AN INVERSE TRANSPORTATION PROBLEM WITH THE LINEAR FRACTIONAL OBJECTIVE FUNCTION

SANJAY JAIN[1]
[1]Department of Mathematical Sciences, Government College Ajmer, Affiliated to M. D. S. University, Ajmer - 305 001, INDIA. drjainsanjay@gmail.com

NITIN ARYA[2]
[2]Department of Mathematics, Government Engineering College, Jhalawar, Affiliated to Rajasthan Technical University, Kota, INDIA. nitin.arya1234@gmail.com

ABSTRACT: This paper presents an inverse optimization model for the transportation problem of optimizing the ratio of linear functions subject to the linear equality constraints and non negative restrictions on the variables. In our discussion, we have considered a feasible solution and in order to make it an optimal one by adjusting the objective coefficients as little as possible, we have proposed an algorithm and finally an example is presented to demonstrate our algorithm.

Keywords: Inverse optimization, Transportation problem, Linear fractional programming, Optimality condition, Feasible solutions

1. INTRODUCTION

In the last 15-20 years, the community of operations research has shown a significant interest in the field of inverse optimization and many applications of inverse optimization have been found in different areas such as: geophysical sciences, traffic equilibrium, isotonic regression, portfolio optimization etc. In an optimization problem, there are some parameters associated with the decision variable in the objective function and constraint's set. When solving the problem, generally it is assumed that all the parameters are known, but in practice, there are many situations where the parameter values are not known with certain, but we may have some estimates of these parameters and also have an optimal solution from the past experience or past practice. In these situations, inverse optimization can be used to adjust the parameter values as little as possible so that the given solution becomes optimal.

Burton and Toint [1] were the first who investigate the inverse optimization for shortest path problem under l_2 norm, since then a lot of work has been done on inverse optimization but most of the work is based on combinatorial optimization problems. Zhang and Liu [2] have first been calculated some inverse linear programming problem and further investigated inverse linear programming problems in [3]. Ahuja and Orlin [4] provide various references in the area of inverse optimization and compile several applications in network flow problems. Huang and Liu [5] and Amin and Emrouznejad [6], have considered applications of inverse problem. Yibing, Tiesong and Zhongping [7] worked on inverse optimal value problem, Zhang and Zhang [8-10] worked on inverse quadratic programming problems, and Wang [11] has given the cutting plane algorithm for inverse integer programming problem. Jain, Arya [12, 13] have presented inverse models for linear fractional programming and quadratic programming problems.

In the real life, there are many problems where we need to optimize profit/cost, profit/ manpower requirement, dept/equity, nurse/patient etc., and then the linear fractional transportation problem comes into picture. The linear fractional programming problem seeks to optimize the objective function of non-negative variables of quotient form with linear functions in numerator and denominator subject to a set of linear and homogeneous constraints. Bajanilov [14] compiled the literature of Linear Fractional Programming: Theory, Methods, Applications and Software in the form of book. Dinkelbach [15], Charnes-Cooper [16], Kantiswarup [17], Jain, Mangal and Parihar [18] and many other researchers worked on linear fractional programming problem.

A classical (traditional) transportation problem is a minimization problem of the cost of transportation from some origins to some other destinations. The minimum cost planning plays an important role for solving the transportation problem from origins to different destinations, such as from factories to warehouses or from warehouses to supermarkets, etc. Here we are considering a class of transportation problem called linear fractional transportation (LFT) problem, which is similar to the classical transportation problem except the objective function is a ratio of two linear functions. These types of problems arise when we want to minimize the cost-to-time ratio or maximize the profit-to-time ratio. Radzik [19] has first been considered combinatorial fractional problem, Joshi and Gupta [20] obtained the initial basic feasible solution and Sirvi et.al. [21] proposed a solution method for linear fractional programming transportation problem. Ramakrishnan [22], Khurana and Arora [23], Monta [24], Jain [25] and many other have worked on different types of transportation problems.

2. THE LINEAR FRACTIONAL TRANSPORTATION (LFT) PROBLEM

We are considering the following transportation problem

$$\min z = \frac{\sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij}}{\sum_{i=1}^{m}\sum_{j=1}^{n} d_{ij}\,x_{ij}}$$

Subject to, $\sum_{j=1}^{n} x_{ij} = a_i$, $\sum_{i=1}^{m} x_{ij} = b_j$, $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$, $x_{ij} \geq 0$,

where a_i is the i^th^ source, b_j is the j^th^ destination, c_{ij} is the unit cost and d_{ij} is the unit preference from i^th^ source to j^th^ destination.

If we introduce the variables u_i', v_j' and u_i'', v_j'' associated with the numerator and denominator of objective as given in [14], where u_i' and u_i'', i = 1, 2,…., m, are corresponding to supply constraints and v_j' and v_j'', j = 1, 2,…., n, are corresponding to demand constraints, and defined as:

$u_i' + v_j' = c_{ij}$, $\qquad (i,j) \in J_B$.

$u_i'' + v_j'' = d_{ij}$, $\qquad (i,j) \in J_B$.

Where J_B is the set of pairs of indices (i,j) of basic variable x_{ij}. The reduced costs Δ_{ij}' and Δ_{ij}'' are defined as:

$\Delta_{ij}' = c_{ij} - (u_i' + v_j')$, i = 1, 2,…., m; j = 1, 2…., n,

$\Delta_{ij}'' = d_{ij} - (u_i'' + v_j'')$, i = 1, 2,…., m; j = 1, 2,…., n,

Further, we define

$U_i(x) = u_i' - zu_i''$, i = 1, 2,…., m,

$V_j(x) = v_j' - zv_j''$, j = 1, 2,…. n,

$Z_{ij}(x) = U_i(x) + V_j(x)$, i = 1, 2,…., m; j = 1, 2,……., n,

$C_{ij}(x) = c_{ij} - zd_{ij}$, i = 1, 2,…., m; j = 1, 2,……., n, and finally

$\Delta_{ij}(x) = C_{ij}(x) - Z_{ij}(x)$, i = 1, 2,…., m; j = 1, 2,……., n,

It can be express as

$\Delta_{ij} - \Delta'_{ij} - z\Delta''_{ij}$, i = 1, 2,…., m; j = 1, 2,……., n,

The optimality condition for LFT problem given in [14] state that a basic feasible solution is optimal if

$\Delta_{ij} \geq 0$, i = 1, 2,…., m; j = 1, 2,…., n,

3. THE INVERSE LFT ALGORITHM

In order to make the given feasible solution an optimal one, we have proposed an algorithm. Our algorithm is divided into two parts: in the first part (i.e. part (a)), we obtained the optimal solution of LFPT using [14] and in the second part (i.e. part (b)), we have converted the given feasible solution to an alternate optimal solution.

The proposed algorithm is as follows:

Part A:

Step 1: Calculate the initial basic feasible solution by using the method given in [24] or by any existing method.

Step 2: Obtain the variables u'_i and v'_j from loaded cells using the relation $c_{ij} = u'_i + v'_j$ for all i = 1, 2,…., m and j = 1, 2,…., n and then calculate $\Delta'_{ij} = c_{ij} \quad (u'_i + v'_j)$ for each unoccupied cell of the transportation table.

Step 3: Obtain the variables u''_i and v''_j from loaded cells using the relation $d_{ij} = u''_i + v''_j$ for all i = 1, 2,…., m and j = 1, 2,…., n and then calculate $\Delta''_{ij} = d_{ij} - (u''_i + v''_j)$ for each unoccupied cell of the transportation table.

Step 4: Calculate $z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij}\, x_{ij} / \sum_{i=1}^{m} \sum_{j=1}^{n} d_{ij}\, x_{ij}$ for the initial basis feasible solution.

Step 5: Calculate $\Delta_{ij} = \Delta'_{ij} - z\Delta''_{ij}$ for all unoccupied cells.

Step 6: If $\Delta_{ij} > 0$ for all i = 1, 2,…., m and j = 1, 2,……., n, then the present solution is optimal, otherwise the variable x_{ij} correspond to most negative Δ_{ij} will enter into the basis.

Step 7: Repeat the procedure until all $\Delta_{ij} \geq 0$ for all i = 1, 2,…., m and j = 1, 2,……., n, and calculate the optimal objective value z^* using step 4.

Part B:

Step 8: If x^0 is the given feasible solution then defining the following sets

A*: set of pairs of indices (*i, j*) of optimal x_{ij}

A^0: set of pairs of indices (*i, j*) of feasible x_{ij}

and A^0/ A*: set of pairs of indices (*i,j*) of A^0 which are not in A*

Step 9: Replace c_{ij} with c'_{ij} and d_{ij} with d'_{ij} for all $(i,j) \in A^0/A^*$, where $c'_{ij} = c_{ij} + \alpha_{ij}$ and $d'_{ij} = d_{ij} + \beta_{ij}$.

Step 10: Calculate $\Delta_{ij} = \Delta'_{ij} - z^*\Delta''_{ij}$ for all $(i,j) \in A^0/A^*$, where z^* is the optimal value of *z*, $\Lambda'_{ij} = c'_{ij} - (u'_i + v'_j)$ and $\Lambda''_{ij} = d'_{ij} - (u''_i + v''_j)$.

Step 11: find the values of α_{ij} and β_{ij} such that all Δ_{ij} calculated in step 10 are equal to zero and the sum of absolute values of α_{ij} and β_{ij} is minimum. Most of the time α_{ij} and β_{ij} can be calculated by inspection only, but for the large problems, we may assume $\alpha_{ij} = e_{ij} - f_{ij}$ and $\beta_{ij} = p_{ij} - q_{ij}$ where $e_{ij}, f_{ij}, p_{ij}, q_{ij} \geq 0$. Using this transformation α_{ij} and β_{ij} can be calculated by solving the following linear programming problem:

Min$\sum_{(i,j)\in A^0/A^*}(e_{ij} + f_{ij} + p_{ij} + q_{ij})$,

Subject to, $\Delta_{ij} = 0$, for all $(i,j) \in A^0/A^*$,

$e_{ij}, f_{ij}, p_{ij}, q_{ij} \geq 0$, for all $(i,j) \in A^0/A^*$.

In the case of maximization, we can use the above algorithm for inverse optimization except the optimality condition replace with $\Delta_{ij} \leq 0$ for all i = 1, 2,…., m and j = 1, 2,……., n, or even we can also use the same optimality condition i.e. $\Delta_{ij} \geq 0$ for all i = 1, 2,…., m and j = 1, 2,…., n, if we redefine Δ'_{ij} and Δ''_{ij} as $\Delta'_{ij} = u'_i + v'_j - c_{ij}$ and $\Delta''_{ij} = u''_i + v''_j - d_{ij}$ for all unoccupied cells.

4. NUMERICAL EXAMPLE

Consider the transportation problem

$$\text{Min } z = \frac{9x_{11} + 12x_{12} + 7x_{13} + 6x_{14} + 11x_{21} + 9x_{22} + 17x_{23} + 6x_{24} + 5x_{31} + 4x_{32} + 3x_{33} + 9x_{34}}{8x_{11} + 10x_{12} + 12x_{13} + 9x_{14} + 6x_{21} + 4x_{22} + 8x_{23} + 11x_{24} + 9x_{31} + 13x_{32} + 11x_{33} + 7x_{34}}$$

Subject to,

$x_{11} + x_{12} + x_{13} + x_{14} = 12$

$x_{21} + x_{22} + x_{23} + x_{24} - 19$

$x_{31} + x_{32} + x_{33} + x_{34} = 17$

$x_{11} + x_{21} + x_{31} = 3$

$x_{12} + x_{22} + x_{32} = 22$

$x_{13} + x_{23} + x_{33} = 18$

$x_{14} + x_{24} + x_{34} = 5$

$x_{ij} \geq 0$ for $i = 1, 2, 3; j = 1, 2, 3, 4$

Placing the data of above problem into the table, we have

Table 1

8 9	10 12	12 (12) 7	9 6	$a_1 = 12$
6 (3) 11	4 (5) 9	8 (6) 17	11 (5) 6	$a_2 = 19$
9 5	13 (17) 4	11 3	7 9	$a_3 = 17$
$b_1 = 3$	$b_2 = 22$	$b_3 = 18$	$b_4 = 5$	Z = 0.715

Where the entries at the top left and bottom right corners of each cell represents d_{ij} and c_{ij}. Now, solving the problem using the algorithm proposed in [21], we obtained the basic feasible solution and the values of x_{ij} are shown in the small brackets in Table 1.

Calculating $u_i', v_j', u_i'', v_j'', \Delta_{ij}'$ and Δ_{ij}'', and place them into the table, we have

Table 2

8 -2 8 9	10 2 13 12	12 (12) 7	9 -6 10 6	$u_1'' = 0$ $u_1' = 0$
6 (3) 11	4 (5+θ) 9	8 (6-θ) 17	11 (5) 6	$u_2'' = -4$ $u_2' = 10$
9 -6 -1 5	13 (17-θ) 4	11 -6 (θ) -9 3	7 -13 8 9	$u_3'' = 5$ $u_3' = 5$
$v_1'' = 10$ $v_1' = 1$	$v_2'' = 8$ $v_2' = -1$	$v_3'' = 12$ $v_3' = 7$	$v_4'' = 15$ $v_4' = -4$	Z = 0.715

The entries written at the bottom left and top right corners of cells represents Δ_{ij}' and Δ_{ij}'' respectively.

Calculating Δ_{ij} for all empty cells, we have

$\Delta_{11} = 8 - 0.715x(-2) = 9.43$

$\Delta_{12} = 13 - 0.715x(2) = 11.57$

$\Delta_{14} = 10 - 0.715x(-6) = 14.29$

$\Delta_{31} = -1 - 0.715x(-6) = 3.29$

$\Delta_{33} = -9 - 0.715x(-6) = -4.71$

$\Delta_{34} = 8 - 0.715x(-13) = 17.295$

Clearly, Δ_{33} is the negative, therefore x_{33} will enter into the basis, if we take $x_{33} = \theta$ (see table 2), then we have $\theta = 6$ and the new transportation table is

Table 3

8 -8 -1 9	10 -4 4 12	12 (12) 7	9 -12 1 6	$u_1'' = 0$ $u_1' = 0$
6 (3) 11	4 (11) 9	8 6 9 17	11 (5) 6	$u_2'' = -10$ $u_2' = 1$
9 -6 -1 5	13 (11) 4	11 (6) 3	7 -13 8 9	$u_3'' = -1$ $u_3' = -4$
$v_1'' = 16$ $v_1' = 10$	$v_2'' = 14$ $v_2' = 8$	$v_3'' = 12$ $v_3' = 7$	$v_4'' = 21$ $v_4' = 5$	Z = 0.655

Further, calculating the values of Δ_{ij}, we have

$\Delta_{11} = -1 - 0.655x(-8) = 4.24$

$\Delta_{12} = 4 - 0.655x(-4) = 6.62$

$\Delta_{14} = 1 - 0.655x(-12) = 8.86$

$\Delta_{23} = 9 - 0.655x(6) = 5.07$

$\Delta_{31} = -1 - 0.655x(-6) = 2.93$

$\Delta_{34} = 8 - 0.655x(-13) = 16.515$

All $\Delta_{ij} \geq 0$ therefore the solution is optimal.

If $x^0 = \{x_{13} = 12,\ x_{22} = 8, x_{23} = 6,\ x_{24} = 5, x_{31} = 3, x_{32} = 14\}$ is the given feasible solution, then we have $A^0 = \{(1,3),(2,2),(2,3),(2,4),(3,1),(3,2)\}$, $A^* = \{(1,3),(2,1),(2,2),(2,4),(3,2),(3,3)\}$ and $A^0/A^* = \{(2,3),(3,1)\}$

Now, we replace c_{23}, d_{23}, c_{31} and d_{31} with $c_{23} + \alpha_{23}, d_{23} + \beta_{23}, c_{31} + \alpha_{31}$ and $d_{31} + \beta_{31}$ and then recalculate Δ_{23} and Δ_{31} with the help of these values

$\Delta_{23} = \Delta'_{23} - z\Delta''_{23}$

$= (c'_{23} - (u'_2 + v'_3)) - z^*(d'_{23} - (u''_2 + v''_3))$

$= (17 + \alpha_{23} - (1 + 7)) - 0.655(8 + \beta_{23} - (-10 + 12))$

$= 5.07 + \alpha_{23} - 0.655\beta_{23},$

and similarly, $\Delta_{31} = 2.93 + \alpha_{31} - 0.655\beta_{31}$. Now we observe that $\alpha_{23} = 5.07$ and $\alpha_{31} = -2.93$ are the values, for which $\Delta_{23} = \Delta_{31} = 0$ and $|\alpha_{23}| + |\beta_{23}| + |\alpha_{31}| + |\beta_{31}|$ is minimum. Therefore the modified values of c_{23} and c_{31} are 11.93 and 2.07.

Now, we can check the optimality of the given feasible solution (x^0) for the modified transportation problem with the help of following table

Table 4

8 4 6.86 9	10 2 7.93 12	12 (12) 7	9 -6 4.93 6	$u_1'' = 0$ $u_1' = 0$
6 6 3.93 11	4 (8) 9	8 (6) 11.93	11 (5) 6	$u_2'' = -4$ $u_2' = 4.93$
9 (3) 2.07	13 (14) 4	11 -6 -3.93 3	7 -13 8 9	$u_3'' = 5$ $u_3' = -0.07$
$v_1'' = 4$ $v_1' - 2.14$	$v_2'' = 8$ $v_2' - 4.07$	$v_3'' = 12$ $v_3' - 7$	$v_4'' = 15$ $v_4' - 1.07$	Z = 0.655

Calculating Δ_{ij} for all unoccupied cells, we have

$$\Delta_{11} = 6.86 - 0.655x(4) = 4{,}24$$

$$\Delta_{12} = 7.93 - 0.655x(2) = 6.62$$

$$\Delta_{14} = 4.93 - 0.655x(-6) = 8.86$$

$$\Delta_{21} = 3.93 - 0.655x(6) = 0$$

$$\Delta_{33} = -3.93 - 0.655x(-6) = 0$$

$$\Delta_{34} = 8 - 0.655x(-13) = 16.515$$

All $\Delta_{ij} \geq 0$, therefore the given solution x^0 is an optimal solution of the modified problem.

5. CONCLUSION

This paper proposed an algorithm for inverse transportation problem of minimizing/maximizing the ratio of two linear functions. The approach can be use to obtain the modified values of objective coefficients in such a way that the given feasible solution becomes optimal. For the future work, this method can be extended to capacitated linear fractional transportation problem, transshipment problem and linear plus linear fractional transportation problem.

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