AMO - Advanced Modeling and Optimization, Volume 15, Number 3, 2013

An active set algorithm for optimal correction of infeasible linear inequalities with nonnegative variables

S. Ketabchi [*], H. Moosaei [†], L. Abedy [‡]

December 31, 2013

Abstract

In this work, we consider the linear inequalities with nonnegative variables that are infeasible. For correcting this system, we are doing the minimal correction using L_2 norm by changing just the right hand vector. We show that solving this problem is equivalent to solving a nonlinear convex problem with nonnegative constraint. We present an active set algorithm (ASA) based on nonmonotone gradient projection step to solving this problem. Our computational results on various randomly generated problems show that this method is efficient.

Keywords: Active set algorithm, Linear inequality, Nonlinear convex problem, Optimal correction.
AMS Subject Classification: 90C30, 90C31, 15A39

1 Introduction

One of the frequently encountered issues in applied science is how to deal with infeasible systems [1, 2, 3].

[*]Faculty of Mathematical Sciences, University of Guilan, P.O. Box 1914, Rasht, Iran, sketabchi@guilan.ac.ir

[†]Corresponding author. Department of Mathematics, University of Bojnord, Bojnord, Iran, hmoosaei@gmail.com

[‡]Faculty of Mathematical Sciences, University of Guilan, P.O. Box 1914, Rasht, Iran

AMO - Advanced Modeling and Optimization. ISSN: 1841-4311

In this paper, we consider the following set of linear inequalities that are infeasible:

$$\begin{aligned} Ax &\leq b, \\ x &\geq 0, \end{aligned} \tag{1}$$

where $A \in R^{m\times n}$ and $b \in R^m$. In other words, there is no $x \in R^n$ for which (1) is feasible. The inconsistency in system (1) might be due to various reasons, such as lack of interaction between different groups who are defining the constraints, wrong or inaccurate estimates, error in data, over optimistic goals, and many others. As remodeling of a problem, finding the errors, and generally removing all the impossible barriers of a system might take remarkable time and expenses, and also we might eventually get to an infeasible system again; we relinquish to do so. Correcting system (1) to a feasible system by minimal changes in its data have been attempted for some time. Up until now several algorithms have been developed [4, 5]. In this work, for correcting system (1), we will make the changes just in the right-hand side vector b and obtain an unconstrained optimization problem with nonnegative variables as follows.

$$\min_{x\geq 0} \frac{1}{2}\|(Ax-b)_+\|^2.$$

We present an active set algorithm (ASA) to solve above problem. The ASA consists of a nonmonotone gradient projection step, an unconstrained optimization step, and a set of rules for branching between the two steps. This paper is organized as follows. Optimal correction of infeasible linear inequality systems is demonstrated in Section 2. In Section 3, ASA is reviewed. In Section 4, some examples on various randomly generated problems are provided to illustrate the efficiency and validity of our proposed method. Concluding remarks are given in Section 5. In this work all vectors will be column vectors and we denote the n-dimensional real space by R^n. We mean $A^\top$, and $\|.\|$, the transpose of matrix A and Euclidean norm respectively. By $(a)_+$ we mean a vector that we obtain from a by replacing the negative component by zero. The gradient $\nabla f(x)$ is a row vector, while $g(x) = \nabla f(x)^T$ is a column vector; here T denotes transpose. The gradient at the iterate x_k is $g_k = g(x_k)$.

2 2-Norm Corrections

The minimal correction using the l_2 norm by changing the right hand side vector is:

$$\min_{x\geq 0,\ r} \frac{1}{2}\|r\|^2, s.t. \quad Ax \leq b + r.$$

In the following theorem we show how we compute optimal x and r values.

Theorem 1. *Let x^* and r^* be the optimal solution of* (2)*. Then $r^* = (Ax^* - b)_+$, and x^* is an optimal solution of*

$$\min_{x\geq 0} \frac{1}{2}\|(Ax-b)_+\|^2. \tag{2}$$

Proof. Let us write (2) as:

$$\begin{aligned} &\min_{x\geq 0} \min_{r} \frac{1}{2}\|r\|^2, \\ &s.t. \quad Ax \leq b + r. \end{aligned} \tag{3}$$

Now for a given $x \in R^n$, let us first consider the inner minimization problem i.e.,

$$\begin{aligned} &\min_{r} \frac{1}{2}\|r\|^2, \\ &s.t. \quad Ax \leq b + r. \end{aligned} \tag{4}$$

It is obvious that problem (4) is a convex minimization problems. The Lagrangian of the problem (4) is given by

$$L(r, \lambda) = \frac{1}{2}\|r\|^2 - \lambda^T(Ax - (b + r)), \quad \lambda \geq 0.$$

The KKT conditions are necessary and sufficient for optimality that are given by:

$$r - \lambda = 0, \tag{5}$$

$$Ax \leq b + r, \tag{6}$$

$$\lambda^T(Ax - b - r) = 0, \tag{7}$$

$$\lambda \geq 0,$$

where the vector λ denotes the lagrange multipliers. From the equation (5) one has $r = \lambda$. From the equations(6) and (7), we have that $r^T(Ax - b - r) = 0$,

$r \geq 0$. Therefore we obtain $r = (Ax - b)_+$ (see [6]). By combining these expressions, we find that the problem (2) can then be written as

$$\min_{x \geq 0} \frac{1}{2} \|(Ax - b)_+\|^2.$$

This completes the proof. □

It is obvious that the KKT conditions for above problem are

$$\begin{aligned} &A^T(Ax - b)_+ \geq 0, \\ &x^T A^T(Ax - b)_+ = 0, \\ &x \geq 0. \end{aligned}$$

3 Hager-Zhang active set algorithm

In this section we apply the Hager-Zhang active set algorithm (HZ-ASA) [7] to solve the following problem:

$$\min_{x \geq 0} \frac{1}{2} \|(Ax - b)_+\|^2.$$

The main reason for selection of this algorithm is its excellent convergence theories in addition to the promising numerical results reported in [7]. Moreover, unlink [8, 9], this method admits the superlinear convergence under the same conditions while it does not need any explicit form of second order information and/or solution of system of linear equations per cycle. These properties makes it an ideal choice to solve large scale problems. Unlike the related methods which use the same interactions in the course of optimization, this method start with a cheap constrained first-order method and after sufficient progress toward a local solution, branches to a (more expensive) higher-order unconstrained solver.

Although the gradient projection scheme of the nonmonotone gradient projection algorithm (NGPA) has an attractive global convergence theory, the convergence rate can be slow in a neighborhood of a local minimizer. In contrast, for unconstrained optimization, the conjugate gradient algorithm often exhibits superlinear convergence in a neighborhood of a local minimizer.

In the following algorithm P denotes the projection onto feasible set:

$$P(x) = \arg\min_{y \in \Omega} \|x - y\| \tag{8}$$

Starting at the current iterate x_k, we compute an initial iterate $\bar{x}_k = x_k - \bar{\alpha}_k g_k$. The only constraint on the initial steplength $\bar{\alpha}_k$ is that $\bar{\alpha}_k \in [\alpha_{min}, \alpha_{max}]$, where

α_{min} and α_{max} are fixed, positive constant, independent of k. Since the nominal iterate may lie outside Ω, we compute its projection $P(\bar{x}_k)$ onto Ω. The search direction is $d_k = P(\bar{x}_k) - x_k$, similar to the choice made in SPG2 [10]. Using a nonmonotone line searche along the line segment connecting x_k and $P(\bar{x}_k)$, we arrive at the new iterate x_{k+1}.
In the statement of the NGPA given below, f_k^r denotes the "reference" function value. A monotone line search corresponds to the choice $f_k^r = f(x_k)$. The nonmonotone GLL scheme takes $f_k^r = f_k^{max}$, where

$$f_k^{max} = \max\{f(x_{k-i}) : 0 \leq i \leq \min(k, M-1)\}. \tag{9}$$

Here $M > 0$ is a fixed integer, the memory. To find a procedure for choosing the refrence function value based on our CBB scheme see [7].
NGPA PARAMETERS.

- $\epsilon \in (0, \infty)$, error tolerance
- $\delta \in (0, 1)$, descent parameter used in Armijo line search
- $\eta \in (0, 1)$, decay factor for stepsize in Armijo line search
- $[\alpha_{min}, \alpha_{max}] \subset (0, \infty)$, interval containing initial stepsiz.

NONMONOTONE GRADIENT PROJECTION ALGORITHM (NGPA). Initialize $k = 0$, $x_0 =$ starting guess, and $f_{-1}^r = f(x_0)$.
While $\|P(x_k - g_k) - x_k\| > \epsilon$

1. Choose $\bar{\alpha}_k \in [\alpha_{min}, \alpha_{max}]$ and set $d_k = P(x_k - \bar{\alpha}_k g_k) - x_k$.
2. Choose f_k^r so that $f(x_k) \leq f_k^r \leq \max\{f_{k-1}^r, f_k^{max}\}$ and $f_k^r \leq f_k^{max}$ infinitely often.
3. Let f_R be either f_k^r or $\min\{f_k^{max}, f_k^r\}$. If $f(x_k + d_k) \leq f_R + \delta g_k^T d_k$, then $\alpha_k = 1$.
4. If $f(x_k + d_k) > f_R + \delta g_k^T d_k$, then $\alpha_k = \eta^j$, where $j > 0$ is the smallest integer such that
$$f(x_k + \eta^j d_k) \leq f_R + \eta^j \delta g_k^T r d_k. \tag{10}$$
5. Set $x_{k+1} = x_k + \alpha_k d_k$ and $k = k + 1$.

End
Further details about NGPA and convergence theory can be found in [7]. Although, the unconstrained algorithm (UA) used in HZ-ASA is the conjugate algorithm CG-DESCENT [11], a broad class of unconstrained optimization algorithms (UAs) can be applied.
We develop an ASA which uses the NGPA to identify active constraints, and which uses an unconstrained optimization algorithm, such as the CG-DESCENT, to optimize f over a face identified by the NGPA.

Now we define some notation for ASA. For any $x \in \Omega$, let $\mathcal{A}(x)$ and $\mathcal{I}(x)$ denote the active and inactive indices, respectively:

$$\begin{aligned} \mathcal{A}(x) &= \{i \in [1,n] \ : \ x_i = 0\}, \\ \mathcal{I}(x) &= \{i \in [1,n] \ : \ x_i > 0\}. \end{aligned}$$

The active indices are further subdivided into those indices satisfying strict complementarity and the degenerate indices:

$$\begin{aligned} \mathcal{A}_+(x) &= \{i \in \mathcal{A} \ : \ g_i(x) > 0\}, \\ \mathcal{A}_0(x) &= \{i \in \mathcal{A} \ : \ g_i(x) = 0\}. \end{aligned}$$

We let $g_I(x)$ denote the vector whose components associated with the set $\mathcal{I}(x)$ are identical to those of $g(x)$, while the components associated with $\mathcal{A}(x)$ are zero:

$$g_{Ii}(x) = \begin{cases} 0 & if x_i = 0, \\ g_i(x) & if x_i \neq 0. \end{cases}$$

An important feature of our algorithm is that we try to distinguish between active constraints satisfying strict complementarity and active constraints that are degenerate using an identification strategy, which is related to the idea of an identification function introduced in [12]. Given fixed parameters $\alpha \in (0,1)$ and $\beta \in (1,2)$, we define the (undecided index) set $\mathcal{U}$ at $x \in \mathcal{B}$ as follows:

$$\mathcal{U}(x) = \{i \in [1,n] \ : \ g_i(x) \geq \|d^1(x)\|^{\alpha} \ \text{and} \ x_i \geq \|d^1(x)\|^{\beta}\}.$$

In the numerical experiments, we take $\alpha = \frac{1}{2}$ and $\beta = \frac{3}{2}$.
The ASA is presented in the following algorithm. In the first step of the algorithm, we execute the NGPA until we feel that the active constraints satisfying strict complementarity have been identified. In step 2, we execute the UA until a subproblem has been solved (step 2a). When new constraints become active

in step 2b, we may decide to restart either the NGPA or the UA. By restarting the NGPA, we mean that x_0 in the NGPA is identified with the current iterate x_k. By restarting the UA, we mean that iterates are generated by the UA using the current iterate as the starting point.

ASA PARAMETERS.

- $\epsilon \in (0, \infty)$, error tolerance, step when $\|d^1(x_k)\| \leq \epsilon$
- $\mu \in (0,1)$, $\|g_I(x_k)\| < \mu\|d^1(x_k)\|$ implies subproblem solved
- $\rho \in (0,1)$,decay factor for μ tolerance
- $n_1 \in [1, n)$, number of repeated $\mathcal{A}(x_k)$ before switch from the NGPA to the UA
- $n_2 \in [1, n)$, used in switch from the UA to the NGPA

ACTIVE SET ALGORITHM (ASA)

1. While $\|d^1(x_k)\| > \epsilon$ execute the NGPA and check the following:

   a. If $\mathcal{U}(x_k) = \emptyset$, then

   If $\|g_I(x_k)\| < \mu\|d^1(x_k)\|$, then $\mu = \rho\mu$.

   Otherwise, go to step 2.

   b. Else if $\mathcal{A}(x_k) = \mathcal{A}(x_{k-1}) = \cdots = \mathcal{A}(x_{k-n_1})$, then if $\|g_I(x_k)\| \geq \mu\|d^1(x_k)\|$, then go to step 2.

   End

2. While $\|d^1(x_k)\| > \epsilon$ execute the UA and check the following:

   a. If $\|g_I(x_k)\| < \mu\|d^1(x_k)\|$, then restart the NGPA (step 1).

   b. If $|\mathcal{A}(x_{k-1)}| < |\mathcal{A}(x_k)|$, then

   If $\mathcal{U}(x_k) = \emptyset$ or $|\mathcal{A}(x_k)| > |\mathcal{A}(x_{k-1)}| + n_2$, restart the UA at x_k.

   Else restart the NGPA.

   End

Furthermore details about ASA and its convergence theory can be found in [7].

4 Numerical testing

In this section we present numerical results to obtain optimal correction of infeasibility system in linear inequality on various randomly generated problems. We used the ASA for solving (2). The algorithm has been tested using MATLAB 7.9.0 on a Core 2 Duo 2.53 GHz with main memory 4 GB.
The stopping condition was

$$\|P(x-g)-x\|_\infty \leq 10^{-8}.$$

In running any of these codes, default values were used for all parameters. In the NGPA, we chose the following parameter values:

$$\alpha_{min} = 10^{-20}, \quad \alpha_{max} = 10^{20}, \quad \eta = .5, \quad \delta = 10^{-4}, \quad M = 8.$$

Here M is the memory used to evaluate f^{max} (see ((9))). In the ASA the parameter values were as follows:

$$\mu = .1, \quad \rho = .5, \quad n_1 = 2, \quad n_2 = 1.$$

In the CBB method, the parameter values were the following:

$$\theta = .975, \quad L = 3, \quad A = 40, \quad m = 4, \quad \gamma_1 = M/L, \quad \gamma_2 = A/M.$$

Test problems are generated infeasible system (1) by using the following MATLAB code:

Sgen: Generate random infeasible system
(Input:m, n, d (density); Output: $A \in R^{m\times n}$ and $b \in R^m$).
```
pl=inline('(abs(x)+x)/2');
m=input('enter m= '); n=input('enter n= ');
d=input('enter d= '); In the numerical experiments, we take d = .1
m1=max(m-round(0.5*m), m-n);
A1=sprand(m1, n, d); A1=1*(A1-0.5*spones(A1));
x=spdiags(rand(n, 1), 0, n, n)*10*(rand(n,1)-rand(n,1));
x=spdiags(ones(n,1)-sign(x), 0, n, n)*10*(rand(n,1)-rand(n,1));
m2=m-m1;
u=randperm(m2); A2=A1(u, :);
b1=A1*x+spdiags((rand(m1,1)), 0, m1, m1)*1*ones(m1,1);
```

```
b2=b1(u)+spdiags((rand(m2,1)), 0, m2, m2)*10*ones(m2,1);
A=100*[A1; –A2];
b=[b1; –b2];
```

In Table 1, we present numerical experiment. In this table the first column indicates the size of matrix A, $\|(Ax^* - b)_+\|$ is the objective function of (2) and $\|x^*\|_\infty$ indicates the norm infinity of solution. The next three columns correspond to the KKT conditions for (2) and the final column indicates time. This results show that the ASA for correcting of system of linear inequalities (1) is efficient with high accuracy in the case where $n \ll m$.

Table 1 illustrates effectiveness and performance behavior of our proposed method.

Table 1: Results of ASA on randomly generated problems

m, n	$\|(Ax^* - b)_+\|$	$\|x^*\|_\infty$	$\|(-x^*)_+\|$	${x^*}^T \nabla f(x^*)$	$\|(-\nabla f(x^*))_+\|_\infty$	time(s)
$50, 10$	$1.9890e+001$	$2.1127e-001$	0	$2.4100e-011$	$8.7539e-012$	0.2239
$200, 100$	$5.7488e+001$	$3.2640e-001$	0	$4.8680e-012$	$4.7834e-011$	2.2114
$300, 100$	$7.2441e+001$	$2.1193e-001$	0	$1.5306e-011$	$8.2068e-012$	1.7255
$500, 250$	$1.1682e+002$	$3.1504e-001$	0	$8.1142e-013$	$5.5451e-011$	8.9036
$700, 300$	$7.5153e+001$	$9.9486e+001$	0	$2.6545e-009$	$9.7741e-011$	10.3966
$800, 400$	$1.6144e+002$	$2.3366e-001$	0	$1.2092e-010$	$7.6071e-011$	24.9134
$1000, 500$	$1.8941e+002$	$3.0364e-001$	0	$2.2778e-011$	$6.7141e-011$	36.6693
$1500, 700$	$1.8486e+002$	$5.1321e+000$	0	$9.3400e-010$	$6.9567e-011$	65.6876

5 Conclusion

In this article, correction of infeasible linear inequalities with nonnegative variable were studied by applying minimal changes right-hand vector, using $2-$norm. In support of predicted theory, several test examples are solved using ASA. Numerical results , show that the suggested algorithm is correct and efficient in the usually case where $n \ll m$. It is natural to ask whether we can adapt the suggested approach to solve (2), in the case where, the number of variables is close to the number of equality.

References

[1] Y. Censor, M.D. Altschuler, W.D. Powlis, A computational solution of the inverse problem in radiation therapy treatment planning, Applied Mathematics and Computation. 25 (1988) 57–87.

[2] Y. Censor, M.D. Altschuler, W.D. Powlis, On the use of Cimminos simultaneous projections method for computing a solution of the inverse problem in radiation therapy treatment planning, Inverse Problems. 4 (1988) 607–623.

[3] Y. Censor, Mathematical optimization for the inverse problem of intensity-modulated radiation therapy, in: J.R. Palta and T.R. Mackie (Editors), Intensity-Modulated Radiation Therapy: The State of The Art, American Association of Physicists in Medicine, Medical Physics Monograph No. 29, Medical Physics Publishing, Madison, Wisconsin, USA. (2003) 25–49.

[4] P. Amaral, M.W. Terosset, P. Barahona, A framework for optimal correction of inconsistent linear constraint, Constraints. 10 (2005) 67–86

[5] P. Amaral, J. Júdice, H.D. Sherali, A reformulation–linearization–convexification algorithm for optimal correction of an inconsistent system of linear constraints, Computers and Operations Research. 35 (2008) 1494–1509.

[6] O. Mangasarian, A Newton method for linear programming, Journal of Optimization Theory and Applications. 121 (2004) 1–18.

[7] W. W. Hager, H. Zhang, A new active set algorithm for box constrained optimization, SIAM J. Optim. 17 (2006) 526-557.

[8] C.J. Lin, J.J. Mońe, Newton's Method for Large Bound-Constrained Optimization Problems, SIAM Journal on Optimization. 9 (1999) 1100–1127.

[9] M. Heinkenschloss, M. Ulbrich, S. Ulbrich, Superlinear and quadratic convergence of affine-scaling interior-point Newton methods for problems with simple bounds without strict complementarity assumption, Mathematical Programming. 86 (1999) 615–635.

[10] E. G. Birgin, J. M. Martínez, M. Raydan, Algorithm 813: SPGsoftware for convexconstrained optimization, ACM Trans. Math. Software. 27 (2001) 340-349.

[11] J. Nocedal, S.J. Wright, Numerical Optimization, Springer Science, 1999.

[12] F. Facchinei, A. Fischer, C. Kanzow, On the accurate identification of active constraints, SIAM J. Optim. 9 (1998) 14-32.