\xi - z|^2}d|\xi|)$$

are two finite quantities and bounded by some constant C_0 *depending on* z.
As we know $(\Phi_N)_N$ *is bounded in* $L^2(\Gamma_1)$, $H^1(\Gamma_2)$, *then* $|\Phi_N| \leq C_0(M + M_1)$. *We deduce that* $(\Phi_N)_N$ *is bounded on all compact set of* D_1. *According the* ***Montel*** *theorem see ([5]):*
there exists $N_{k'}$ *and* Φ_0 *such that* $\Phi_{N_{k'}}$ *uniformly converges to* Φ_0 *on all compact set of* D_1. *Further we see that* Φ_0 *is a holomorphic function on* D_1 . *Taking* $N_{k_i} = \sup(N_k, N_{k'})$ *which denoted by* N_k *we have :*
(Φ_{N_k}) *uniformly converges to* Φ_0 *on all compact set of* D_1. (Φ_{N_k}) *weakly converges to* Φ_2 *in* $H^1(\Gamma_2)$. (Φ_{N_k}) *weakly converges to* Φ_1 *in* $L^2(\Gamma_1)$.

As $\frac{1}{\xi - z} \in L^2(\Gamma_1) \cup L^2(\Gamma_2)$, $z \in D_1$, *according to* ***Rellich*** *theorem* (Φ_{N_k}) *strongly converges to*

$$\frac{1}{2i\pi}\int_{\Gamma_2} \frac{\Phi_2(\xi)}{(\xi - z)}d\xi + \frac{1}{2i\pi}\int_{\Gamma_1} \frac{\Phi_1(\xi)}{(\xi - z)}d\xi.$$

We deduce

$$\Phi_0(z) = \frac{1}{2i\pi}\int_{\Gamma_2} \frac{\Phi_2(\xi)}{(\xi - z)}d\xi + \frac{1}{2i\pi}\int_{\Gamma_1} \frac{\Phi_1(\xi)}{(\xi - z)}d\xi \ \forall z \in D_1.$$

The power expansion series of the function $\frac{1}{\xi - z}$ *for* $\xi \in \Gamma_2$ *and for* $\xi \in \Gamma_1$ *gives*

$$\begin{aligned} \frac{1}{\xi - z} &= \frac{1}{\xi(1 - \frac{z}{\xi})} \\ \frac{1}{\xi - z} &= \frac{1}{\xi}\sum_{n\geq 0}(\frac{z}{\xi})^n \end{aligned}$$

We have

$$\begin{aligned} \Phi_0(z) &= \frac{1}{2i\pi}(\sum_{n\geq 0}\int_{\Gamma_2}\frac{\Phi_2(\xi)z^n}{\xi^n\xi}d\xi + \sum_{n\geq 1}\int_{\Gamma_1}\frac{\Phi_1(\xi)z^n}{\xi^n\xi}d\xi) \\ &= \frac{1}{2i\pi}\sum_{n\geq 0}\int_{\Gamma_2}\frac{\Phi_2(\xi)}{\xi^{n+1}}d\xi z^n + \frac{1}{2i\pi}\sum_{n\geq 0}\int_{\Gamma_1}\frac{\Phi_1(\xi)}{\xi^{n+1}}d\xi z^n \\ \Phi_0(z) &= \sum_{n=0}^{+\infty} a_n z^n \end{aligned}$$

Where the coefficients are

$$\begin{aligned} a_n &= \frac{1}{2i\pi}\int_{\Gamma_2}\frac{\Phi_2(\xi)}{\xi^{n+1}}d\xi, \ \ \theta \in [0;\alpha\pi] \\ a_n &= \frac{1}{2i\pi}\int_{\Gamma_1}\frac{\Phi_1(\xi)}{\xi^{n+1}}d\xi, \ \ \theta \in [\alpha\pi;2\pi] \end{aligned}$$

□

We have the following corollary

Corollary 4.1 *Let' take* $(,)$ *the scalar product in* L^2*.*

$$\begin{aligned} a_n &= (\Phi_2, e^{in\theta})_{L^2(\Gamma_2)\times L^2(\Gamma_2)}, \ \ \theta \in [0;\alpha\pi] \\ a_n &= (\Phi_1, e^{in\theta})_{L^2(\Gamma_1)\times L^2(\Gamma_1)}, \ \ \theta \in [\alpha\pi;2\pi] \\ (\Phi_2, e^{in\theta})_{L^2(\Gamma_2)\times L^2(\Gamma_2)} &= (\Phi_0, e^{in\theta})_{L^2(\Gamma_2)\times L^2(\Gamma_2)}, \ \ \theta \in [0;\alpha\pi] \\ (\Phi_1, e^{in\theta})_{L^2(\Gamma_1)\times L^2(\Gamma_1)} &= (\Phi_0, e^{in\theta})_{L^2(\Gamma_1)\times L^2(\Gamma_1)}, \ \ \theta \in [\alpha\pi;2\pi] \end{aligned}$$

, $\Phi_0(e^{i\theta})$ *is defined p.p on* $[0;2\pi]$ *and for almost all* $\theta \in [0;2\pi]$*. Precisely we have:*

$$\Phi_0(e^{i\theta}) = \sum_{n=0}^{+\infty} a_n z^n \ \text{ for almost all } \ \theta \in \ [0;2\pi]$$

Remark 4.1 *In this corollary, our difficulty is to extend the function* Φ_0 *on* Γ_{D_1}*, to prove this corollary , we need the following result see [24].*

Proposition 4.4 *Lest ' s take* D_1 *a unit disk.*

1. *A function* $f \in H(D_1)$ *defined by* $f(z) = \sum\limits_{n=0}^{+\infty} a_n z^n, z \in D_1$ *belongs to* H^2 *if only if* $\sum\limits_{n\geq 1} |a_n|^2 < \infty$, H^2 *is a* ***Hardy*** *space. In this case* $||f||_2 = \{\sum\limits_{n=1}^{+\infty} |a_n|^2\}^{\frac{1}{2}}$.

2. *If* $f \in H^2$, f *admits the radial limits* $f^*(e^{i\theta})$ *for almost all points of* Γ_{D_1}*; the function* $f^* \in L^2(\Gamma_{D_1})$ *and the nth* ***Fourier*** *coefficient of* f^* *is* a_n, *if* $n \geq 0$ *and* 0 *if* $n < 0$. *The* L^2*'norm approximative of* $\lim\limits_{r\to 1} \frac{1}{2\pi}\int_0^{2\pi} |f^*(e^{i\theta}) - f(re^{i\theta})|^2 d\theta = 0$ *is exact.*

3. *The map which* f *associates* f^* *is an isometry of* H^2 *under the subspace of the functions* l *of* $L^2(\Gamma_{D_1})$ *such that:* $\widehat{l}(n) = 0, \ \forall \ n < 0$.

$$\textit{Pour}, \ z = e^{i\theta}, \quad f(z) = \frac{1}{2\pi}\int_0^{2\pi} P_r(\theta - t) f^*(e^{it}) dt = \frac{1}{2\pi}\int_{\Gamma_{D_1}} \frac{f^*(\xi)}{\xi - z} d\xi$$

where $P_r(\theta - t) = \Re(\frac{e^{it}+z}{e^{it}-z})$ *is the* ***Poisson*** *kernel and the boundary* Γ_{D_1} *is counter clockwise orientation.*
The norm of the ***Hardy*** *space* H^2 *is define by*

$$||u||_{H^2} = \lim_{r\to 1}\{\frac{1}{2\pi}\int_0^{2\pi} |u(re^{i\theta})|^2 d\theta\}^{\frac{1}{2}}$$

Note that H^2 *is a Hilbert'space which is identified with a subspace of* $L^2(\Gamma_{D_1})$, *where* Γ_{D_1} *is the unit circle.*

Preuve du corollaire 4.1 *Let' take* $\Phi_N(z) = \sum\limits_{n=0}^{N} a_n^N z^n$ *where*

$$\begin{cases} a_n^N = \frac{1}{2i\pi}\int_{\Gamma_2} \frac{\Phi_N(\xi)}{\xi^{n+1}} d\xi, \ \ \theta \in [0; \alpha\pi] \\ \\ a_n^N = \frac{1}{2i\pi}\int_{\Gamma_1} \frac{\Phi_N(\xi)}{\xi^{n+1}} d\xi, \ \ \theta \in [\alpha\pi; 2\pi] \end{cases}$$

For $n \in \mathbb{N}$ *fixed integer:*

$$a_n^N \quad \textit{converges to} \ \frac{1}{2i\pi}(\int_{\Gamma_1} \frac{\Phi_1(\xi)}{\xi^{n+1}} d\xi + \int_{\Gamma_2} \frac{\Phi_2(\xi)}{\xi^{n+1}} d\xi)$$

This limit is exactly the coefficient a_n.
Let' take $\xi = e^{i\theta}$, *we obtain*

$$a_n = \frac{1}{2\pi}(\int_0^{\alpha\pi} \frac{\Phi_2(e^{i\theta})}{e^{in\theta}} d\theta. \ \theta \in [0; \alpha\pi]$$

$$a_n = \frac{1}{2\pi}(\int_{\alpha\pi}^{2\pi} \frac{\Phi_1(e^{i\theta})}{e^{in\theta}} \ \theta \in [\alpha\pi; 2\pi]$$

we deduce

$$a_n = (\Phi_2, e^{in\theta})_{L^2(\Gamma_2)\times L^2(\Gamma_2)},\ \ \theta \in [0;\alpha\pi]$$
$$a_n = (\Phi_1, e^{in\theta})_{L^2(\Gamma_1)\times L^2(\Gamma_1)},\ \ \theta \in [\alpha\pi;2\pi]$$

In one part we know that $\Phi_0(z) = \sum_{n=1}^{+\infty} a_n z^n$ *for all* $z \in D_1$

In either part as Φ_0 *is holomorphic function on* D_1, *we have :* $\Phi_0(z) = \frac{1}{2i\pi}\int_{\Gamma_{D_1}} \frac{\Phi_0(\xi)}{\xi - z} d\xi$, *for all* $z \in D_1$ *this implies* $\Phi_0(z) = \frac{1}{2i\pi}\sum_{n=1}^{+\infty}\int_{\Gamma_{D_1}} \frac{\Phi_0(\xi)}{\xi^{n+1}} d\xi z^n$.

we deduce

$$\begin{aligned} a_n &= \frac{1}{2i\pi}\int_{\Gamma_{D_1}} \frac{\Phi_0(\xi)}{\xi^{n+1}} d\xi, \quad \textit{for } \xi = e^{i\theta}, \ \textit{we obtain} \\ a_n &= \frac{1}{2\pi}\Big(\int_0^{\alpha\pi} \frac{\Phi_0(e^{i\theta})}{e^{in\theta}} d\theta + \int_{\alpha\pi}^{2\pi} \frac{\Phi_0(e^{i\theta})}{e^{in\theta}} d\theta\Big). \end{aligned}$$

In conclusion, we have the result desired :

$$\begin{aligned} (\Phi_2, e^{in\theta})_{L^2(\Gamma_2)\times L^2(\Gamma_2)} &= (\Phi_0, e^{in\theta})_{L^2(\Gamma_2)\times L^2(\Gamma_2)} \quad \textit{and} \\ (\Phi_1, e^{in\theta})_{L^2(\Gamma_1)\times L^2(\Gamma_1)} &= (\Phi_0, e^{in\theta})_{L^2(\Gamma_1)\times L^2(\Gamma_1)} \end{aligned}$$

□

For $z \in D_1$, Φ_0, can be write see ([5]) as

$$\begin{aligned} \Phi_0(z) &= f_1(z) + f_2(z) \ \text{ where} \\ f_1(z) &= \sum_{n=0}^{+\infty} a_n z^n \ \in \ \mathbf{H}(D_1) \ \text{ and} \\ f_2(z) &= 0 \ \text{ if not} \end{aligned}$$

For $z = e^{i\theta}$, we have in one part f_1 is define everywhere on Γ_{D_1}, in either part Φ_2 belongs to $L^2(\Gamma_2)$ and Φ_1 belongs to $L^2(\Gamma_1)$ then we get

$$\sum_{n=0}^{+\infty} |a_n|^2 < +\infty$$

According to the theorem (4.4), we have either $\sum_{n=0}^{+\infty} a_n e^{in\theta}$ is defined p.p on $[0, 2\pi]$ in this case $\Phi_0(e^{i\theta}) = \sum_{n=0}^{+\infty} a_n e^{in\theta}$ is defined p.p on $[0, 2\pi]$.According to the corollary (4.1) we have $\Phi_0 = \Phi_2$ *p.p* on Γ_2 and $\Phi_0 = \Phi_1$ *p.p* on Γ_1. We deduce

- (Φ_{N_k}) weakly converges to Φ_1 in $L^2(\Gamma_1)$. As $L^2(\Gamma_1)$ is separable Hilbert space then there exists an orthonormal Hilbert space basis $(e_n)_{k\geq n\geq 1}$. The best known example of an orthonormal basis in an infinite Hilbert space is the set of functions $(e^{in\theta})_{k\geq n\geq 1}$

  $\Phi_1 \in L^2(\Gamma_1)$, we have: $\Phi_1(e^{i\theta}) = \sum_{n=1}^{k} a_n e^{in\theta}$

- $\Phi_2 \in L^2(\Gamma_2)$, we have : $\Phi_2(e^{i\theta}) = \sum\limits_{n=k+1}^{+\infty} a_n e^{in\theta}$

  where a_n are **Fourier** coefficients, $a_n = \frac{1}{2\pi}(\int_{\Gamma_2} \frac{\Phi_2(e^{i\theta})}{e^i n\theta} d\theta + \int_{\Gamma_1} \frac{\Phi_1(e^{i\theta})}{e^i n\theta} d\theta)$.

Φ_2 and Φ_1 are **Fourier** series respectively defined on Γ_2 and Γ_1 then holomorphic on $\overline{D}_1$.
The subsequence (Φ_{N_k}) weakly converges to converge Φ_2 in $L^2([0;\alpha\pi])$ then for all $\psi \in \mathcal{D}([0;\alpha\pi])$, we have :

$$|\int_{\Gamma_2} L(|h(\Phi_{N_k}(e^{i\theta}))|^2 |(F_{N_k})'(e^{i\theta})|^2 - C^2)\psi d\theta|$$
$$\leq ||\psi||_{L^2(\Gamma_2)^{\frac{1}{2}}} (\int_{\Gamma_2} L(|h(\Phi_{N_k}(e^{i\theta}))|^2 |(F_{N_k})'(e^{i\theta})|^2 - C^2) d\theta)^{\frac{1}{2}}$$
$$\leq (||\psi||_{L^2(\Gamma_2)})^{\frac{1}{2}} (J(\Phi_{N_k}))^{\frac{1}{2}}$$

First $J(\Phi_{N_k})$ goes to zero when k goes to zero then $L(|h(\Phi_{N_k}(e^{i\theta}))|^2 |(F_{N_k})'(e^{i\theta})|^2 - C^2)$ goes to zero in $\mathcal{D}'([0;\alpha\pi])$.
Second as (Φ_{N_k}) weakly converges to (Φ_2) in $L^2(\Gamma_2)$, then for all $\psi \in \mathcal{D}([0;\alpha\pi])$, we get

$$\int_{\Gamma_2} L(|h(\Phi_{N_k}(e^{i\theta}))|^2 |(F_{N_k})'(e^{i\theta})|^2 - C^2)\psi d\theta \to \int_{\Gamma_2} L(|h(\Phi_2(e^{i\theta}))|^2 |(F_2)'(e^{i\theta})|^2 - C^2)\psi d\theta,$$

Either $\int_{\Gamma_2} L(|h(\Phi_2(e^{i\theta}))|^2 |(F_2)'(e^{i\theta})|^2 - C^2) d\theta = 0$ for all $\psi \in \mathcal{D}([0;\alpha\pi])$ then $|h(\Phi_2(e^{i\theta}))|^2 |(F_2)'(e^{i\theta})|^2 = C^2$ in $L^2(\Gamma_2)$ this implies that

$$|h(\Phi_2(e^{i\theta}))|^2 |(F_2)'(e^{i\theta})|^2 = C^2$$

p.p on Γ_2.
We have proved $\Phi_2 = \Phi_0 \, p.p$ on Γ_2 this implies that

$$|h(\Phi_0(e^{i\theta}))|^2 |(F_0)'(e^{i\theta})|^2 = C^2$$

p.p on Γ_2.
Let us look at what happens when h is constant negative.

5. Numerical examples

We consider the first eigenvalue $\lambda_1 = l_{0_1}^2$ and we suppose that $h = C_0$, on Γ_0 with C_0, negative constant.

Then we have the following proposition

Proposition 5.1 *Consider following problem*

$$\begin{cases} -\Delta u & = & \lambda_1 u & in \ \Omega, \\ u & = & 0 & on \ \partial\Omega \\ \frac{\partial u}{\partial \nu} = C_0 & on \ \Gamma_0 \end{cases} \tag{15}$$

Where Ω unknown, C_0 given negative constant, λ_1 the first eigenvalue of $-\triangle$, $\Gamma_0 \subset \Omega$ such that $\Gamma_0 \cup \Gamma = \Omega$ and $\Gamma_0 \neq \emptyset$.
Then Ω is a disk centered on origin and radius R, $D = (0, R)$, with $R \in \mathbb{R}_+^*$.

Preuve de la proposition 5.1 *We know that*

$$F^{'}(e^{i\theta}) = \frac{A_{kn} l_{kn} J_k^{'}(l_{k_n}) e^{ik\theta}}{h(\Phi(e^{i\theta}))}$$

and

$$\frac{\partial u}{\partial \nu} = h = C_0 \ \ on \ \Gamma_0$$

Then

$$\Phi^{'}(e^{i\theta}) = \frac{A_{0_1} l_{0_1} J_0^{'}(l_{0_1})}{C_0}$$

We take $K = \frac{A_{0_1} l_{0_1} J_0^{'}(l_{0_1})}{C_0}$
we have therefore $\Phi(z) = Kz + b$ *and* $\forall \ z \in \partial D_1$.
Since $\Phi(\partial D_1) = \partial\Omega$ *then* $\forall \ (x, y) \in \partial\Omega$ *we have*

$$\begin{cases} x = K\cos\theta + b_1 \\ y = K\sin\theta + b_2 \end{cases}$$

with $b = b_1 + ib_2$
So we have

$$(x - b_1)^2 + (y - b_2)^2 = K^2$$

We deduce that $\partial\Omega$ *is circle centered on* $B(b_1, b_2)$ *and radius* $|K|$.
We conclude that Ω *is a disk centered on* $B(b_1, b_2)$ *and radius* $R = |K| = |\frac{A_{0_1} l_{0_1} J_0^{'}(l_{0_1})}{C_0}|$.

□

In this section we present the results of some numerical simulations in order to show the accuracy and effectiveness of the reconstruction method in this paper. The algorithm is to minimize the functional J defined above on $\mathbb{R}^{2N}$ because to find Φ_N it is good enough to know $\overrightarrow{X}$, where $\overrightarrow{X} = (\alpha_1, ..., \alpha_N, \beta_1, ..., \beta_N) \in \mathbb{R}^{2N}$. The coordinates of $\overrightarrow{X}$ are the coefficients of trigonometric polynomial Φ_N

$$\Re(\Phi_N) = \sum_{n=1}^{N} \alpha_n \cos n\theta - \beta_n \sin n\theta$$
$$\Im(\Phi_N) = \rho, \ \text{taking into account equation } (10)$$

As Cauchy data we have

$g(x, y) = 0 \ \ h(x, y) = -1 - x^2 - y^2$ and secondly $g(x, y) = 0, \quad h(x, y) = -1$ such that the problem (1)-(2)-(3) be satisfied.

5.1 Case where $g(x,y) = 0$ and $h(x,y) = -1$

In this case

$$F^{'}(e^{i\theta}) \simeq -\frac{\frac{9\pi}{2}\sqrt{2\pi}}{\sqrt{9\pi^2-4}} \text{ on } \Gamma_2$$

We represent the disk centered on origin and radius $\frac{\frac{9\pi}{2}\sqrt{2\pi}}{\sqrt{9\pi^2-4}}$.

In the figure 1, we choose $b_1 = 0, b_2 = 0$ and $\alpha = 0.5$. The part of boundary Γ_2 is represented by the red part of the graph and Γ_1 the blue part.

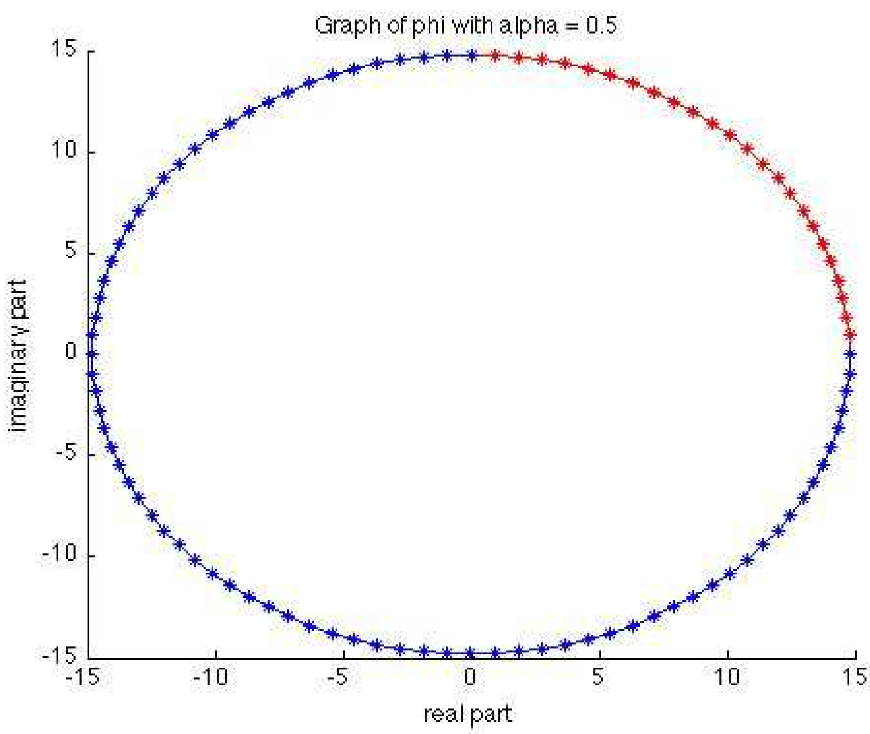

Figure 1:

Therefore for all case where h negative constant and f constant we have Ω a disk.

5.2 Case where $g(x,y)=0$ and $h(x,y)=-1-x^2-y^2$

We minimize the functional $J(\Phi_N)$ defined as follow:

$$J(\Phi_N)=\int_0^{\alpha\pi}(|1+(\sum_{n=0}^{N}\alpha_n\cos n\theta-\beta_n\sin n\theta)^2+\rho^2|^2|\sum_{n=0}^{N}n(\alpha_n\cos(n-1)\theta-\beta_n\sin(n-1)\theta)|^2-C^2)^2d\theta$$

with $C=\frac{\frac{9\pi}{2}\sqrt{2\pi}}{\sqrt{9\pi^2-4}}$.

And with the minimum of the functional we reconstruct the unknown part Γ_0 and the accessible part Γ of the boundary of the approximate domain.

We show with $\alpha=1$ and $\alpha=0.5$

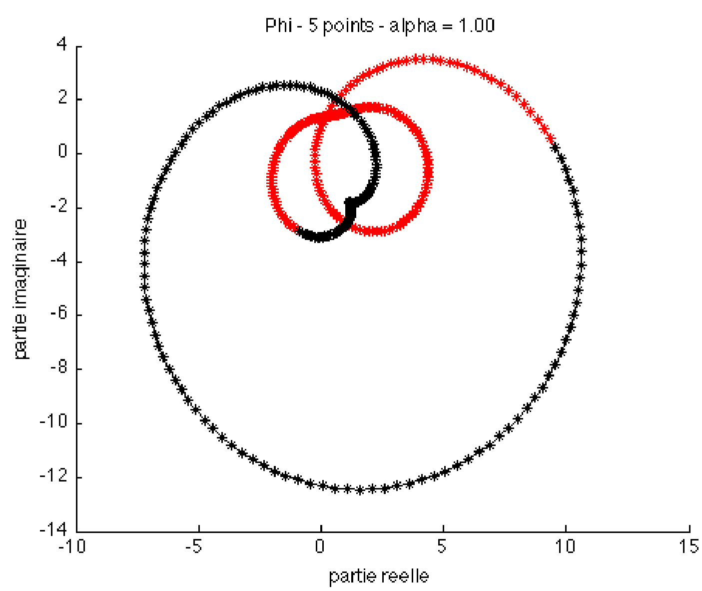

Figure 2:

The part of the boundary Γ_2 is represented by the red part of the graph and Γ_1 the black part.

The numerical experiments are executed on a computer: Mac OS X, version 10.7.5, CPU 2.2 GHz Intel Core i7, 8.0Gb of RAM.

The optimization problem for minimizing J is solved by the software IPOPT (Interior Point OPTimization) 3.9 stable [29, 30], running with linear solver ma27. Matlab software

[22] is used to plot the function $J(\Phi(\theta))$.
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