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Recursive Least Square Estimations of AR(q) Series

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Abstract. This paper is concerned with the recursive least square estimation in a time varying AR(q) process. The estimates are given and recursive relations are proposed. Some applications about the rate of convergence to the true value of parameters and change point detection is also studied. Finally a conclusion section is given.

Keywords: Auto-regressive; Decay factor; Recursive least square; Time varying parameters

1 Introduction. There are many unknown parameters while modeling a physical phenomena by a stochastic model. These parameters may be time varying, in practice. In this case, related models are referred as dynamic. There are many interesting estimation methods such as Kalman filter and recursive least square (RLS). The RLS has been applied in engineering (Kung (1978)), statistics (Brown *et al.* (1975)) and finance (JP Morgan company (1996)). For a more comprehensive review see Sinha and Rao (1991). Mathematically, this method provides a set of recursive relations between least square estimates based on $n-1$ and n sample sizes.

One of traditional time series is auto-regressive AR(p) model. Under some certain conditions, this model is stationary. However, when coefficients are time varying, this model is changed to dynamic model which is non-stationary. Therefore, the RLS may be applied here, which is the point of the current paper. This paper is organized as follows. in the next section, we propose the theoretical results. Recursive relations are given in section 3. Some applications are given in section 4. Conclusions are presented in section 5.

2 RLS in AR(p). Let X_t be an AR(p) process with time varying parameters, i.e., for $t \geq 1$

$$X_t = \sum_{j=1}^{q} \phi_{t-j} X_{t-j} + \varepsilon_t,$$

where ε_t is a white noise process with zero mean and variance σ^2 independent of $\{X_1, ..., X_{t-1}\}$. Here, $\{\phi_j\}_{-\infty}^{\infty}$ is a sequence of constants which should be estimated.

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Here, we want to estimate ϕ_t by

$$\widehat{\phi}_t = \sum_{k=0}^{p} w_k x_{t-k} = \mathbf{w}_t^T \mathbf{X}_t,$$

where the filter coefficients are $\mathbf{w}_t = (w_1, ..., w_p)^T$ and the p-recent samples of $X_s, s \geq 1$, that is $\mathbf{X}_t = (X_t, X_{t-1}, ..., X_{t-p})^T$, for some suitable p. The notation T stands for transpose of a vector. The RLS estimate of $\mathbf{w}_t$ is a weight vector which minimizes the cost function

$$C(\mathbf{w}_t) = \sum_{i=0}^{t} \lambda^{t-i} e_i^2,$$

where $e_i = \widehat{\phi}_i - \phi_i$, $i = 0, 1, ..., t$ and $0 < \lambda \leq 1$ is the decay factor. By differentiating of $C(\mathbf{w}_t)$ with respect to w_k, the k-th component of vector $\mathbf{w}_t$, we find that

$$\frac{\partial C}{\partial w_k} = 2 \sum_{i=0}^{t} \lambda^{t-i} e_i X_{i-k} = 0,$$

$k = 1, 2, ..., q$. These values of w_k, $k = 1, 2, ..., q$ are minimizers of cost function. Therefore,

$$\sum_{i=0}^{t} \lambda^{t-i} [\phi_i - \sum_{l=0}^{p} w_l X_{i-l}] X_{i-k} = 0,$$

or equivalently,

$$\sum_{l=0}^{p} w_l [\sum_{i=0}^{t} \lambda^{t-i} X_{i-l} X_{i-k}] = \sum_{i=0}^{t} \lambda^{t-i} \phi_i X_{i-k}.$$

However, the above q-equations may be summarized in a matrix equation as

$$R_t \mathbf{w}_t = \mathbf{r}_t,$$

where R_t and $\mathbf{r}_t$ are the weighted sample correlation and cross correlation between ϕ and $\mathbf{X}_t$, respectively. Therefore, $\mathbf{w}_t = R_t^{-1} \mathbf{r}_t$. They are defined as

$$\begin{aligned} R_t &= \sum_{i=0}^{t} \lambda^{t-i} \mathbf{X}_i \mathbf{X}_i^T, \\ \mathbf{r}_t &= \sum_{i=0}^{t} \lambda^{t-i} \phi_i \mathbf{X}_i. \end{aligned}$$

3 Recursive algorithm. Using the Woodbury matrix identity (see Chow and Wang (1994)) and some algebraic manipulations, we find that

$$\begin{aligned} \mathbf{w}_t &= \mathbf{w}_{t-1} + \alpha_t \mathbf{g}_t, \\ P_t &= \lambda^{-1} P_{t-1} + \mathbf{g}_t \mathbf{X}_t^T \lambda^{-1} P_{t-1}. \end{aligned}$$

The gain vector $\mathbf{g}_t$ is defined by

$$\mathbf{g}_t = P_{t-1}\mathbf{X}_t^T\{\lambda + \mathbf{X}_t^T P_{t-1}\mathbf{X}_t\}^{-1},$$

and the prior error is given by

$$\alpha_t = \phi_t - \mathbf{X}_t^T\mathbf{w}_{t-1}.$$

Note that the recursions for P_t satisfies in Riccarti equations and thus have parallel relations with Kalman filters. For run this recursive algorithm, we consider the initial values as $X_t = 0$ for $k = -p, ..., -1$ and $\mathbf{w}_0 = 0$ and $P_0 = \delta I_{(p+1)\times(p+1)}$.

4 Applications. In this section, we propose two applications of this method in the style of two examples.

Example 1. In this example, we survey the convergence of the RLS estimate to the true value of parameter. To this end, we consider an AR(1) process defined by

$$X_t = \pi X_{t-1} + \varepsilon_t,$$

$t = 1, 2, ..., 1000$ where ε_t is zero mean white noise process with variance 0.15 and $\pi = -0.48$. We have plotted the RLS estimate of π in Figure 1 (page 4) for $t = 200, ..., 1000$. It is seen that π_t converges to π.

Example 2. In this example, we study the change point detection. Consider the AR process of the previous example, but here, assume that parameter π changes from -0.48 to 0.01 after $t = 500$. We have plotted the RLS estimate for $t = 100, ..., 1000$ in Figure 2 (page 4). It is seen that the change point is detected correctly.

5 Conclusions. We considered the RLS estimates of coefficients of a time varying AR(q) process. The recursive algorithm is proposed and some applications are given.

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Figure 1: RLS in AR(1), with no change point

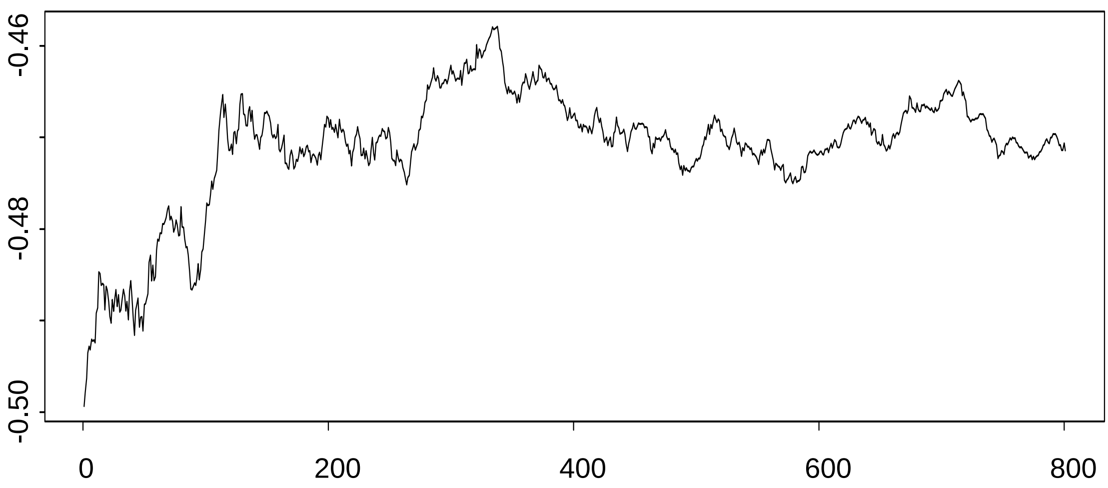

Figure 2: RLS in AR(1), with a change point