Generalized (H_p, r)-invexity in multiobjective programming problems

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Abstract

In the present paper, we move forward in the study of multiobjective programming problems and establish sufficient optimality conditions under generalized (H_p, r)-invexity assumptions. Weak, strong and strict converse duality theorems are also derived for Mond-Weir type dual model in order to relate efficient solutions of primal and dual problems.

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1. Introduction

The field of multiobjective programming problems, also known as vector optimization problems, has grown remarkably in different directions in the setting of optimality conditions and duality theory. Such problems can arise in practically every field of science, engineering and business, and the need for efficient and reliable solution methods is increasing. Multiobjective optimization has been applied in many cases where optimal decisions need to be taken in the presence of trade-offs between two or more conflicting ob-

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jectives. A decision maker is needed to provide additional preference information and to identify the most satisfactory solution. Minimizing weight while maximizing the strength of a particular component, and maximizing performance whilst minimizing fuel consumption and emission of pollutants of a vehicle are examples of multiobjective optimization problems involving two and three objectives, respectively.

Convex analysis has been linked with the theory of multiobjective mathematical programming problem since it evolved from multiobjective linear programming problem. In order to ensure that necessary conditions for optimality are sufficient under convexity, we establish optimality conditions, duality theorems, saddle point analysis *etc.*. This development in multiobjective programming problems was originated by the growth of generalizations of invexity, introduced by Hanson [9]. The definition of invexity in the sense of Hanson reduces to the notion of convexity when $\eta(x, x_0) = x - x_0$, which were extended by many of the authors in recent past see, [2, 7, 12, 18, 19, 20] and the references cited therein.

Egudo and Hanson [8] discussed differentiable multiobjective duality results between primal problems and its Wolfe type dual problems under invexity assumptions. Liang *et al.* [15] introduced (F, α, ρ, d)-convexity as extension of several concepts of generalized convexity and obtained some corresponding optimality conditions and duality results for the multiobjective fractional programming problem. Agarwal *et al.* [1] proposed a new class of generalized $d - \rho - (\eta, \theta)$ type I invex functions for a nondifferentiable multiobjective programming problem and obtained optimality conditions and duality results for efficient/weak efficient solutions. Jayswal *et al.* [11] considered $(p, r) - \rho - (\eta, \theta)$-invex functions to establish sufficient conditions and duality theorems for a class of multiobjective fractional programming problems. Recently, Lai and Ho [13] discussed the optimality conditions and duality results for a subdifferentiable multiobjective fractional programming problem involving exponential V-r-invex Lipschitz functions.

In the course of generalization of convex functions, Avriel [6] first introduced the definition of r-convex functions and established some characterizations and the relations between r-convexity and other generalization of convexity. Antczak [3] introduced the concept of a class of r-preinvex functions which is a generalization of r-convex functions and preinvex functions, and obtained some optimality results under r-preinvexity assumption for constrained optimization problems. Lee and Ho [14] established necessary and sufficient conditions for efficiency of multiobjective fractional programming problems involving r-invex functions and investigated the parametric, Wolfe and Mond-Weir type dual for multiobjective fractional programming problems concerning r-invexity. In order to generalize the notion of invex and pre-invex functions, Antczak [4] introduced p-invex sets and (p, r)-invex functions and derived sufficient optimality conditions for a nonlinear programming problem involving (p, r)-invex functions.

Yuan *et al.* [20] introduced the concept of locally (H_p, r, α)-pre-invex functions and locally H_p-invex sets, respectively and derived necessary optimality conditions, sufficient optimality conditions and duality theorems for nonlinear programming problems. Liu *et al*. [16] proposed the concept of (H_p, r)-invex function and proved sufficient optimality conditions to multiple objective programming problem and mulitiobjective fractional programming problem but no steps are taken to establish duality theorems for the considered problems.

(H_p, r)-invex functions are extension of (p, r)-invex functions and r-invex functions. Since many practical and real situations give rise to exponential and logarithmic functions, so a powerful tool is needed to tackle such problem. Up to some extent (H_p, r)-invexity can handle such types of problems. In the present paper, our aim is to establish sufficient optimality conditions and duality results to meet the above mentioned demand.

The paper is organized as follows. In Section 2, we recall some notations and definitions needed in the sequel of the paper. In Section 3, we derive sufficient optimality conditions under generalized (H_p, r)-invexity assumptions for a class of multiobjective programming problems (MOP). Weak, strong and strict converse duality theorems for Mond-Weir dual problem (MOP) are discussed in Section 4. In the last Section 5, we hae discussed conclusions and future possible works that might take in this direction.

2. Notations and Preliminaries

Throughout the paper, let R^n be the n-dimensional Euclidean space, $R^n_+ = \{x \in R^n \mid x \geq 0\}$ and $\dot{R}^n_+ = \{x \in R^n \mid x > 0\}$. If $x, y \in R^n$, then $x \preceq y$ is used to denote the case $x_i \leq y_i, i = 1, 2, \cdots, n$ and $x \neq y$.

Definition 2.1 [4] Let $a_1, a_2 > 0, \lambda \in (0, 1)$ and $r \in R$. Then the weighted r-mean of a_1

and a_2 is given by

$$M_r(a_1, a_2; \lambda) = \begin{cases} (\lambda {a_1}^r + (1-\lambda)a_2^r)^{\frac{1}{r}}, & for\ r \neq 0, \\ a_1^\lambda a_2^{(1-\lambda)}, & for\ r = 0, \end{cases}$$

where $\lambda \in (0, 1)$ and $r \in R$.

Definition 2.2 [16] A subset $X \subset R^n$ is said to be H_p-invex set, if for any x,$u \in X$, there exists a vector function $H_p : X \times X \times [0, 1] \to R^n$, such that

$$H_p(x, u; 0) = e^u,\ H_p(x, u; \lambda) \in \dot{R}^n_+,$$

$$\ln(H_p(x, u; \lambda)) \in X,\ \forall\lambda \in [0, 1],\ p\ \in R.$$

In the above definitions, the logarithm and the exponentials appearing in the expressions are understood to be taken componentwise.

Now, we illustrate an example to show the existence of an H_p-invex set but not p-invex.

Example 2.1 Let $S \subset R^2$ defined by $S = \{x = (x_1, x_2) : x_2 = f(x_1)\}$, where $f : [0, 1] \to R$ is given by $f(t) = \frac{t^2}{1+t^2}$. For any given $x, u \in S$, the function H_p is defined by

$$H_p(x, u; \lambda) = \begin{cases} h_p(x; \lambda), & u = (0, 0)^T,\ \lambda \in [0, 1], \\ (e^{u_1}, e^{\frac{u_1^2}{1+u_1^2}}), & u \neq (0, 0)^T,\ \lambda \in [0, 1], \end{cases}$$

where $x = (x_1, x_2)^T,\ u = (u_1, u_2)^T,\ 0 < p < 1$ and

$$h_p(x; \lambda) = \begin{cases} (1, 1)^T, & \lambda = 0, \\ (e^\lambda, e^{\frac{\lambda^2}{1+\lambda^2}}), & \lambda \in (0, 1]. \end{cases}$$

Clearly, the set S defined above is H_p-invex set but not p-invex as $\ln(M_p(e^{\eta(x,u)+u}, e^u; \lambda)) \notin S,\ \forall\lambda \in [0, 1]$.

For convenience throughout the paper, we assume that X be a H_p-invex set, H_p is right differentiable at 0 with respect to the variable λ for each given pair $x, u \in X$, and $f : X \to R$ is differential on X. The symbol

$$H_p^{'}(x, u; 0+) \triangleq (H_{p1}^{'}(x, u; 0+), ..., H_{pn}^{'}(x, u; 0+))^T$$

denotes the right derivative of H_p at 0 with respect to the variable λ for each given pair $x, u \in X;\ \nabla f(x) \triangleq (\nabla_1 f(x), ..., \nabla_n f(x))^T$ denotes the differential of f at x, and so $\frac{\nabla f(u)}{e^u}$ denotes $(\frac{\nabla_1 f(u)}{e^{u_1}}, ..., \frac{\nabla_n f(u)}{e^{u_n}})^T$.

Definition 2.4 [16] A differentiable function $f : X \to R$ is said to be (strictly) (H_p, r)-invex at $u \in X$, if for all $x \in X$, we have

$$\frac{1}{r}e^{rf(x)} \geq \frac{1}{r}e^{rf(u)}\left[1 + r\frac{\nabla f(u)^T}{e^u}H_p^{'}(x, u; 0+)\right] \quad (>) \quad \text{for} \quad r \neq 0,$$

$$f(x) - f(u) \geq \frac{\nabla f(u)^T}{e^u}H_p^{'}(x, u; 0+) \qquad (>) \quad \text{for} \quad r = 0.$$

If the above inequalities are satisfied at any point $u \in X$, then f is said to be (H_p, r)-invex (strictly (H_p, r)-invex) on X.

The existence of (H_p, r)-invex functions is revealed by an example given in Jayswal et al. [10].

Special cases:

(i) If $H_p(x, u; \lambda) = M_p(e^{\eta(x,u)+u}, e^u; \lambda)$ and $a(x, u) = 1$ for all $x, u \in X$, then the above Definition 2.4 becomes (p, r)-invex with respect to η on X given in Antczak [4].

(ii) In addition to (i), if we take $p = 0$, then Definition 2.4 reduces to r-invex with respect to η on X given in Antczak [5].

(iii) In addition to (i) and (ii), if we take $r = 0$, then Definition 2.4 reduces to definition of invex function given in Hanson [9].

Definition 2.5 [10] A differentiable function $f : X \to R$ is said to be (H_p, r)-pseudoinvex at $u \in X$, if for all $x \in X$, the relations

$$\frac{\nabla f(u)^T}{e^u}H_p^{'}(x, u; 0+) \geq 0 \quad \Rightarrow \quad \frac{1}{r}[e^{r(f(x)-f(u)} - 1] \geq 0, \quad \text{for} \quad r \neq 0,$$

$$\frac{\nabla f(u)^T}{e^u}H_p^{'}(x, u; 0+) \geq 0 \quad \Rightarrow \quad f(x) - f(u) \geq 0, \quad \text{for } r = 0,$$

hold. If the above inequalities are satisfied at any point $u \in X$, then f is said to be (H_p, r)-pseudoinvex on X.

Definition 2.6 A differentiable function $f : X \to R$ is said to be strict (H_p, r)-pseudoinvex at $u \in X$, if for all $x \in X$, the relations

$$\frac{\nabla f(u)^T}{e^u}H_p^{'}(x, u; 0+) \geq 0 \quad \Rightarrow \quad \frac{1}{r}[e^{r(f(x)-f(u)} - 1] > 0, \quad \text{for } r \neq 0,$$

$$\frac{\nabla f(u)^T}{e^u}H_p^{'}(x, u; 0+) \geq 0 \quad \Rightarrow \quad f(x) - f(u) > 0, \quad \text{for } r = 0,$$

hold. If the above inequalities are satisfied at any point $u \in X$, then f is said to be strict (H_p, r)-pseudoinvex on X.

It revealed by an example given in Jayswal et al. [10] that there exist (H_p, r)-pseudoinvex functions but not (H_p, r)-invex.

Definition 2.7 [10] A differentiable function $f : X \to R$ is said to be (H_p, r)-quasiinvex at $u \in X$, if for all $x \in X$, the relations

$$\frac{1}{r}[e^{r(f(x)-f(u)} - 1] \leq \ 0 \quad \Rightarrow \quad \frac{\nabla f(u)^T}{e^u} H_p^{'}(x, u; 0+) \leq 0, \quad \text{for } r \neq 0,$$

$$f(x) - f(u) \leq \ 0 \quad \Rightarrow \quad \frac{\nabla f(u)^T}{e^u} H_p^{'}(x, u; 0+) \leq 0, \quad \text{for } r = 0,$$

hold. If the above inequalities are satisfied at any point $u \in X$, then f is said to be (H_p, r)-quasiinvex on X.

It revealed by an example given in Jayswal et al. [10] that there exist (H_p, r)-quasiinvex functions but not (H_p, r)-pseudoinvex.

Remark 2.1 All the theorems in the subsequent parts of this paper will be proved only in the the case when $r \neq 0$. The proofs in other case are easier than in this one. Moreover, without loss of generality, we shall assume that $r > 0$ (in the case when $r < 0$, the direction some of the inequalities in the proof of the theorems should be changed to the opposite one).

Consider the following multiobjective programming problem:

(MOP) Minimize $f(x) \triangleq (f_1(x), f_2(x), ..., f_q(x))$

subject to

$$g(x) \leq 0, x \in X,$$

where $f : X \to R^q$ and $g : X \to R^m$ are differentiable functions on a nonempty H_p-invex set X. Let S denotes the set of all feasible solution to (MOP), i.e. $S = \{x \in X : g(x) \leq 0\}$.

Definition 2.8 A feasible solution $u \in S$ of (MOP) is said to be an (weak) efficient solution if there exist no other feasible solution $x \in S$ such that

$$f(x) \preceq (<) \ f(u).$$

3. Sufficient optimality conditions

In the present section, we establish some sufficient optimality conditions under generalized (H_p, r)-invex functions introduced in previous sections.

Theorem 3.1 (Sufficient optimality conditions). *Let S be a H_p-invex set with the respect* *to the same* H_p. *Let* $u \in X$ *be a feasible solution of* (MOP), *and there exist scalars* $\lambda \in R^q,\ \lambda > 0,\ \mu \in R^m,\ \mu \geq 0$ *such that the following conditions hold:*

$$\sum_{i=1}^{q} \lambda_i \nabla f_i(u) + \sum_{j=1}^{m} \mu_j \nabla g_j(u) = 0, \tag{1}$$

$$\sum_{j=1}^{m} \mu_j g_j(u) = 0. \tag{2}$$

If $\sum_{i=1}^{q} \lambda_i f_i$ *is* (H_p, r)*-pseudoinvex and* $\sum_{j=1}^{m} \mu_j g_j$ *is* (H_p, r)*-quasiinvex at* $u \in X$. *Then* u *is an efficient solution to* (MOP).

Proof. Suppose contrary to the result that u is not an efficient solution to (MOP). Then there exists $x \in S$ such that

$$f(x) \preceq f(u).$$

Since $\lambda > 0$, after some algebraic transformations, the above inequality yields

$$\frac{1}{r}[e^{r(\sum_{i=1}^{q} \lambda_i f_i(x) - \sum_{i=1}^{q} \lambda_i f_i(u))} - 1] < 0.$$

From the assumption that $\sum_{i=1}^{q} \lambda_i f_i$ is (H_p, r)-pseudoinvex at $u \in X$, we have

$$\Big(\frac{\sum_{i=1}^{q} \lambda_i \nabla f_i(u)}{e^u}\Big)^T H_p^{'}(x, u; 0+) < 0. \tag{3}$$

Since $\mu \geq 0$, from the feasibility of x and (2), we have

$$\sum_{j=1}^{m} \mu_j g_j(x) \leq \sum_{j=1}^{m} \mu_j g_j(u),$$

which in turn after some algebraic transformations yields

$$\frac{1}{r}(e^{r(\sum_{j=1}^{m} \mu_j g_j(x) - \sum_{j=1}^{m} \mu_j g_j(u))} - 1) \leq 0.$$

Using the quasiinvexity of $\sum_{j=1}^{m} \mu_j g_j$ at $u \in X$, we get

$$\Big(\frac{\sum_{j=1}^{m} \mu_j \nabla g_j(u)}{e^u}\Big)^T H_p^{'}(x, u; 0+) \leq 0. \tag{4}$$

Adding (3) and (4), we obtain

$$\Big(\frac{\sum_{i=1}^{q} \lambda_i \nabla f_i(u) + \sum_{j=1}^{m} \mu_j \nabla g_j(u)}{e^u}\Big)^T H_p^{'}(x, u; 0+) < 0,$$

which contradicts (1). This completes the proof.

The proof of the following theorems along the similar lines of Theorem 2.1, and hence being omitted.

Theorem 3.2 (Sufficient optimality conditions). *Let S be a H_p-invex set with the respect to the same H_p. Let $u \in X$ be a feasible solution to* (MOP), *and there exists scalars $\lambda \in R^q,\ \lambda \geq 0,\ \mu \in R^m,\ \mu \geq 0$ such that the following conditions hold:*

$$\sum_{i=1}^{q} \lambda_i \nabla f_i(u) + \sum_{j=1}^{m} \mu_j \nabla g_j(u) = 0, \tag{5}$$

$$\sum_{j=1}^{m} \mu_j g_j(u) = 0. \tag{6}$$

If $\sum_{i=1}^{q} \lambda_i f_i$ is strict (H_p, r)-pseudoinvex and $\sum_{j=1}^{m} \mu_j g_j$ is (H_p, r)-quasiinvex at $u \in X$. Then u is an efficient solution to (MOP).

Theorem 3.3 (Sufficient optimality conditions). *Let S be a H_p-invex set with the respect to the same H_p. Let $u \in X$ be a feasible solution to* (MOP), *and there exists scalars $\lambda \in R^q,\ \lambda \succeq 0,\ \mu \in R^m,\ \mu \geq 0$ such that the following conditions hold:*

$$\sum_{i=1}^{q} \lambda_i \nabla f_i(u) + \sum_{j=1}^{m} \mu_j \nabla g_j(u) = 0, \tag{7}$$

$$\sum_{j=1}^{m} \mu_j g_j(u) = 0. \tag{8}$$

If $\sum_{i=1}^{q} \lambda_i f_i$ is (H_p, r)-pseudoinvex and $\sum_{j=1}^{m} \mu_j g_j$ is (H_p, r)-quasiinvex at $u \in X$. Then u is a weakly efficient solution to (MOP).

4. Mond-Weir type duality

We now consider the following Mond-Weir type dual problem related to (MOP):

(MWD) Maximize $f(y) \triangleq (f_1(y), f_2(y), \ldots, f_p(y))$

subject to

$$\sum_{i=1}^{q} \lambda_i \nabla f_i(y) + \sum_{j=1}^{m} \mu_j \nabla g_j(y) = 0, \tag{9}$$

$$\sum_{j=1}^{m} \mu_j g_j(y) \geq 0, \tag{10}$$

$$y \in X,\ \lambda \geq 0, \mu \geq 0. \tag{11}$$

Theorem 4.1 (Weak duality). *Let x and (y, λ, μ) be feasible solutions to* (MOP) *and* (MWD), *respectively. Assume that $\lambda > 0$, $\sum_{i=1}^{q} \lambda_i f_i$ is (H_p, r)-pseudoinvex and*

$\sum_{j=1}^{m} \mu_j g_j$ *is* (H_p, r)*-quasiinvex. Then*

$$f(x) \npreceq f(y).$$

Proof. Suppose contrary to the result that

$$f(x) \preceq f(y).$$

Since $\lambda > 0$, after some algebraic transformations, the above inequality yields

$$\frac{1}{r}[e^{r(\sum_{i=1}^{q} \lambda_i f_i(x) - \sum_{i=1}^{q} \lambda_i f_i(y))} - 1] < 0.$$

From the assumption that $\sum_{i=1}^{q} \lambda_i f_i$ is (H_p, r)-pseudoinvex at y,

$$\Big(\frac{\sum_{i=1}^{q} \lambda_i \nabla f_i(y)}{e^y}\Big)^T H_p'(x, y; 0+) < 0. \tag{12}$$

Since $\mu \geq 0$, from the feasibility of x and (10), we have

$$\sum_{j=1}^{m} \mu_j g_j(x) \leq \sum_{j=1}^{m} \mu_j g_j(y),$$

which in turn after some algebraic transformations yield

$$\frac{1}{r}(e^{r(\sum_{j=1}^{m} \mu_j g_j(x) - \sum_{j=1}^{m} \mu_j g_j(y))} - 1) \leq 0.$$

Using the quasiinvexity of $\sum_{j=1}^{m} \mu_j g_j$ at y, we obtain

$$\Big(\frac{\sum_{j=1}^{m} \mu_j \nabla g_j(y)}{e^y}\Big)^T H_p'(x, y; 0+) \leq 0. \tag{13}$$

Adding (12) and (13), we get

$$\Big(\frac{\sum_{i=1}^{q} \lambda_i \nabla f_i(y) + \sum_{j=1}^{m} \mu_j \nabla g_j(y)}{e^y}\Big)^T H_p'(x, y; 0+) < 0,$$

which contradicts the dual constraint (9). This completes the proof.

The proof of the following theorem along the similar lines of Theorem 4.1, and hence being omitted.

Theorem 4.2 (Weak duality). *Let x and (y, λ, μ) be feasible solutions to* (MOP) *and* (MWD), *respectively. Assume that $\lambda \geq 0$, $\sum_{i=1}^{q} \lambda_i f_i$ is strict (H_p, r)-pseudoinvex and $\sum_{j=1}^{m} \mu_j g_j$ is (H_p, r)-quasiinvex. Then*

$$f(x) \npreceq f(y).$$

Theorem 4.3 (Strong duality). *Let $\bar{x}$ be an efficient solution for* (MOP)*and $\bar{x}$ satisfies a constraints qualification for* (MOP) *in Marusciac [17]. Then there exist $\bar{\lambda} \in R^q$, $\bar{\mu} \in R^m$ such that $(\bar{x}, \bar{\lambda}, \bar{\mu})$ is feasible for* (MWD). *If any of the weak duality in Theorms 4.1-4.2 also hold, then $(\bar{x}, \bar{\lambda}, \bar{\mu})$ is an efficient solution to* (MWD).

Proof. Since $\bar{x}$ is an efficient solution to (MOP) and satisfies the constraint qualification for (MOP), then from Kuhn-Tucker necessary optimality condition, there exist $\bar{\lambda} > 0$, $\bar{\mu} \geq 0$ such that

$$\sum_{i=1}^{q} \bar{\lambda}_i \nabla f_i(\bar{x}) + \sum_{j=1}^{m} \bar{\mu}_j \nabla g_j(\bar{x}) = 0,$$

$$\sum_{j=1}^{m} \bar{\mu}_j g_j(\bar{x}) = 0,$$

which yields that $(\bar{x}, \bar{\lambda}, \bar{\mu})$ is feasible for (MWD). The efficiency $(\bar{x}, \bar{\lambda}, \bar{\mu})$ for (MWD) follows from weak duality theorems. This completes the proof.

Theorem 4.4 (Strict converse duality). *Let $\bar{x}$ be a feasible solution for* (MOP) *and $(\bar{y}, \bar{\lambda}, \bar{\mu})$ be feasible solution for* (MWD). *Let $\sum_{i=1}^{q} \bar{\lambda}_i f_i$ be strict (H_p, r)-pseudoinvex and $\sum_{j=1}^{m} \bar{\mu}_j g_j$ be (H_p, r)-quasiinvex such that*

$$\sum_{i=1}^{q} \bar{\lambda}_i f_i(\bar{x}) \leq \sum_{i=1}^{q} \bar{\lambda}_i f_i(\bar{y}). \tag{14}$$

Then $\bar{x} = \bar{y}$, i.e. $\bar{y}$ is an efficient solution for (MOP).

Proof. We assume the contarary that $\bar{x} \neq \bar{y}$, and exhibit a contradiction. Since $\bar{\mu} \geq 0$, from the feasibility of $\bar{x}$ and $(\bar{y}, \bar{\lambda}, \bar{\mu})$ for (MOP) and (MWD), respectively, we obtain

$$\sum_{j=1}^{m} \bar{\mu}_j g_j(\bar{x}) \leq \sum_{j=1}^{m} \bar{\mu}_j g_j(\bar{y}),$$

which in turn after some algebraic transformations yield

$$\frac{1}{r}\left(e^{r(\sum_{j=1}^{m} \bar{\mu}_j g_j(\bar{x}) - \sum_{j=1}^{m} \bar{\mu}_j g_j(\bar{y}))} - 1\right) \leq 0.$$

Using the quasiinvexity of $\sum_{j=1}^{m} \bar{\mu}_j g_j$ at $\bar{y}$, we get

$$\Big(\frac{\sum_{j=1}^{m} \bar{\mu}_j \nabla g_j(\bar{y})}{e^{\bar{y}}}\Big)^T H_p'(\bar{x}, \bar{y}; 0+) \leq 0.$$

The above inequality together with (9) yields

$$\Big(\frac{\sum_{i=1}^{q} \bar{\lambda}_i \nabla f_i(\bar{y})}{e^{\bar{y}}}\Big)^T H_p'(\bar{x}, \bar{y}; 0+) \geq 0.$$

From the assumption that $\sum_{i=1}^{q} \bar{\lambda}_i f_i$ is strict (H_p, r)-pseudoinvex at $\bar{y}$, we have

$$\frac{1}{r}[e^{r(\sum_{i=1}^{q} \bar{\lambda}_i f_i(\bar{x}) - \sum_{i=1}^{q} \bar{\lambda}_i f_i(\bar{y}))} - 1] > 0.$$

Using the fundamental property of exponential functions, we have

$$\sum_{i=1}^{q} \bar{\lambda}_i f_i(\bar{x}) > \sum_{i=1}^{q} \bar{\lambda}_i f_i(\bar{y}),$$

which contradicts the the assumptions (14). This completes the proof.

6. Conclusion

In this paper, we have established sufficient optimality conditions for a class of multiobective programming problems by using the concept of (strict) (H_p, r)-pseudoinvex and (H_p, r)-quasiinvex functions and derived weak, strong and strict converse duality theorems for Mond-Weir type dual problem in order to relate efficient solutions of primal and dual problems. It will be interest to obtain optimality conditions and duality results for a class of minimax programming problems. This will orient the future research of the authors.

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