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ON THE MAXIMUM OF AVERAGE DISPLACEMENT OF AN ELASTIC MEMBRANE SUBJECT TO THE SHAPE DEFORMATION

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ABSTRACT. In this paper, we prove that the average displacement of the elastic membrane, subject to a constant vertical force is a monotone function on a family of neighborhoods satisfying certain properties. The main idea is motivated by [5]. Two numerical examples have been selected to verify and illustrate this property.

Key Words: Schwarz symmetrization; Elastic membrane; Increasing; Maximum
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1. INTRODUCTION

Let $1 \leq p \leq \infty$, $|.|_p$ be the standard p-norm on $\mathbb{R}^2$, and $|.|_{p,t}$ be the corresponding weighted norm with weights t and t^{-1}, $t > 0$, in x_1 and x_2 directions respectively. Let $r > 0$ and $c = (c_1, c_2) \in \mathbb{R}^2$. Consider

$$\Omega_{p,t} = \{x = (x_1, x_2) \in \mathbb{R}^2 : |x - c|_{p,t} < r\},$$

the r-neighborhood of the point c. Explicitly, for $1 \leq p < \infty$

$$\Omega_{p,t} = \{x = (x_1, x_2) \in \mathbb{R}^2 : [|t(x_1 - c_1)|^p + |t^{-1}(x_2 - c_2)|^p]^{\frac{1}{p}} < r\},$$

and for $p = \infty$

$$\Omega_{\infty,t} = \{x = (x_1, x_2) \in \mathbb{R}^2 : \max(|t(x_1 - c_1)|, |t^{-1}(x_2 - c_2)|) < r\}.$$

It is apparent that the area of each neighborhood is

$$|\Omega_{p,t}| = \begin{cases} \dfrac{4r^2\ \left(\Gamma(\frac{p+1}{p})\right)^2}{\Gamma(\frac{p+2}{p})} & p \geq 1, \\ \\ 4r^2 & p = \infty, \end{cases}$$

for all $t > 0$, see [7, p. 208]. From hereon, we assume that p is fixed and for simplicity of the notations, we will drop p when it is unnecessary.

Let $\alpha \geq 0$ and $\beta > 0$. Consider the boundary value problem

$$\begin{cases} -\Delta u + \alpha u = \beta & \text{in } \Omega_t, \\ u = 0 & \text{on } \partial\Omega_t. \end{cases} \tag{1.1}$$

The affine transformation

$$\zeta := \frac{1}{r}(x - c),$$

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transforms the boundary value problem (1.1) to

$$\begin{cases} -\Delta u + \hat{\alpha} u = \hat{\beta} & \text{in } \hat{\Omega}_t, \\ u = 0 & \text{on } \partial\hat{\Omega}_t, \end{cases}$$

where $\hat{\alpha} = \alpha r^2$, $\hat{\beta} = \beta r^2$ and

$$\hat{\Omega}_t = \{x \in \mathbb{R}^2 : |x|_{p,t} < 1\}.$$

Therefore, without loss of generality we can assume that $c = (0, 0)$, $r = 1$ and Ω_t is a scaled unit ball. For each $t > 0$, we denote the unique solution, see [1], of the boundary value problem (1.1) by u_t. It is well known that

$$\int_{\Omega_t} \nabla u_t \cdot \nabla v \ \mathrm{d}x + \alpha \int_{\Omega_t} u_t v \ \mathrm{d}x = \beta \int_{\Omega_t} v \ \mathrm{d}x, \quad \text{for all } v \in H_0^1(\Omega_t). \tag{1.2}$$

We define the functional $J_t : H_0^1(\Omega_t) \to \mathbb{R}$ by

$$J_t(u) = \beta \int_{\Omega_t} u \ \mathrm{d}x - \frac{\alpha}{2} \int_{\Omega_t} u^2 \ \mathrm{d}x - \frac{1}{2} \int_{\Omega_t} |\nabla u|^2 \ \mathrm{d}x,$$

so, u_t is the unique maximizer of J_t on $H_0^1(\Omega_t)$;

$$J_t(u_t) = \max_{u \in H_0^1(\Omega_t)} J_t(u). \tag{1.3}$$

From (1.2), for $v = u_t$, we conclude that

$$J_t(u_t) = \frac{\beta}{2} \int_{\Omega_t} u_t \ \mathrm{d}x = \frac{1}{2} \int_{\Omega_t} (|\nabla u_t|^2 + \alpha u_t^2) \ \mathrm{d}x. \tag{1.4}$$

Let $0 < a < 1$, we define the function $\xi : [a, \frac{1}{a}] \to \mathbb{R}$ by $\xi(t) = \int_{\Omega_t} u_t \ \mathrm{d}x$. We prove that the function ξ is increasing on $[a, 1]$.

2. Elastic membrane

Suppose Ω_t is a planar region occupied by an elastic membrane fixed around the boundary. We assume the membrane is made from several materials with same densities and is subject to a constant vertical force such as a load distribution. These assumptions justify the role of the two constant functions α and β in (1.1) respectively. In the mathematical modeling (1.1), the solution u_t denotes the displacement of the membrane. For each $t > 0$, $\xi(t)$ is the average displacement. In the following we prove that for any $0 < a < 1$, the average displacement increases on $[a, 1]$ and then decreases on $[1, \frac{1}{a}]$. Thus, the average displacement is maximum on the domain Ω_1.

3. Main results

The main result in this paper is the following theorem.

Theorem 3.1. *For each $0 < a < 1$, the function ξ is increasing on $[a, 1]$.*

Proof. Let $0 < a < 1$. For $(x_1, x_2) \in \mathbb{R}^2$ and $t \in [a, 1]$ we consider the mapping

$$\mathbf{\Phi}_t(x_1, x_2) = \frac{a}{t} x_1 \vec{i} + \frac{t}{a} x_2 \vec{j},$$

where $\vec{i}$ and $\vec{j}$ stand for the standard unit vectors in $\mathbb{R}^2$. It is apparent that $\mathbf{\Phi}_t(\Omega_a) = \Omega_t$ for each $t \in [a, 1]$. Therefore, the map

$$[a, 1] \ni t \to \mathbf{\Phi}_t(\Omega_a),$$

is a continuous Schwarz symmetrization of Ω_a. Thus, the continuous Schwarz symmetrization of u_a, see [2, 3], is the unique function $u_a^t \in H_0^1(\Omega_t)$ such that

$$\{(x_1,x_2) : u_a^t(x_1,x_2) > \gamma\} = \mathbf{\Phi}_t(\{(x_1,x_2) : u_a(x_1,x_2) > \gamma\}),$$

for all $\gamma \in \mathbb{R}$ and $t \in [a,1]$. From [4] we have the following results

(i) $\int_{\Omega_a} |u_a|^s \, \mathrm{d}x = \int_{\Omega_t} |u_a^t|^s \, \mathrm{d}x, \quad s \geq 1;$

(ii) $\int_{\Omega_a} |\nabla u_a|^2 \, \mathrm{d}x \geq \int_{\Omega_t} |\nabla u_a^t|^2 \, \mathrm{d}x, \quad t \in [a,1].$

Now, From (i), (ii) and (1.3) we deduce that

$$\frac{\beta}{2}\xi(t) = J_t(u_t) \geq J_t(u_a^t) \geq J_a(u_a) = \frac{\beta}{2}\xi(a).$$

Thus

$$\xi(a) \leq \xi(t), \quad \text{for all } t \in [a,1].$$

Since the above argument can be started by any intermediate design, using $\Omega_{t'}$ instead of Ω_a, we infer that ξ is an increasing function on $[a,1]$. □

Corollary 3.2. *By symmetrization we infer that the function ξ is decreasing on $[1, \frac{1}{a}]$. Therefore,*

$$\xi(1) = \max_{t\in[a,\frac{1}{a}]} \xi(t).$$

It is well known that the norm

$$\|u\| = \left(\int_{\Omega_t} (|\nabla u|^2 + \alpha u^2) \, \mathrm{d}x\right)^{\frac{1}{2}}, \quad u \in H_0^1(\Omega_t).$$

is equivalent to the standard norm in $H_0^1(\Omega_t)$. Now, we can state the following corollary.

Corollary 3.3. *If $0 < a < 1$, then*

$$\max_{t\in[a,\frac{1}{a}]} \|u_t\|^2 = \beta\xi(1).$$

Proof. This result is a consequence of (1.4) and the Corollary 3.2. □

4. Numerical results

In this section, we examine two numerical examples corresponding to a simply connected domains with smooth $(C^\infty(\partial\Omega_t))$ and non-smooth$(C^0(\partial\Omega_t))$ boundaries. To demonstrate the accuracy of computations, the residuals, in some sense, are evaluated and shown in tables.

***Example* 1.** Consider the boundary value problem

$$\begin{cases} -\Delta u + u = 1 & \text{in } \Omega_t, \\ u = 0 & \text{on } \partial\Omega_t. \end{cases} \tag{4.1}$$

Here $p = 2$ and

$$\Omega_t = \{x = (x_1,x_2) \in \mathbb{R}^2 : |tx_1|^2 + |t^{-1}x_2|^2 < 1\}.$$

Let

$$\xi(t) = \int_{\Omega_t} u_t(x,y) \, \mathrm{d}x \, \mathrm{d}y, \quad t > 0$$

where u_t is the unique solution of the BVP (4.1).The graph of $\xi(t)$ on the interval $[4/5, 5/4]$ is shown in Figure 1. The function $\xi(t)$ attains its maximum at $t = 1$.

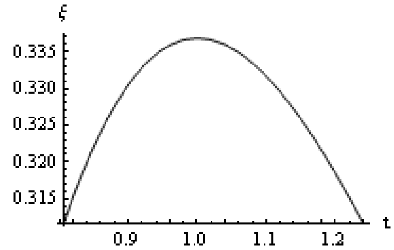

FIGURE 1. The graph of $\xi(t)$.

Let u^h denotes the finite element solution of the problem (4.1). The values of the function $\xi(t)$ for different values of t are evaluated and shown in Table 1. Let

t	0.877193	0.909091	0.943396	0.961538	1	1.04	1.08	1.12
u^h	0.327151	0.331657	0.334889	0.335955	0.336842	0.335955	0.333447	0.329549

TABLE 1. Results of Example 1.

$t_i = 0.877193$ and $t_f = 1.12$ denote the initial and final values of t. The graph of domains corresponding to the values of t_i and t_f are illustrated in Figure 2 and the graph of graph displacement of the membrane corresponding to the values of t_i and t_f are illustrated in Figure 3.

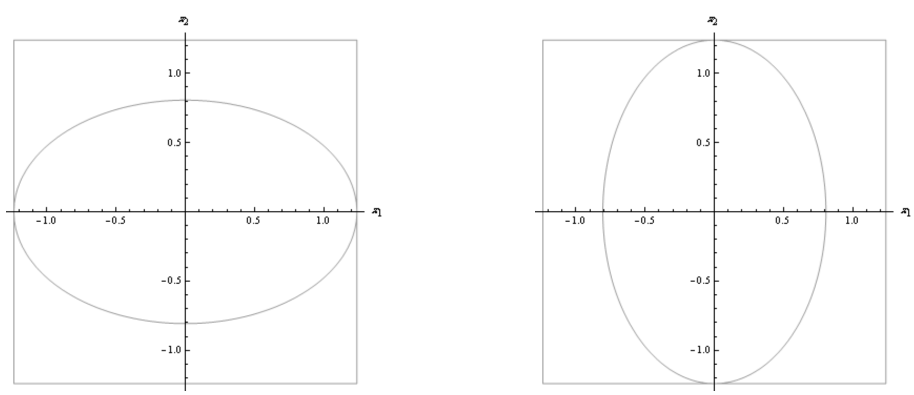

FIGURE 2. The graph of domains.

Example 2. Consider the boundary value problem (4.1) where $p = \infty$,

$$\Omega_t = \{x = (x_1, x_2) \in \mathbb{R}^2 : \max\{|tx_1|, |t^{-1}x_2|\} < 1\}.$$

Let

$$\xi(t) = \int_{\Omega_t} u_t(x, y)\ \mathrm{d}x\ \mathrm{d}y, \quad t > 0$$

where u_t is the unique solution of the BVP (4.1). The graph of $\xi(t)$ on the interval $[4/5, 5/4]$ is shown in Figure 4. The function $\xi(t)$ attains its maximum at $t = 1$.

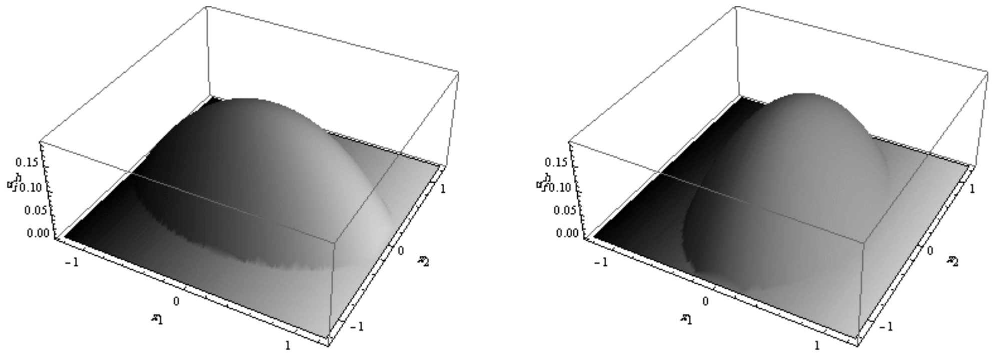

FIGURE 3. The graph of graph displacement of the membrane.

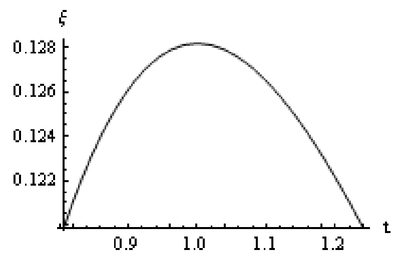

FIGURE 4. The graph of $\xi(t)$

Numerical values of the function $\xi(t)$ for different values of t are shown in Table 2. The graph of domains corresponding to the values of t_i and t_f are illustrated in Figure 5. The graph of graph displacement of the membrane corresponding to the values of t_i and t_f are illustrated in Figure 6.

t	0.877193	0.909091	0.943396	0.961538	1	1.04	1.08	1.12
$\xi(t)$	0.125011	0.12649	0.127548	0.127896	0.128185	0.127896	0.127076	0.125799

TABLE 2. Results of Example 2.

In this paper, numerical examples are solved using the finite element package in Matlab 7 software. Programs are implemented in a cluster environment at Laboratory of Scientific Computation in Institute for Studies in Theoretical Physics and Mathematics visit

http://math.ipm.ac.ir/mcc/

More animated results in two dimension are available online as for check the maximum properties average displacement of the membrane with respect to shape deformations. To see please browse

http://www.camera-ac.ir/

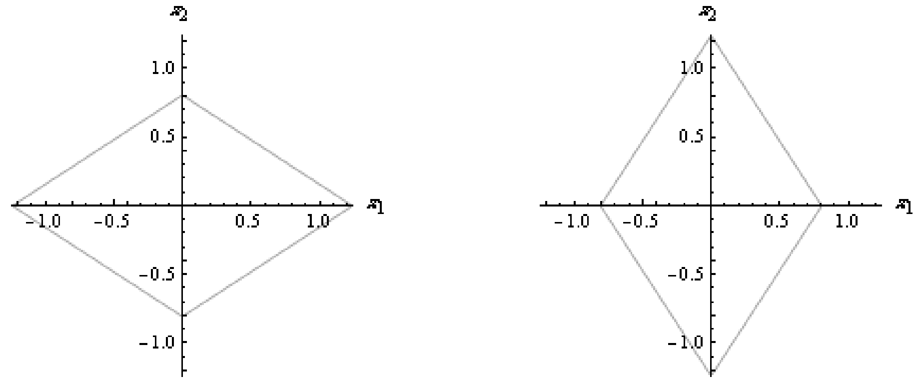

FIGURE 5. The graph of domains.

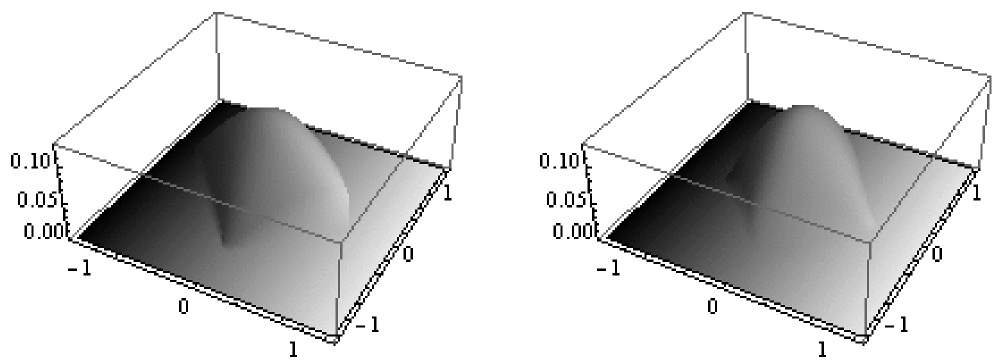

FIGURE 6. The graph of graph displacement of the membrane.

5. CONCLUSION

The qualitative and quantitative study of the average displacement and least eigenvalue of an elastic membrane on variable domains are interesting problems in the fields of pure and applied mathematics see for example [5, 6, 8]. In this paper, we restricted ourselves to the variation of the average displacement on a family of domains. Analysis of the least eigenvalue of a membrane on these domains are postponed to future work.

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