

Application of Uncertain Programming to an Inventory Model for Imperfect Quantity under Time Varying Demand

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Abstract

In the present paper we investigate a EOQ model for imperfect items under time variable linear demand. The defective items is being screened out by a 100 % screening process and then sold in a single batch by the end of the 100% screening process with a salvage price. The unsold perfect item during the cycle is sold at a different salvage price at the end of cycle. Two different types for cost parameters are considered namely crisp constant and uncertain variable. For each case, optimal policy is obtained. We have considered expected value model and uncertain chance constraint programming for the Uncertain EOQ model. Numerical example and sensitivity analysis are provided to illustrate the effectiveness of the above models.

keywords

Inventory, Imperfect quality; Economic Order Quantity

1 Introduction

Inventory management is the one of the important branch in management sciences. The main key of a successful business is to provide the customer his demand within shortest possible time, with the best quality, and all at a competitive price. Using proper Inventory Management this is quite possible to achieve. The classical economic order quantity (EOQ) model and its variants are popular among researchers and management professionals for their simplicity. Applying the various modifications on the traditional EOQ model, academicians and professionals are always interested to develop a model which can cope with real life situations in much better way. Every year hundreds of research papers are being published in this area in various national and international journals. The fundamental EOQ model developed by Harris(1915) involved the assumption of constant demand and perfect quality items ; both of these conditions fail to cope up with the realistic situations in the Business scenario. In reality, the production process is not always free of defects. Imperfect quality items are unavoidable in an inventory system due to imperfect production process, natural disasters, damages, or many other reasons. During last decades lot-of research work were published in the area of EOQ and EPQ of imperfect quality items. Rosenblatt and Lee [19] discussed an EPQ model where they assumed that the defective items could be reworked instantaneously at a cost and found that the presence of a fraction of defective products motivates smaller lot sizes. Shwaller

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[20] presented a procedure and assumed that imperfect quality items are present in a known proportions and considered fixed and variable inspection costs for finding and removing the defective items. Zhang and Gherchak [21] considered a joint lot sizing and inspection policy with the assumption that a random proportion of lot size are defective. They assumed that defective items are not reworkable and used the concept of replacement of defective items by good quality items. Salameh and Jaber[22] assumed that the defective items could be sold at a discounted price in a single batch by the end of the 100 % screening process and found that the economic lot size quantity tends to increase as the average percentage of imperfect quality items increases. Goyal et al.[23] made some modifications in the model of Salameh et al.[22] to calculate actual cost and actual order quantity. Chang [24] considered inventory problem for items received with imperfect quality, where, upon the arrival of order lot, 100% screening process is performed and the items of imperfect quality are sold as a single batch at a discounted price, prior to receiving the next shipment. He assumed defective rate as a fuzzy number. He also presented a model with fuzzy defective rate and fuzzy demand. Papachristos et al[25] pointed out that the sufficient conditions to prevent shortages given in Salameh et al.[22] may not really prevent their occurrence and considering the timing of withdrawing the imperfect quality items from stock, they clarified a point not clearly stated in Salameh et al.[22]. Wee et al [34] developed a optimal inventory model for items with imperfect quantity and shortages were backordered. They allowed 100% screening of items in which screening rate is greater than the demand rate. Chung et al.[27] considered an inventory model with imperfect quality items under the condition of two warehouses for storing items. Jaber et al. [28] incorporated the concept of entropy cost in the extension of the inventory model with imperfect quality and they assumed two different types of holding costs for the items with perfect quality and imperfect quality. Jaber et al.[29] also considered the assumptions of learning curve and shown that percentage of defective lot size reduces according to the learning curve. A detailed survey of the recent inventory models with imperfect items are provided by [30]. Along a different viewpoint, [31] considered a production-inventory model under a process-quality design approach and obtained optimal policy using mathematica software . They claimed their model enhanced high-quality product at a minimal total cost. [33] considered imperfect production system with allowable shortages due to regular preventive maintenance for products sold with free minimal repair warranty to attract customers and determines the optimum buffer level and production run time to minimize per unit production cost. The cost minimization optimal policy was considered by [32] when the produced item of imperfect production system obeys general distribution pattern, with its quality being either perfect, imperfect or defective. The fractions of such items were restricted to constants and they also established that their model becomes classical EPQ model in case imperfect quality percentage is zero or even close to zero. Recently, [34] used renewal reward theorem to construct economic production quantity model for imperfect items with shortage and screening constraint using time interval as decision variable and shown the robustness of the model.

Demand is the one of the important characteristic of an inventory system .Actually, the inventory system exist because there are demands. Demands provide the revenue in the organization. Higher is the demand larger will be the profit and performance of organization will be better. To satisfy the demand and avoid shortages, inventories are kept as buffer; then the organization becomes an inventory system. Demand is not usually constant but have increasing or decreasing trend with time. The role of demand in an inventory system is something like money in the wallet; without money wallet is of no use. The traditional EOQ assumes that demand is constant; which is quite incompatible in the real business scenario. Many authors have worked in the area of time varying demand. Donaldson [8] was the first to developed an EOQ model with linearly increasing demand over a finite time horizon. Wagner and Whitin[13] gave Dynamic Programming (DP) algorithm for the determination of an EOQ by treating time to be a discrete variable. Silver [12] obtained a simple solution for Donaldsons problem[8] using the Silver-Meal heuristic. Ritchie [11] developed the exact solution for a linearly increasing demand, for Donaldsons

problem[8]. Mitra, Cox and Jesse [10] presented an algorithm for adjusting the EOQ model for the case of demand patterns having linearly increasing or decreasing trends. Goswami et al.[14] considered a finite replenishment rate inventory model under linear time varying demand allowing shortages.

The world we live is not deterministic, namely it is not the one in which past events fully determine future ones. So one has to understand that nature of world is full of uncertainty and many theories are developed so far to handel uncertainty in mathematical models. Wherever we talk about measurement or some calculation, uncertainty comes naturally. For example,an error in a scientific measurement or calculation means the inevitable uncertainty that is present in all measurements and they cannot be eliminated. A suitable mathematical approach to deal with uncertainty in decision-making should also take into consideration the human subjectivity, rather than employing only objective probability measures. This kind of uncertainty of human behavior led to the development of a new areas in decision -analysis such as Probability theory,Fuzzy Logic and Uncertainty theory. The probability theory is based on the results of the previous experiments. By performing an experiment infinitely large number of times and obtaining the ratio of the number of occurrence of an event to the total number of experiments performed; the probability of occurrence of that event are calculated. Also for calculating probability one must have complete information of sample space and to obtain probability a condition of equally likeliness must be imposed. The concept of a fuzzy set was coined by Zadeh [7] to represent the classes whose boundaries are ill-defined, or flexible. To denote the level of evidence in support of belongingness or occurrences, membership functions are defined whose values lie in the interval $[0, 1]$. Fuzzy methodologies are mainly useful for approximate reasoning, mainly for the systems in which there is some vagueness and imprecision to deal with the linguistic variables. In the fundamental paper Zadeh[7] clearly stated that fuzziness deals with vagueness but it shouldnot be considered identical with uncertainty. So, Probability theory and fuzzy theory are the explicitly the theories of different kind of uncertainties Also the fuzzy logic is not self-dual. This implies an event occurs and does not occur at the same time with different level of evidences which is contradictory to human reasoning. So it is quite clear that both probability and fuzzy theory have their limitations in modeling the uncertainties of human behavior.

Various unpredictable incidents may occur in the market place; such as- the outbreak of war, terrorist attacks, companys bankruptcy, discount offers, merger or consolidation between companies and so on. When such events appear, historical data (or forecast) are not able to fully determine the prices. Also the factors like petrol price hikes, vehicle or labor strikes, inflation, hike in road tax or custom tax occurs suddenly and they highly effect the carrying costs, labor cost and all other associated costs. Liu[1] in 2007 developed Uncertainty theory which is quite able to deal with such kind subjective uncertainty. The uncertainty theory concerns an incomplete or imperfect knowledge of something which is necessary to solve the problem. Peng and Iwamura [3] gave a sufficient and necessary condition of uncertainty distribution. Gao [4] provided some mathematical properties of uncertain measure. You [16] proved some convergence theorems of uncertain sequences. In addition, Liu and Ha [6] proved some expected value formulas for functions of uncertain variables. As an application of uncertainty theory, Liu [2] proposed a spectrum of uncertain programming that is a type of mathematical programming involving uncertain variables. For our investigation, we have considered all the costs associated with our inventory model as Uncertain variables. We have developed expected value model and uncertain chance constrained programming model for the uncertain cost function to obtain the optimal inventory policy.

In the present paper we have developed first a deterministic EOQ model with imperfect item under time varying declining demand; then we have modified it to one which consider all the cost parameters as uncertain variables. Since uncertain model cannot be solved directly we have used two different approaches to transform the uncertain profit function into its crisp equivalent and then solution procedure is provided. In order to incorporate intuitionistic implications, we have also discussed the condition under which our model can be transformed into simpler models.

2 Definitions and Basics of Uncertainty Theory

Before presenting the inventory model in uncertain environment, let us introduce some useful definitions and fundamental results about Uncertainty theory in brief.

Definition 2.1 (Liu [1]) Let Γ be a nonempty set, and let \mathbf{A} be a σ -algebra over Γ . Each $\Lambda \in \mathbf{A}$ is called an event . In order to provide an axiomatic definition of uncertain measure, it is necessary to assign to each event Λ ; a number $M\{\Lambda\}$ which indicates the level that Λ will occur. In order to ensure that the number $M\{\Lambda\}$ has uncertain mathematical properties, Liu [1] proposed the following four axioms:

Axiom 1 (Normality) $M\{\Gamma\} = 1$.

Axiom 2 (Self-Duality) $M\{\Lambda\} + M\{\Lambda^c\} = 1$ for any event Λ .

Axiom 3 (Countable Subadditivity) For every countable sequence of events $\{\Lambda_i\}$, we have

$$M\left\{\bigcup_{i=1}^{\infty} \Lambda_i\right\} \leq \sum_{i=1}^{\infty} M\{\Lambda_i\} \tag{2.1}$$

Axiom 4 (Product Measure Axiom) Let $\Lambda_1 \times \Lambda_2 \cdots \times \Lambda_n$ be an uncertain event on the universe $\Gamma_1 \times \Gamma_2 \cdots \times \Gamma_n$ Then,

$$M\{\Lambda_1 \times \Lambda_2 \cdots \times \Lambda_n\} = \text{Min}\{M\{\Lambda_1\}, M\{\Lambda_2\} \cdots M\{\Lambda_n\}\} \tag{2.2}$$

Definition 2.2 (Liu [1]) The set function M is called an uncertain measure if it satisfies the normality, self-duality, Countable subadditivity and Product Measure axioms. The triplet (Γ, \mathbf{A}, M) is called uncertainty space.

Definition 2.3 (Liu [1]) An uncertain variable is measurable function ξ from an uncertainty space (Γ, \mathbf{A}, M) to the set of Real numbers \mathbb{R} , i.e, for any Borel set \mathbf{B} of real numbers, the set

$$\{\xi \in \mathbf{B}\} = \{\gamma \in \Gamma | \xi\{\gamma\} \in \mathbf{B}\} \text{ is an event.}$$

Definition 2.4 (Liu [1]) The uncertainty distribution $\Phi : \mathbb{R} \rightarrow [0, 1]$ of an uncertain variable ξ is defined by $\Phi(x) = M\{\xi \leq x\}$

Definition 2.5 (Liu [1]) Let ξ be an uncertain variable. Then the expected value of ξ is defined by $E[\xi] = \int_0^{\infty} M\{\xi \geq r\}dr - \int_{-\infty}^0 M\{\xi \leq r\}dr$

Definition 2.6 (Liu [1]) The uncertainty distribution Φ is said to be Regular if its inverse function $\Phi^{-1}(\alpha)$ exists and is unique for each $\alpha \in (0, 1)$.

Definition 2.7 (Liu [1]) Let ξ be an uncertain variable with Regular uncertainty distribution Φ . Then the inverse function Φ^{-1} is called the inverse uncertainty distribution of ξ .

Definition 2.8 (Liu [1]) Let ξ be an uncertain variable, and $\alpha \in (0, 1]$. Then $\xi_{inf}(\alpha) = \text{inf}\{r | M\{\xi \leq r\} \geq \alpha\}$ is called the α -pessimistic value to ξ .

Definition 2.9 (Liu [1]) Let ξ and η be two uncertain variables, we say that $\xi > \eta$ iff $E[\xi] > E[\eta]$ holds. This is called Expected value criterion for comparing two uncertain variables.

Definition 2.10 (Liu [1]) Let ξ and η be two uncertain variables, we say that $\xi > \eta$ iff $\xi_{inf}(\alpha) > \eta_{inf}(\alpha)$ holds $\forall \alpha \in [0,1]$. This is called Pessimistic Value criterion for comparing two uncertain variables.

2.1 Useful Results

Theorem 2.1 (Liu [1]) Let ξ be an uncertain variable with uncertainty distribution Φ . If the expected value exists, then

$$E[\xi] = \int_0^1 \phi^{-1}(\alpha) d\alpha \quad (2.3)$$

Theorem 2.2 (Liu [1]) Let ξ be an uncertain variable with uncertainty distribution Φ . Then its α -pessimistic value is

$$\xi_{inf}(\alpha) = \phi^{-1}(\alpha) \quad (2.4)$$

Theorem 2.3(Liu [1]) Let $\xi_1, \xi_2, \xi_3, \dots, \xi_n$ be independent uncertain variables with uncertainty distributions $\Phi_1, \Phi_2, \Phi_3, \dots, \Phi_n$ respectively. If f is a strictly increasing function. Then

$$\xi = f(\xi_1, \xi_2, \xi_3, \dots, \xi_n) \quad (2.5)$$

is an uncertain variable with inverse uncertainty distribution

$$\phi^{-1}(\alpha) = f(\phi_1^{-1}(\alpha), \phi_2^{-1}(\alpha), \phi_3^{-1}(\alpha), \dots, \phi_n^{-1}(\alpha)) \quad (2.6)$$

3 Mathematical Formulation of the Model

3.1 Derivation of the Crisp Model

Notations and Assumptions

Following notations are used in the Crisp model-

$\lambda(t)$: variable demand rate;

s : screening rate;

c_h : holding cost per unit, per unit time;

c_o : ordering cost ;

c_p : purchasing cost per unit;

c_s : screening cost per unit;

v_1 : selling price for the good items during selling season.

v_2 : salvage price for the good items after selling season.

v_3 : salvage price for the imperfect items.

y : lot size;

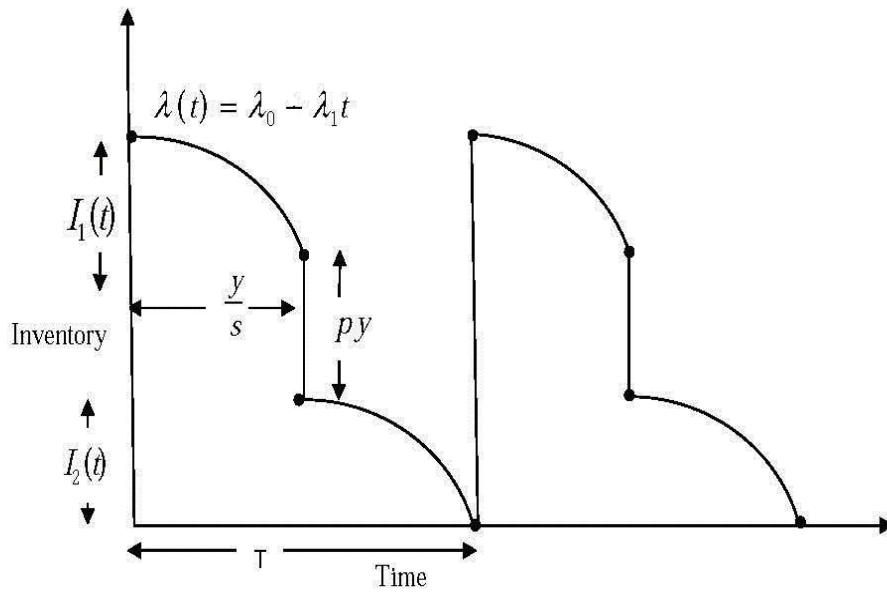
t_s : time taken for screening the lot size;

$I_1(t)$: inventory level at time t ; $t \in [0, t_s]$

$I_2(t)$:inventory level at time t ; $t \in (t_s, T]$

While developing the crisp model, the following assumptions are made:

1. Replenishment is instantaneous; Lead time is zero.
2. The time horizon is infinite.
3. Demand is a linear function of time t ; given by $\lambda(t) = \lambda_0 - \lambda_1 t$; where, $\lambda_0, \lambda_1 \in \mathbb{R}$ such that $\lambda_0 > 0$ and $\lambda_1 < \frac{\lambda_0}{T}$.
4. The lot size y is the only decision variable in the model.
5. The defective percentage p is a fixed constant for a lot size.
6. The screening process and demand proceeds simultaneously, but the screening rate is greater than maximum demand rate, i.e. $s > \lambda_0$; where λ_0 is the maximum demand rate.
7. A single product is considered.
8. All type of costs are considered as constants.
9. To avoid the cost of lost sales, shortage is ignored.
10. The selling price of perfect items during selling season is v_1 . The price of unsold perfect item is v_2 and that of imperfect item is v_3 . In general $v_1 \geq v_2 \geq v_3$



Inventory-Time Graph for Imperfect Quantity under Time Varying Demand

The differential equations governing the inventory level at time $t \in [0, t_s]$

$$\frac{dI_1}{dt} = -\lambda(t) \text{ for } (0, t_s] \quad (3.1)$$

subject to the initial condition: $I_1(0) = y$.

Solving and applying the initial condition,

$$I_1(t) = y - \lambda_0 t + \frac{\lambda_1 t^2}{2} \quad (3.2)$$

The initial inventory at time t_s , $I_{s1} = y - \lambda_0 \frac{y}{s} + \lambda_1 \frac{y^2}{2s^2}$

The final inventory at time t_s , $I_{s2} = I_{s1} - py = y(1-p) - \lambda_0 \frac{y}{s} + \lambda_1 \frac{y^2}{2s^2}$

The differential equations governing the inventory level at time $t \in (t_s, T]$

$$\frac{dI_2}{dt} = -\lambda(t) \text{ for } t \in (t_s, T] \tag{3.3}$$

subject to the initial condition: $I_2(t_s) = I_{s2}$.

Solution of (3.3) is

$$I_2(t) = y(1-p) - \lambda_0 \frac{y}{s} + \lambda_1 \frac{y^2}{2s^2} - (\lambda_0 t - \frac{\lambda_1 t^2}{2}) \tag{3.4}$$

since $y = \frac{\lambda_0}{1-p} T = KT$ (say)

where $K = \frac{\lambda_0}{1-p}$

$$\text{Holding cost} = h \left[\int_0^{y/s} I_1(t) dt + \int_{y/s}^T I_2(t) dt \right] = \frac{1}{6} \frac{T(\lambda_1 s T - 3 \lambda_0 s + 6 s K - 6 p s K + 6 p K^2) c_h}{s}$$

So the total cost becomes

$$TC(y) = (\text{Ordering Cost}) + (\text{Purchasing Cost}) + (\text{Screening Cost}) + (\text{Holding Cost});$$

$$\text{or, } TC(y) = c_o + c_p y + c_s y + h \left[T y - \lambda_0 T^2 + \lambda_1 \frac{T^3}{6} - p y (T - \frac{y}{s}) \right]$$

Now, in order to prevent shortage, we must have $T = \frac{y(1-p)}{\lambda_0}$ where $\lambda_0 = \lambda(0)$ where $\lambda_0 = \max\{\lambda(t)\}$ Clearly $\lambda_0 \geq \lambda_0$

Sold Items of good quality

$$= \int_0^T (\lambda_0 + \lambda_1 t) dt = \lambda_0 T - \frac{1}{2} \lambda_1 T^2$$

Unsold Items of good quality:

$$= y(1-p) - \int_0^T (\lambda_0 + \lambda_1 t) dt = y(1-p) - \lambda_0 T - \frac{1}{2} \lambda_1 T^2$$

$$\text{Total Revenue } TR = v_1 (\lambda_0 T - \frac{1}{2} \lambda_1 T^2) + v_2 (y(1-p) - \lambda_0 T + \frac{1}{2} \lambda_1 T^2) + v_3 p y$$

The total average profit is given by the following expression:

$$\begin{aligned} TAP(y) &= \frac{TR-TC}{T} \\ &= (v_1 - v_2)(\lambda_0 - \frac{1}{2} \lambda_1 T) - \frac{1}{6} T (\lambda_1 T - 3 \lambda_0 + 6 s K - 6 p s K + 6 \frac{p K^2}{s}) c_h + K(v_3 p \\ &\quad + v_2 (1-p) - c_p - c_s) - \frac{c_o}{T} \end{aligned} \tag{3.5}$$

Differentiating with respect to T successively twice, we get,

$$TAP'(T) = \frac{-1}{6} \frac{(\lambda_1 s T - 3 \lambda_0 s + 6 s K - 6 p s K + 6 p K^2) c_h}{s} + \frac{1}{2} (v_2 - v_1) \lambda_1 + \frac{c_o}{T^2} = 0$$

$$TAP''(T) = -\frac{1}{3} \lambda_1 c_h - 2 \frac{c_o}{T^3}$$

Since $TAP''(T) < 0$; $TAP(T)$ is concave.

Thus the optimal solution is given by the equation: $TAP'(y) = 0$; which means to solve the cubic obtained from

following equation:

$$\frac{-1}{6} \frac{(\lambda_1 s T - 3 \lambda_0 s + 6 s K - 6 p s K + 6 p K^2) c_h}{s} + \frac{1}{2} (v_2 - v_1) \lambda_1 + \frac{c_o}{T^2} = 0 \tag{3.6}$$

$$T^3 + [3(v_1 - v_2) + \frac{6}{\lambda_1} (\frac{\lambda_0}{2} + \frac{p}{s} \frac{\lambda_0^2}{(1-p)^2})] T^2 - \frac{6c_o}{\lambda_1 c_h} = 0 \tag{3.7}$$

We first prove the existence and uniqueness of unique positive root of above cubic equation, which guarantees the optimal solution to our problem.

Theorem 3.1 *There exist a positive root of equation (3.7) and it is unique for all possible parameter values.*

Proof : Let us consider the equation (3.7) as, $T^3 + A_1 T^2 - A_2 = 0$ (3.8)

where $A_1 = 3(v_1 - v_2) + \frac{6}{\lambda_1} (\frac{\lambda_0}{2} + \frac{p}{s} \frac{\lambda_0^2}{(1-p)^2})$; and $A_2 = \frac{6c_o}{\lambda_1 c_h}$

Now it is obvious that $A_1 > 0$ since each term in the ()brackets is positive. Further $A_2 > 0$ since $c_o > 0, \lambda_1 > 0$ and $c_h > 0$.

Let $A(T) = T^3 + A_1 T^2 - A_2$.

Then it is clearly observed that $A(T)$ has 1 change of sign, (and $A(-T)$ has 2 change of sign.) By the Descartes' rule of sign "the number of positive roots of a polynomial with real coefficients is either equal to the number of sign changes between consecutive nonzero coefficients, or is less than it by a multiple of 2." This condition is trivially satisfied. Thus, we can conclude that $A(T)$ has exactly 1 positive root. (and $A(-T)$ has 2 positive roots. \Rightarrow $A(T)$ has 2 negative roots.) Hence, the existence and uniqueness follows.

Solution Procedure

In equation (3.8) substituting $T = \tau - \frac{A_1}{3}$, we obtain a new cubic equation $\tau^3 - B_1 \tau + B_2 = 0$

$$B_1 = \frac{-A_1^2}{3} \text{ and } B_2 = \frac{2A_1^3 - 27A_2}{27}$$

Using Cardano's procedure of solving cubic equation, the solution is given by

$$\tau^* = \left(\frac{-B_2}{2} + \sqrt{\frac{B_1^3}{27} + \frac{B_2^2}{4}} \right)^{\frac{1}{3}} + \left(\frac{-B_2}{2} - \sqrt{\frac{B_1^3}{27} + \frac{B_2^2}{4}} \right)^{\frac{1}{3}}$$

Optimal cycle time $T^* = \tau^* - \frac{A_1}{3}$ It can be shown that above EOQ for imperfect items with time varying demand can be converted to a EOQ model for imperfect items with constant demand and further to traditional EOQ by some suitable substitutions.

Case 1:

Taking $\lambda_1 = 0$ in the equation (3.6), we get an EOQ for imperfect items with constant demand.

Governing equation becomes $\frac{c_o}{T^2} - (\frac{pK^2}{s} + K(1-p) - \frac{\lambda_0}{2})c_h = 0$

$$\frac{c_o}{T^2} - (p \frac{\lambda_0}{(1-p)^2 s} + \frac{\lambda_0}{2})c_h = 0$$

$$T = \sqrt{\frac{c_o}{(\frac{p}{s(1-p)^2} + \frac{1}{2})\lambda_0 c_h}}$$

which gives $y = \frac{\lambda_0 T}{1-p} = \sqrt{\frac{c_o \lambda_0}{(\frac{p}{s} + \frac{(1-p)^2}{2})c_h}}$

Case 2:

Further putting $p = 0$,

we get:

$$T = \sqrt{\frac{2c_o}{\lambda_0 c_h}}$$

This gives,

$$y = \lambda_0 T = \sqrt{\frac{2c_o \lambda_0}{c_h}}$$

3.2 Modification of crisp model using Uncertainty Theory

Beside the assumptions of Crisp Model following modifications are made:

1. All the cost parameters are independent uncertain variables.
2. Each of the uncertain cost variable has regular uncertainty distribution.
3. For each uncertain cost variable the expected value and pessimistic value exist.

Let:

ξ_1 : Ordering cost; an uncertain variable.

ξ_2 : Purchase cost per unit; an uncertain variable.

ξ_3 : Screening cost per unit; an uncertain variable.

ξ_4 : Holding cost per unit per unit time; an uncertain variable.

Φ_i : Uncertainty distribution of ξ_i ; ($i = 1, 2, 3, 4$)

α : a crisp constant such that $\alpha \in [0, 1]$

Then uncertain total average profit is represent by

$$f(\xi_1, \xi_2, \xi_3, \xi_4; y) = (v_1 - v_2)(\lambda_0 - \frac{1}{2}\lambda_1 T) - \frac{1}{6}T \left(\lambda_1 T - 3\lambda_0 + 6sK - 6pK + 6\frac{pK^2}{s} \right) \xi_4 + K(v_3p + v_2(1-p) - \xi_2 - \xi_3) - \frac{\xi_1}{T} \quad (3.9)$$

Then the uncertain total average profit f being the function of uncertain variables is itself an uncertain variable with inverse uncertainty distribution $\Psi^{-1}(\alpha)$ where

$$\Psi^{-1}(\alpha) = (v_1 - v_2)(\lambda_0 - \frac{1}{2}\lambda_1 T) - \frac{1}{6}T \left(\lambda_1 T - 3\lambda_0 + 6sK - 6pK + 6\frac{pK^2}{s} \right) \Phi_4^{-1}(\alpha) + K(v_3p + v_2(1-p) - \Phi_2^{-1}(\alpha) - \Phi_3^{-1}(\alpha)) - \frac{\Phi_1^{-1}(\alpha)}{T} \quad (3.10)$$

3.3 Derivation of the Uncertain Expected Value Model

The maximization of an uncertain variable is meaningless. Thus, one has to convert the uncertain variable or some function of uncertain variable to its expected value so that some mathematical programming method can be applied to obtain a optimum value for which expected value is minimum.

Let as assume that expectations of the uncertain variables ξ_i ;($i = 1, 2, 3...n$)exist and so that of $f(\xi_1, \xi_2, \xi_3, \dots, \xi_n)$ also exists. Now we can develop expected value model as follows:

Maximize $E[f(\xi_1, \xi_2, \xi_3, \dots, \xi_n)]$
 subject to (some suitable constraints).

Above problem is a crisp optimization problem which can be solved using traditional methods.

Here uncertain total average profit function is an uncertain variable, which should be converted to its equivalent crisp value to obtain the optimal policy of the uncertain model.

Since expectations of the variables ξ_i exist and so that of $f(\xi_1, \xi_2, \xi_3, \xi_4; y)$ also exists.

Now we can define the Uncertain Expected Value Model (UEVM) as follows:

Maximize $E[f(\xi_1, \xi_2, \xi_3, \xi_4; y)]$

Subject to:

$$y > 0$$

Here y is the decision variable.

Using the **Theorem 2.1** we get ,

$$E[f(\xi_1, \xi_2, \xi_3, \xi_4; y)] = \int_0^1 \Psi^{-1}(\alpha) d\alpha$$

$$\Psi^{-1}(\alpha) = (v_1 - v_2)(\lambda_0 - \frac{1}{2}\lambda_1 T) - \frac{1}{6}T \left(\lambda_1 T - 3\lambda_0 + 6sK - 6pK + 6\frac{pK^2}{s} \right) \int_0^1 \Phi_4^{-1}(\alpha) d\alpha$$

$$+ K(v_3 p + v_2(1-p) - \int_0^1 \Phi_2^{-1}(\alpha) d\alpha - \int_0^1 \Phi_3^{-1}(\alpha) d\alpha) - \frac{\int_0^1 \Phi_1^{-1}(\alpha) d\alpha}{T} \tag{3.11}$$

Equation (3.11) is function of y only and let it be $e(y)$. Thus the transformed crisp expected value model can be written as-

Maximize $e(y)$

Subject to

$$y > 0$$

3.4 Derivation of the Linear Uncertain Expected Value Model

Let us suppose $\xi_i; (i = 1, 2, 3, 4)$ be Linear uncertain variables. Then, $\xi_i \sim L(a_i, b_i)$ where a_i, b_i for $(i = 1, 2, 3, 4)$ are real numbers.

Now the Linear Uncertainty distribution functions are given by

$$\Phi_i(x) = \begin{cases} 0 & x \leq a_i \\ \frac{x-a_i}{b_i-a_i} & a_i \leq x \leq b_i \\ 1 & x \geq b_i \end{cases}$$

So the corresponding inverse uncertainty distribution functions are given by :

$$\Phi_i^{-1}(\alpha) = (1 - \alpha) a_i + \alpha b_i; \quad \text{for } (i = 1, 2, 3, 4)$$

Thus, $E[\xi_i] = \frac{a_i+b_i}{2}$

Now, if we apply the above result to the expected value model developed in section 3.2; we get expected total profit of the model as

$$E[f(\xi_1, \xi_2, \xi_3, \xi_4; y)] = (v_1 - v_2)(\lambda_0 - \frac{1}{2}\lambda_1 T) - \frac{1}{6}T \left(\lambda_1 T - 3\lambda_0 + 6sK - 6pK + 6\frac{pK^2}{s} \right) \frac{a_4+b_4}{2} +$$

$$+ K(v_3p + v_2(1-p) - \frac{a_2+b_2}{2} - \frac{a_3+b_3}{2}) - \frac{a_1+b_1}{T}$$

where, $e_i = \frac{(a_i+b_i)}{2}$ for $i = 1, 2, 3, 4$.

Special Cases: Case 1:

Taking $\lambda_1 = 0$ in the equation (3.6), we get an EOQ for imperfect items with constant demand.

Governing equation becomes $\frac{\frac{a_1+b_1}{2}}{T^2} - (\frac{pK^2}{s} + K(1-p) - \frac{\lambda_0}{2}) \frac{a_4+b_4}{2} = 0$

$$\frac{\frac{a_1+b_1}{2}}{T^2} - (p\frac{\lambda_0}{(1-p)^2s} + \frac{\lambda_0}{2}) \frac{a_4+b_4}{2} = 0$$

$$T = \sqrt{\frac{\frac{a_1+b_1}{2}}{(\frac{p}{s(1-p)^2} + \frac{1}{2})\lambda_0 \frac{a_4+b_4}{2}}}$$

which gives $y = \frac{\lambda_0 T}{1-p} = \sqrt{\frac{\frac{a_1+b_1}{2} \lambda_0}{(\frac{p}{s} + \frac{(1-p)^2}{2}) \frac{a_4+b_4}{2}}}$

Case 2:

Further putting $p = 0$,

we get:

$$T = \sqrt{\frac{2\frac{a_1+b_1}{2}}{\lambda_0 \frac{a_4+b_4}{2}}}$$

This gives,

$$y = \lambda_0 T = \sqrt{\frac{2\frac{a_1+b_1}{2} \lambda_0}{\frac{a_4+b_4}{2}}}$$

3.5 Derivation of the Normal Uncertain Expected Value Model

Let ξ_i for ($i = 1, 2, 3, 4$) be Normal uncertain variables.

ie. $\xi_i \sim N(e_i, \sigma_i)$ where e_i, σ_i for ($i = 1, 2, 3, 4$) are real numbers and $\sigma_i > 0$ for each i .

Now the normal Uncertainty distribution functions are given by

$$\Phi_i(x) = \left(1 + \exp\left(\frac{\pi(e_i-x)}{\sqrt{3}\sigma_i}\right) \right)^{-1} \quad x \in \mathbb{R}$$

Then the corresponding inverse uncertainty distribution functions given by-

$$\Phi_i^{-1}(\alpha) = e_i + \sigma_i \frac{\sqrt{3}}{\pi} \ln\left(\frac{\alpha}{1-\alpha}\right)$$

Thus

$$E[\xi_i] = \int_0^1 \Phi_i^{-1}(\alpha) d\alpha = e_i; \quad \text{for } (i = 1, 2, 3, 4)$$

$$E[f(\xi_1, \xi_2, \xi_3, \xi_4; y)] = (v_1 - v_2)(\lambda_0 - \frac{1}{2}\lambda_1 T) - \frac{1}{6}T \left(\lambda_1 T - 3\lambda_0 + 6sK - 6pK + 6\frac{pK^2}{s} \right) e_4 + K(v_3p +$$

$$v_2(1-p) - e_2 - e_3) - \frac{e_1}{T}$$

Using **Theorem (3.1)** and subsequent solution procedure, we obtain the optimal solution.

Special Cases:

Case 1:

Taking $\lambda_1 = 0$ in the equation (3.6), we get an EOQ for imperfect items with constant demand:

Governing equation becomes:

$$\frac{e_1}{T^2} - (\frac{pK^2}{s} + K(1-p) - \frac{\lambda_0}{2})e_4 = 0$$

$$\frac{e_1}{T^2} - (p\frac{\lambda_0}{(1-p)^2s} + \frac{\lambda_0}{2})e_4 = 0$$

$$T = \sqrt{\frac{e_1}{(\frac{p}{s(1-p)^2} + \frac{1}{2})\lambda_0 e_4}}$$

which gives $y = \frac{\lambda_0 T}{1-p} = \sqrt{\frac{e_1 \lambda_0}{(\frac{p}{s} + \frac{(1-p)^2}{2})e_4}}$

Case 2:

Further putting $p = 0$,

we get:

$$T = \sqrt{\frac{2e_1}{\lambda_0 e_4}}$$

This gives,

$$y = \lambda_0 T = \sqrt{\frac{2e_1 \lambda_0}{e_4}} \text{ -which is the required EOQ .}$$

It can be shown that above Uncertain EOQ model for imperfect items with time varying demand in both Linear and Normal variable case can be converted to corresponding Crisp EOQ models and also to an Uncertain and Crisp EOQ models for imperfect items with constant demand and further to traditional EOQ by suitable substitutions.

3.6 Derivation of the Uncertain Chance-Constrained Programming model

Introduced by Charnes and Cooper in 1959 , chance constrained programming (CCP) has been one of the successful methods for solving optimization problems under uncertainty. The main idea of CCP is that inequality constraints in uncertain model will hold with a pre-specified uncertainty value to ensure a certain level of satisfaction. That means the decision maker will provide the value of satisfaction level α such that the constraint will hold atleast α of time. Here we choose the pessimistic value α as the satisfaction criteria of the model.

Assume that the pessimistic value criterion holds. If we want to Maximize the pessimistic value of the objective function subject to some uncertain chance constraints, the corresponding uncertain chance-constrained programming problem becomes,

$$\text{Maximize } \bar{f}$$

Subject to:

$$M\{f(\xi_1, \xi_2, \xi_3, \dots, \xi_n; y) \leq \bar{f}\} \geq \alpha$$

where α is a predetermined confidence level and $\min \bar{f}$ is α - pessimistic return.

We have:

$$\begin{aligned} \xi_{inf}(\alpha) &= \Psi^{-1}(\alpha) \\ &= f(\phi_1^{-1}(\alpha), \phi_2^{-1}(\alpha), \phi_3^{-1}(\alpha), \dots, \phi_n^{-1}(\alpha)) \\ &= f_c(\alpha) \text{ (say).} \end{aligned}$$

So, we obtain a crisp chance constraining problem as follows:

$$\text{Maximize } f_c(\alpha)$$

Subject to:

(some equality or inequality constraints.)

This crisp problem also can be solved by the method which is based on Theorem(3.1).

$$\begin{aligned} \xi_{inf}(\alpha) &= \Psi^{-1}(\alpha) \\ \Psi^{-1}(\alpha) &= \left\{ \frac{\Phi_1^{-1}(\alpha)}{y} + (\Phi_2^{-1}(\alpha) + \Phi_3^{-1}(\alpha)) \frac{\lambda_0}{(1-p)} + \Phi_4^{-1}(\alpha) \left[y - \lambda_0 \frac{y(1-p)}{2\lambda_0} + \lambda_1 \frac{y^2(1-p)^2}{6\lambda^*2} - py \left(1 - \frac{\lambda_0}{(1-p)s} \right) \right] \right\} \end{aligned}$$

The corresponding crisp chance constraining problem as follows:

$$\text{Maximize } \Psi^{-1}(\alpha)$$

$$\text{Subject to: } y > 0$$

Remark:

for $\alpha = 0.5$, the result of linear or normal uncertain chance constrained programming become identical with corresponding expected value model.

3.7 Derivation of the Linear uncertain chance constrained programming model

For solving the chance constrained programming model, we substitute the pessimistic value of linear uncertain variable :

$$\Phi_i^{-1}(\alpha) = (1 - \alpha) a_i + \alpha b_i = c_i(\text{say})$$

which gives

$$f_c(\alpha) = (v_1 - v_2)(\lambda_0 - \frac{1}{2}\lambda_1 T) - \frac{1}{6}T \left(\lambda_1 T - 3\lambda_0 + 6sK - 6pK + 6\frac{pK^2}{s} \right) c_4 + K(v_3 p + v_2(1 - p) - c_2 - c_3) - \frac{c_1}{T}$$

Special Cases

Case 1:

Taking $\lambda_1 = 0$ in the equation (3.6), we get an EOQ for imperfect items with constant demand:

Governing equation becomes $\frac{c_1}{T^2} - (\frac{pK^2}{s} + K(1 - p) - \frac{\lambda_0}{2})c_4 = 0$

or, $\frac{c_1}{T^2} - (p\frac{\lambda_0}{(1-p)^2 s} + \frac{\lambda_0}{2})c_4 = 0$

which gives,

$$T = \sqrt{\frac{c_1}{(\frac{p}{s(1-p)^2} + \frac{1}{2})\lambda_0 c_4}}$$

In particular, for $\alpha = 0$ we have, $T_0 = \sqrt{\frac{a_1}{(\frac{p}{s(1-p)^2} + \frac{1}{2})\lambda_0 a_4}}$

which gives $y_0 = \frac{\lambda_0 T}{1-p} = \sqrt{\frac{a_1 \lambda_0}{(\frac{p}{s} + \frac{(1-p)^2}{2})a_4}}$

Again, for $\alpha = 1$ we have,

$$T_1 = \sqrt{\frac{b_1}{(\frac{p}{s(1-p)^2} + \frac{1}{2})\lambda_0 b_4}}$$

which gives $y_1 = \frac{\lambda_0 T}{1-p} = \sqrt{\frac{b_1 \lambda_0}{(\frac{p}{s} + \frac{(1-p)^2}{2})b_4}}$

Thus we have for all $\alpha \in [0, 1]$;

$$\min\{T_0, T_1\} \leq T \leq \max\{T_0, T_1\}$$

Case 2:

Further putting $p = 0$,

we get:

$$T = \sqrt{\frac{2c_1}{\lambda_0 c_4}}$$

This gives,

$y = \lambda_0 T = \sqrt{\frac{2c_1 \lambda_0}{c_4}}$ -which is the crisp EOQ for imperfect quality items. similar implications hold as of previous case.

3.8 Derivation of the Normal uncertain chance constrained programming model

For solving the chance constrained programming model, like previous case, we substitute the pessimistic value of normal uncertain variable

$$\Phi_i^{-1}(\alpha) = c_i \text{ where, } c_i = e_i + \sigma_i \frac{\sqrt{3}}{\pi} \ln \left(\frac{\alpha}{1-\alpha} \right)$$

which gives

$$f_c(\alpha) = (v_1 - v_2)(\lambda_0 - \frac{1}{2}\lambda_1 T) - \frac{1}{6}T \left(\lambda_1 T - 3\lambda_0 + 6sK - 6pK + 6\frac{pK^2}{s} \right) c_4 + K(v_3 p + v_2(1-p) - c_2 - c_3) - \frac{c_1}{T}$$

$$y^* = \sqrt{\frac{2c_1\lambda_0}{(1-p) \left[c_4 \left(2 - \frac{\lambda_0}{\lambda_0} (1-p) - 2p \left(1 - \frac{\lambda_0}{(1-p)s} \right) \right) - 2\frac{\lambda_1}{\lambda_0} (1-p)(s-v_1) \right]}}$$

$$y^* = \sqrt{\frac{2(e_1 + \sigma_1 \frac{\sqrt{3}}{\pi} \ln \left(\frac{\alpha}{1-\alpha} \right) \lambda_0}{(1-p) \left[(e_4 + \sigma_4 \frac{\sqrt{3}}{\pi} \ln \left(\frac{\alpha}{1-\alpha} \right) \left(2 - \frac{\lambda_0}{\lambda_0} (1-p) - 2p \left(1 - \frac{\lambda_0}{(1-p)s} \right) - 2\frac{\lambda_1}{\lambda_0} (1-p)(s-v_1) \right) \right]}}$$

-which is the crisp EOQ for imperfect quality items.

For, $\alpha = .5$, we obtain

$$y^* = \sqrt{\frac{2e_1\lambda_0}{(1-p) \left[e_4 \left(2 - \frac{\lambda_0}{\lambda_0} (1-p) - 2p \left(1 - \frac{\lambda_0}{(1-p)s} \right) \right) - 2\frac{\lambda_1}{\lambda_0} (1-p)(s-v_1) \right]}}$$

which is the expression for optimal EOQ.

3.9 Numerical Examples

Example-1.

We consider following data for our model:

Demand per unit time is given by $\lambda(t) = 350 - 2t$ per month; which means $\lambda_0 = 350, \lambda_1 = 2$

Defective item percentage $p = 5\%$ items per lot i.e. $p = 0.05$,

Screening rate $s = 370$ / month,

Purchase cost $c = 5$ \$per unit ,

Holding cost $c_h = 5$ \$ per unit per month,

Ordering cost $c_o = 100$ \$,

Screening cost $c_s = .5$ \$ per unit,

Selling price of perfect items $v_1 = 40$ \$ per unit,

Salvage price for unsold perfect items $v_2 = 25$ \$ per unit,

Salvage price for imperfect items $v_3 = 5$ \$ per unit

Using equation (3.7),

$$3.333 T^3 + 996.712 T^2 - 100.0 = 0$$

which gives the optimal scheduling period $T = 0.319$ month = 9.6 days(approximately) ,

optimal lot size $y^* = 111.92$ unit, and optimal profit $TAP^* = 3214.52$ \$ per cycle.

The sensitivity of decision variable and total average profit for changes in costs namely ordering cost c_o , holding cost c_h , purchase cost c_p and screening cost c_s are shown in table 1. It is observed that :

1. Optimal lot size decreases with decreasing ordering cost, but optimal profit increases gradually; which is natural because the declining in ordering cost motivates decision maker to order less because of cost involved in storage etc can be reduced. This suggest the hike in optimal profit.
2. Optimal lot size decreases with increasing holding cost and optimal profit decreases accordingly; which is due

c_o	c_h	c_p	c_s	T^*	y^*	TAP^*
-40%				.247	87.32	9738.18
-20%				.285	100.75	9663.06
+20%				.349	123.38	9537.02
+40%				.377	133.28	9481.99
	-40%			.410	144.95	9735.65
	-40%			.356	125.86	9661.96
	+20%			.292	103.23	9537.93
	+40%			.272	96.16	9483.68
		-40%		.319	112.78	11070.55
		-20%		.319	112.78	10333.71
		+20%		.319	112.78	8860.02
		+40%		.319	112.78	8123.18
			-40%	.319	112.78	9670.55
			-20%	.319	112.78	9633.71
			+20%	.319	112.78	9560.02
			+40%	.319	112.78	9523.18

Table 1: Effect of various cost parameters on the optimal policy

to the fact that increment in holding cost motivates decision maker to order less because of cost involved in storage can be reduced. This results in profit decrement.

3. Purchase and screening costs have no tendency to change the lot size but the increase in these costs have slightly reduction effects in optimal profit.

Example-2.

If the Ordering cost ξ_1 , Purchase cost ξ_2 , Screening cost ξ_3 and Holding cost ξ_4 have Linear uncertainty distribution. Let them be

$$\xi_1 \sim L(90, 110), \xi_2 \sim L(4, 6), \xi_3 \sim L(0, 1), \xi_4 \sim L(4.5, 5.5),$$

Then, using the formula derived in the **UEVM** for Linear Uncertain variables, we get:

the optimal scheduling period $T = 0.319$ month = 9.6 days(approximately),

optimal lot size $y^* = 111.92$ unit, and

optimal profit $TAP^* = 3214.52$ per cycle.

The reason behind identical result is that we have adjusted our parameters of linear uncertain variables in such a way that the expected value of each uncertain variable is the corresponding crisp value.

Then, using the formula derived in the **UCCP** for Linear Uncertain variables, for $\alpha = 0.9$

we obtain the optimal scheduling period $T = 0.264$ month = 7.9 days(approximately),

optimal lot size $y^* = 93.33$ unit, and

optimal profit $TAP^* = 2846.31$ per cycle.

Example-3

Let the Ordering cost ξ_1 , Purchase cost ξ_2 , Screening cost ξ_3 and Holding cost ξ_4 follow Normal Uncertainty

Distribution. Let them be

$$\xi_1 \sim N(100, 4), \xi_2 \sim N(10, 2), \xi_3 \sim N(0.5, .1), \xi_4 \sim N(5, 3)$$

Now using the formula derived in the **UEVM** for Normal uncertain variables; we get

the optimal scheduling period $T = 0.319$ month = 9.6 days(approximately) ,

optimal lot size $y^* = 111.92$ unit, and

optimal profit $TAP^* = 3214.52$ \$ per cycle.

(Here too, we have considered normal uncertain variables in such a way that the expected value of each uncertain variable is the corresponding crisp value.)

Then, using the formula derived in the **UCCP** for Normal Uncertain variables, for $\alpha = 0.9$, we get

the optimal scheduling period $T = 0.256$ month = 7.7 days(approximately) ,

optimal lot size $y^* = 90.50$ unit, and optimal profit $TAP^* = 2751.52$ \$ per cycle.

4 Conclusion

We have first developed a crisp inventory model for linearly decreasing time varying demand for imperfect quality items. We have considered that the imperfect items are screened out after a 100 % screening process and sold at the reduced price. Also the unsold items after the selling season is sold at a reduced price. The Uncertain programming approach is applied to modify the aforementioned model into one with uncertain cost variables. We developed the Uncertain Expected value model as an approach of Uncertain Programming and we have shown that above model can be reduced to corresponding crisp model and traditional EOQ model subjected to certain conditions. Using the numerical examples we have first considered the crisp case and then uncertain case. Using sensitivity analysis, we have observed the effect of various crisp cost parameters in our model and discussed its significance. The uncertain cases are illustrated with the help of two numerical examples.

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