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An extragradient iterative method for approximating the common solutions of a variational inequality, a system of variational inequalities and a hierarchical fixed point problem

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Abstract. In this paper, we suggest and analyze an iterative scheme for finding the approximate element of the common set of solutions of a variational inequality , a system of variational inequalities and a hierarchical fixed point problem in a real Hilbert space. Strong convergence of the proposed method is proved under some conditions. Results proved in this paper may be viewed as an improvement and refinement of the previously known results.

Key word. System of variational inequalities, variational inequality problem, hierarchical fixed point problem, projection method, strictly pseudo-contractive mapping.

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1 Introduction

Let H be a real Hilbert space, whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$. Let C be a nonempty closed convex subset of H and A is a mapping from C into H. A classical variational inequality problem, denoted by VI(A, C), is to find a vector $u \in C$ such that

$$\langle v - u, Au \rangle \geq 0, \quad \forall v \in C. \tag{1.1}$$

The solution of VI(A, C) is denoted by Ω^*. It is easy to observe that

$$u^* \in \Omega^* \Longleftrightarrow u^* = P_C[u^* - \rho A u^*], \quad \text{where} \quad \rho > 0.$$

Variational inequality problems are of fundamental importance in a wide range of mathematical and applied sciences problems, such as mathematical programming, traffic engineering, economics and equilibrium problems, see [1–20]. The ideas and techniques of the variational inequalities are being applied in a variety of diverse areas of sciences and proved to be productive and innovative. It has been shown that this theory provides a simple, natural and unified framework for a general treatment of unrelated problems. The projection and contraction method and its invariant forms represent an important tool for finding the approximation solution of various types of variational inequalities and complementarity problems.

We introduce the following definitions which are useful in the following analysis.

Definition 1.1 The mapping $T : C \to H$ is said to be

(a) monotone, if

$$\langle Tx - Ty, x - y \rangle \geq 0, \qquad \forall x, y \in C;$$

(b) strongly monotone, if there exists an $\alpha > 0$ such that

$$\langle Tx - Ty, x - y \rangle \geq \alpha \|x - y\|^2, \qquad \forall x, y \in C;$$

(c) α-inverse strongly monotone, if there exists an $\alpha > 0$ such that

$$\langle Tx - Ty, x - y \rangle \geq \alpha \|Tx - Ty\|^2, \qquad \forall x, y \in C;$$

(d) nonexpansive, if

$$\|Tx - Ty\| \leq \|x - y\|, \qquad \forall x, y \in C;$$

(e) k-Lipschitz continuous, if there exists a constant $k > 0$ such that

$$\|Tx - Ty\| \leq k\|x - y\|, \qquad \forall x, y \in C;$$

(f) contraction on C, if there exists a constant $0 \leq k < 1$ such that

$$\|Tx - Ty\| \leq k\|x - y\|, \qquad \forall x, y \in C;$$

It is easy to observe that every α-inverse strongly monotone T is monotone and Lipschitz continuous. A mapping $T : C \to H$ is called k-strict pseudo-contraction, if there exists a constant $0 \leq k < 1$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2, \qquad \forall x, y \in C. \tag{1.2}$$

The fixed point problem for the mapping T is to find $x \in C$ such that

$$Tx = x. \tag{1.3}$$

We denote $F(T)$ the set of solutions of (1.3). It is well-known that the class of strict pseudo-contractions strictly includes the class of nonexpansive mappings, then $F(T)$ is closed and convex and $P_{F(T)}$ is well defined (see [1]).

We consider the system of variational inequalities of finding $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \mu_1 B_1 y^* + x^* - y^*, x - x^* \rangle \geq 0; & \forall x \in C \quad \text{and} \quad \mu_1 > 0, \\ \langle \mu_2 B_2 x^* + y^* - x^*, x - y^* \rangle \geq 0; & \forall x \in C \quad \text{and} \quad \mu_2 > 0, \end{cases} \tag{1.4}$$

where $B_i : C \to C$ is a nonlinear mapping for each $i = 1, 2$. The solution set of (1.4) is denoted by S^*.

If $B_1 = B_2 = B$, then the problem (1.4) reduces of finding $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \mu_1 B y^* + x^* - y^*, x - x^* \rangle \geq 0; & \forall x \in C \quad \text{and} \quad \mu_1 > 0, \\ \langle \mu_2 B x^* + y^* - x^*, x - y^* \rangle \geq 0; & \forall x \in C \quad \text{and} \quad \mu_2 > 0, \end{cases} \tag{1.5}$$

which has been introduced and studied by Verma [2, 3].
If $x^* = y^*$ and $\mu_1 = \mu_2$, then the problem (1.5) collapses to the classical variational inequality finding $x^* \in C$, such that

$$\langle Bx^*, x - x^* \rangle \geq 0, \quad \forall x \in C.$$

We now have a variety of techniques to suggest and analyze various iterative algorithms for solving the system of variational inequalities see [2, 3, 4, 5, 6, 7, 8, 9, 10].

Let $S : C \to H$ be a nonexpansive mapping. The following problem is called a hierarchical fixed point problem: Find $x \in F(T)$ such that

$$\langle x - Sx, y - x \rangle \geq 0, \quad \forall y \in F(T). \tag{1.6}$$

It is known that the hierarchical fixed point problem (1.6) links with some monotone variational inequalities and convex programming problems; see [11, 12, 13]. Various methods have been proposed to solve the hierarchical fixed point problem; see Moudafi [14], Mainge and Moudafi in [15], Marino and Xu in [16] and Cianciaruso et al. [17]. Very recently, Yao et al. [12] introduced the following strong convergence iterative algorithm to solve the problem (1.6):

$$\begin{aligned} y_n &= \beta_n S x_n + (1 - \beta_n) x_n \qquad (1.7) \\ x_{n+1} &= P_C[\alpha_n f(x_n) + (1 - \alpha_n) T y_n], \quad \forall n \geq 0 \end{aligned}$$

where $f : C \to H$ is a contraction mapping and $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $(0, 1)$. Under some certain restrictions on parameters, Yao et al. proved that the sequence $\{x_n\}$ generated by (1.7) converges strongly to $z \in F(T)$, which is the unique solution of the following variational inequality:

$$\langle (I - f)z, y - z \rangle \geq 0, \quad \forall y \in F(T). \tag{1.8}$$

By changing the restrictions on parameters, the authors obtained another result on the iterative scheme (1.7), the sequence $\{x_n\}$ generated by (1.7) converges strongly to a point $z \in F(T)$, which is the unique solution of the following variational inequality:

$$\langle \frac{1}{\tau}(I - f)z + (I - S)z, y - z \rangle \geq 0, \quad \forall y \in F(T). \tag{1.9}$$

Let $S : C \to H$ be a nonexpansive mapping and $\{T_i\}_{i=1}^{\infty} : C \to C$ is a countable family of nonexpansive mappings. Very recently, Gu et al. [11] introduced the following iterative algorithm:

$$y_n = P_C[\beta_n S x_n + (1-\beta_n)x_n] \tag{1.10}$$
$$x_{n+1} = P_C[\alpha_n f(x_n) + \sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)T_i y_n], \quad \forall n \geq 1$$

where $\alpha_0 = 1$, $\{\alpha_n\}$ is a strictly decreasing sequence in $(0,1)$ and $\{\beta_n\}$ is a sequence in $(0,1)$. Under some certain conditions on parameters, Gu et al. proved that the sequence $\{x_n\}$ generated by (1.10) converges strongly to $z \in \bigcap_{i=1}^{\infty} F(T_i)$, which is unique solution of one of the variational inequalities (1.8) and (1.9).

In this paper, motivated by the work of Yao et al. [12] and Gu et al. [11] and by the recent work going in this direction, we give an iterative method for finding the approximate element of the common set of solutions of (1.1), (1.4) and (1.6) for a strictly pseudo-contraction mapping in real Hilbert space. We establish a strong convergence theorem based on this method. The presented method improve and generalize many known results for solving system of variational inequality problems, variational inequality problems and hierarchical fixed point problems, see, e.g. [11, 12, 15, 17] and relevant references cited therein.

2 Preliminaries

In this section, we list some fundamental lemmas that are useful in the consequent analysis. The first lemma provides some basic properties of projection onto C.

Lemma 2.1. Let P_C denote the projection of H onto C. Then, we have the following inequalities.

$$\langle z - P_C[z], P_C[z] - v\rangle \geq 0, \qquad \forall z \in H, v \in C; \tag{2.1}$$

$$\langle u - v, P_C[u] - P_C[v]\rangle \geq \|P_C[u] - P_C[v]\|^2, \qquad \forall u, v \in H; \tag{2.2}$$

$$\|P_C[u] - P_C[v]\| \leq \|u - v\|, \qquad \forall u, v \in H; \tag{2.3}$$

$$\|u - P_C[z]\|^2 \leq \|z - u\|^2 - \|z - P_C[z]\|^2, \qquad \forall z \in H, u \in C. \tag{2.4}$$

Lemma 2.2[4] For any $(x^*, y^*) \in C \times C, (x^*, y^*)$ is a solution of (1.4) if and only if x^* is a fixed point of the mapping $Q : C \to C$ de fined by

$$Q(x) = P_C[P_C[x - \mu_2 B_2 x] - \mu_1 B_1 P_C[x - \mu_2 B_2 x]], \quad \forall x \in C, \tag{2.5}$$

where $y^* = P_C[x^* - \mu_2 B_2 x^*], \mu_i \in (0, 2\theta_i)$ and $B_i : C \to C$ is θ_i-inverse strongly monotone mappings for each $i = 1, 2$.

Lemma 2.3[18] Let C be a nonempty closed convex subset of a real Hilbert space H. If $T : C \to C$ is a k-strict pseudo-contraction, then

(i) The mapping $I - T$ is demiclosed at 0, i.e., if $\{x_n\}$ is a sequence in C weakly converging to x and if $\{(I - T)x_n\}$ converges strongly to 0, then $(I - T)x = 0$;

(ii) The set $F(T)$ of T is closed and convex so that the projection $P_{F(T)}$ is well defined.

Lemma 2.4[19] Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that

$$a_{n+1} \leq (1 - \gamma_n)a_n + \delta_n,$$

where $\{\gamma_n\}$ is a sequence in $(0, 1)$ and δ_n is a sequence such that

(1) $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(2) $\limsup_{n\to\infty} \delta_n/\gamma_n \leq 0$ or $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.5[20] Let C be a closed convex subset of H. Let $\{x_n\}$ be a bounded sequence in H. Assume that

(i) The weak w-limit set $w_n(x_n) \subset C$ where $w_n(x_n) = \{x : x_{n_i} \rightharpoonup x\}$.

(ii) For each $z \in C, \lim_{n\to\infty} \|x_n - z\|$ exists.

Then $\{x_n\}$ is weakly convergent to a point in C.

Lemma 2.6[11] Let H be a Hilbert space, C be a closed and convex subset of H, and $T : C \to C$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. Then

$$\|Tx - x\|^2 \leq 2\langle x - Tx, x - x'\rangle, \quad \forall x' \in F(T), \forall x \in C.$$

3 The proposed method and some properties

In this Section, we suggest and analyze our method for finding the common solutions of the variational inequality (1.1), the system of variational inequality problem (1.4) and the hierarchical fixed point problem (1.6).

Let C be a nonempty closed convex subset of a real Hilbert space H. Let $A, B_i : C \to H$ be α, θ_i-inverse strongly monotone mappings for each $i = 1, 2$, respectively. Let $S : C \to H$ be a nonexpansive mapping and $\{T_i\}_{i=1}^{\infty} : C \to C$ is a countable family of k_i-strict pseudo-contraction mappings such that $\Omega^* \cap S^* \cap F(T) \neq \emptyset$, where $F(T) = \bigcap_{i=1}^{\infty} F(T_i)$. Let f be a ρ-contraction mapping.

Algorithm 3.1. For a given $x_0 \in C$ arbitrarily, let the iterative sequences $\{u_n\}, \{x_n\}, \{y_n\}$ and $\{z_n\}$ be generated by

$$\begin{cases} u_n = P_C[x_n - \lambda_n A x_n]; \\ z_n = P_C[P_C[u_n - \mu_2 B_2 u_n] - \mu_1 B_1 P_C[u_n - \mu_2 B_2 u_n]]; \\ y_n = P_C[\beta_n S x_n + (1 - \beta_n) z_n]; \\ x_{n+1} = P_C[\alpha_n f(x_n) + \sum_{i=1}^{n} (\alpha_{i-1} - \alpha_i) T_i y_n], \quad \forall n \geq 0, \end{cases} \tag{3.1}$$

where $\mu_i \in (0, 2\theta_i)$ for each $i = 1, 2, \{\lambda_n\} \subset (0, 2\alpha)$, $\alpha_0 = 1$, $\{\alpha_n\}$ is a strictly decreasing sequence in $(0, 1)$ and $\{\beta_n\}$ is a sequence in $(0, 1)$ satisfying the following conditions:

(a) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(b) $\lim_{n\to\infty} (\beta_n/\alpha_n) = 0$,

(c) $\sum_{n=1}^{\infty} |\alpha_{n-1} - \alpha_n| < \infty$ and $\sum_{n=1}^{\infty} |\beta_{n-1} - \beta_n| < \infty$,

(d) $\liminf_{n\to\infty} \lambda_n < \limsup_{n\to\infty} \lambda_n < 2\alpha$ and $\sum_{n=1}^{\infty} |\lambda_{n-1} - \lambda_n| < \infty$.

Remark 3.1 Our method can be viewed as extension and improvement for some well known results for example.

- If $A = 0$, we obtain an extension and improvement of the method of of Gu et al. [11] for finding the approximate element of the common set of solutions of a system of variational inequality problem and a hierarchical fixed point problem in a real Hilbert space.

- If $A = 0$ and $T_i = T$ $\forall i \geq 1$ we obtain an extension and improvement of the method of Yao et al. [12] for finding the approximate element of the common set of solutions of a system of variational inequality problem and a hierarchical fixed point problem in a real Hilbert space.

Lemma 3.1 Let $x^* \in \Omega^* \cap S^* \cap F(T)$. Then $\{x_n\}$ $\{u_n\}, \{z_n\}$ and $\{y_n\}$ are bounded.

Proof. First, we show that the mapping $(I - \lambda_n A)$ is nonexpansive. For any $x, y \in C$,

$$\begin{aligned}
\|(I-\lambda_n A)x - (I-\lambda_n A)y\|^2 &= \|(x-y) - \lambda_n(Ax - Ay)\|^2 \\
&= \|x-y\|^2 - 2\lambda_n\langle x-y, Ax-Ay\rangle + \lambda_n^2\|Ax-Ay\|^2 \\
&\leq \|x-y\|^2 - \lambda_n(2\alpha - \lambda_n)\|Ax-Ay\|^2 \\
&\leq \|x-y\|^2.
\end{aligned}$$

Similarly we can show that the mapping $(I - \mu_i B_i)$ are nonexpansive for each $i = 1, 2$.
Let $x^* \in \Omega^* \cap S^* \cap F(T)$, we have

$$x^* = P_C[y^* - \mu_1 B_1 y^*]$$

where

$$y^* = P_C[x^* - \mu_2 B_2 x^*].$$

Since the mapping A is α-inverse strongly monotone, we have

$$\begin{aligned}
\|u_n - x^*\|^2 &= \|P_C[x_n - \lambda_n Ax_n) - P_C[x^* - \lambda_n Ax^*]\|^2 \\
&\leq \|x_n - x^* - \lambda_n(Ax_n - Ax^*)\|^2 \\
&\leq \|x_n - x^*\|^2 - \lambda_n(2\alpha - \lambda_n)\|Ax_n - Ax^*\|^2 \qquad (3.2)\\
&\leq \|x_n - x^*\|^2.
\end{aligned}$$

Setting $v_n := P_C[u_n - \mu_2 B_2 u_n]$. Since B_2 is θ_2-inverse strongly monotone mapping, it follows that

$$\begin{aligned}
\|v_n - y^*\|^2 &= \|P_C[u_n - \mu_2 B_2 u_n] - P_C[x^* - \mu_2 B_2 x^*]\|^2 \\
&\leq \|u_n - u^* - \mu_2(B_2 u_n - B_2 x^*)\|^2 \\
&\leq \|x_n - x^*\|^2 - \mu_2(2\theta - \mu_2)\|B_2 u_n - B_2 x^*\|^2 \qquad (3.3)\\
&\leq \|x_n - x^*\|^2.
\end{aligned}$$

Since B_i is θ_i-inverse strongly monotone mappings for each $i = 1, 2$, we get

$$\begin{aligned}
\|z_n - x^*\|^2 &= \|P_C[P_C[u_n - \mu_2 B_2 u_n] - \mu_1 B_1 P_C[u_n - \mu_2 B_2 u_n]] \\
&\quad - P_C[P_C[x^* - \mu_2 B_2 x^*] - \mu_1 B_1 P_C[x^* - \mu_2 B_2 x^*]]\|^2 \\
&\leq \|P_C[u_n - \mu_2 B_2 u_n] - \mu_1 B_1 P_C[u_n - \mu_2 B_2 u_n] \\
&\quad - (P_C[x^* - \mu_2 B_2 x^*] - \mu_1 B_1 P_C[x^* - \mu_2 B_2 x^*])\|^2 \\
&= \|P_C[u_n - \mu_2 B_2 u_n] - P_C[x^* - \mu_2 B_2 x^*] \\
&\quad - \mu_1 (B_1 P_C[u_n - \mu_2 B_2 u_n] - B_1 P_C[x^* - \mu_2 B_2 x^*])\|^2 \\
&\leq \|P_C[u_n - \mu_2 B_2 u_n] - P_C[x^* - \mu_2 B_2 x^*]\|^2 \\
&\quad - \mu_1 (2\theta_1 - \mu_1) \|B_1 P_C[u_n - \mu_2 B_2 u_n] - B_1 P_C[x^* - \mu_2 B_2 x^*]\|^2 \\
&\leq \|(u_n - \mu_2 B_2 u_n) - (x^* - \mu_2 B_2 x^*)\|^2 \\
&\quad - \mu_1 (2\theta_1 - \mu_1) \|B_1 P_C[u_n - \mu_2 B_2 u_n] - B_1 P_C[x^* - \mu_2 B_2 x^*]\|^2 \\
&\leq \|u_n - x^*\|^2 - \mu_2 (2\theta_2 - \mu_2) \|B_2 u_n - B_2 x^*\|^2 \\
&\quad - \mu_1 (2\theta_1 - \mu_1) \|B_1 v_n - B_1 y^*\|^2 \qquad (3.4) \\
&\leq \|u_n - x^*\|^2 \\
&\leq \|x_n - x^*\|^2.
\end{aligned}$$

Next, we prove that the sequence $\{x_n\}$ is bounded, without loss of generality we can assume that $\beta_n \leq \alpha_n$ for all $n \geq 1$. From (3.1), we have

$$\begin{aligned}
\|x_{n+1}-x^*\| &\leq \|\alpha_n f(x_n)+\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)T_iy_n-\alpha_n x^*-\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)T_ix^*\| \\
&\leq \alpha_n\|f(x_n)-f(x^*)\|+\alpha_n\|f(x^*)-x^*\|+\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\|T_iy_n-T_ix^*\| \\
&\leq \alpha_n\|f(x_n)-f(x^*)\|+\alpha_n\|f(x^*)-x^*\|+\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\|y_n-x^*\| \\
&= \alpha_n\|f(x_n)-f(x^*)\|+\alpha_n\|f(x^*)-x^*\|+(1-\alpha_n)\|\beta_nSx_n+(1-\beta_n)z_n-x^*\| \\
&\leq \alpha_n\|f(x_n)-f(x^*)\|+\alpha_n\|f(x^*)-x^*\|+(1-\alpha_n)(\beta_n\|Sx_n-Sx^*\|+\beta_n\|Sx^*-x^*\| \\
&\quad +(1-\beta_n)\|z_n-x^*\|) \qquad (3.5)\\
&\leq \alpha_n\rho\|x_n-x^*\|+\alpha_n\|f(x^*)-x^*\|+(1-\alpha_n)(\beta_n\|x_n-x^*\|+\beta_n\|Sx^*-x^*\| \\
&\quad +(1-\beta_n)\|x_n-x^*\| \\
&= (1-\alpha_n(1-\rho))\|x_n-x^*\|+\alpha_n\|f(x^*)-x^*\|+(1-\alpha_n)\beta_n\|Sx^*-x^*\| \\
&\leq (1-\alpha_n(1-\rho))\|x_n-x^*\|+\alpha_n\|f(x^*)-x^*\|+\beta_n\|Sx^*-x^*\| \\
&\leq (1-\alpha_n(1-\rho))\|x_n-x^*\|+\alpha_n(\|f(x^*)-x^*\|+\|Sx^*-x^*\|) \\
&= (1-\alpha_n(1-\rho))\|x_n-x^*\|+\frac{\alpha_n(1-\rho)}{1-\rho}(\|f(x^*)-x^*\|+\|Sx^*-x^*\|) \\
&\leq \max\{\|x_n-x^*\|,\frac{1}{1-\rho}(\|f(x^*)-x^*\|+\|Sx^*-x^*\|)\}.
\end{aligned}$$

By induction on n, we obtain $\|x_n-x^*\| \leq \max\{\|x_0-x^*\|, \frac{1}{1-\rho}(\|f(x^*)-x^*\|+\|Sx^*-x^*\|)\}$, for $n \geq 0$ and $x_0 \in C$. Hence $\{x_n\}$ is bounded and consequently, we deduce that $\{u_n\}, \{z_n\}, \{v_n\}$ and $\{y_n\}$ are bounded. □

Lemma 3.2 Let $x^* \in \Omega^* \cap S^* \cap F(T)$ and $\{x_n\}$ be the sequence generated by the Algorithm 3.1. Then we have

(a) $\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0$.

(b) The weak w-limit set $w_w(x_n) \subset F(T)$ where$w_w(x_n) = \{x : x_{n_i} \rightharpoonup x\}$.

Proof. From the nonexpansivity of the mapping $(I-\lambda_n A)$ and P_C, we have

$$\begin{aligned}\|u_n-u_{n-1}\| &\leq \|(x_n-\lambda_n Ax_n)-(x_{n-1}-\lambda_{n-1}Ax_{n-1})\| \\ &= \|(x_n-x_{n-1})-\lambda_n(Ax_n-Ax_{n-1})-(\lambda_n-\lambda_{n-1})Ax_{n-1}\| \\ &\leq \|(x_n-x_{n-1})-\lambda_n(Ax_n-Ax_{n-1})\|+|\lambda_n-\lambda_{n-1}|\|Ax_{n-1}\| \\ &\leq \|x_n-x_{n-1}\|+|\lambda_n-\lambda_{n-1}|\|Ax_{n-1}\|. \end{aligned} \tag{3.6}$$

Next, we estimate

$$\begin{aligned}\|z_n-z_{n-1}\|^2 &= \|P_C[P_C[u_n-\mu_2B_2u_n]-\mu_1B_1P_C[u_n-\mu_2B_2u_n]] \\ &\quad -P_C[P_C[u_{n-1}-\mu_2B_2u_{n-1}]-\mu_1B_1P_C[u_{n-1}-\mu_2B_2u_{n-1}]]\|^2 \\ &\leq \|P_C[u_n-\mu_2B_2u_n]-\mu_1B_1P_C[u_n-\mu_2B_2u_n] \\ &\quad -(P_C[u_{n-1}-\mu_2B_2u_{n-1}]-\mu_1B_1P_C[u_{n-1}-\mu_2B_2u_{n-1}])\|^2 \\ &= \|P_C[u_n-\mu_2B_2u_n]-P_C[u_{n-1}-\mu_2B_2u_{n-1}] \\ &\quad -\mu_1(B_1P_C[u_n-\mu_2B_2u_n]-B_1P_C[u_{n-1}-\mu_2B_2u_{n-1}])\|^2 \\ &\leq \|P_C[u_n-\mu_2B_2u_n]-P_C[u_{n-1}-\mu_2B_2u_{n-1}]\|^2 \\ &\quad -\mu_1(2\theta_1-\mu_1)\|B_1P_C[u_n-\mu_2B_2u_n]-B_1P_C[u_{n-1}-\mu_2B_2u_{n-1}]\|^2 \\ &\leq \|P_C[u_n-\mu_2B_2u_n]-P_C[u_{n-1}-\mu_2B_2u_{n-1}]\|^2 \\ &\leq \|(u_n-u_{n-1})-\mu_2(B_2u_n-B_2u_{n-1})\|^2 \\ &\leq \|u_n-u_{n-1}\|^2-\mu_2(2\theta_2-\mu_2)\|B_2u_n-B_2u_{n-1}]\|^2 \\ &\leq \|u_n-u_{n-1}\|^2. \end{aligned} \tag{3.7}$$

It follows from (3.6) and (3.7) that

$$\|z_n-z_{n-1}\| \leq \|x_n-x_{n-1}\|+|\lambda_n-\lambda_{n-1}|\|Ax_{n-1}\|.$$

From (3.1) and the above inequality, we get

$$\begin{aligned}
\|y_n - y_{n-1}\| &\leq \|\beta_n Sx_n + (1-\beta_n)z_n - (\beta_{n-1}Sx_{n-1} + (1-\beta_{n-1})z_{n-1})\| \\
&= \|\beta_n(Sx_n - Sx_{n-1}) + (\beta_n - \beta_{n-1})Sx_{n-1} + (1-\beta_n)(z_n - z_{n-1}) + (\beta_{n-1} - \beta_n)z_{n-1}\| \\
&\leq \beta_n\|x_n - x_{n-1}\| + (1-\beta_n)\|z_n - z_{n-1}\| + |\beta_n - \beta_{n-1}|(\|Sx_{n-1}\| + \|z_{n-1}\|) \\
&\leq \beta_n\|x_n - x_{n-1}\| + (1-\beta_n)\{\|x_n - x_{n-1}\| + |\lambda_n - \lambda_{n-1}|\|Ax_{n-1}\|\} \\
&\quad + |\beta_n - \beta_{n-1}|(\|Sx_{n-1}\| + \|z_{n-1}\|) \\
&\leq \|x_n - x_{n-1}\| + |\lambda_n - \lambda_{n-1}|\|Ax_{n-1}\| + |\beta_n - \beta_{n-1}|(\|Sx_{n-1}\| + \|z_{n-1}\|). \qquad (3.8)
\end{aligned}$$

Next, we estimate

$$\begin{aligned}
\|x_{n+1} - x_n\| &\leq \|\alpha_n f(x_n) + \sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)T_i y_n - (\alpha_{n-1}f(x_{n-1}) + \sum_{i=1}^{n-1}(\alpha_{i-1} - \alpha_i)T_i y_{n-1})\| \\
&= \|\alpha_n(f(x_n) - f(x_{n-1})) + (\alpha_n - \alpha_{n-1})f(x_{n-1}) + \sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)(T_i y_n - T_i y_{n-1}) \\
&\quad + (\alpha_{n-1} - \alpha_n)T_n y_{n-1}\| \\
&\leq \alpha_n\|f(x_n) - f(x_{n-1})\| + \sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)\|T_i y_n - T_i y_{n-1}\| \\
&\quad + |\alpha_n - \alpha_{n-1}|(\|f(x_{n-1})\| + \|T_n y_{n-1}\|) \\
&\leq \alpha_n\rho\|x_n - x_{n-1}\| + \sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)\|y_n - y_{n-1}\| + |\alpha_n - \alpha_{n-1}|(\|f(x_{n-1})\| + \|T_n y_{n-1}\|) \\
&= \alpha_n\rho\|x_n - x_{n-1}\| + (1-\alpha_n)\|y_n - y_{n-1}\| + |\alpha_n - \alpha_{n-1}|(\|f(x_{n-1})\| + \|T_n y_{n-1}\|).
\end{aligned} \qquad (3.9)$$

From (3.8) and (3.9), we have

$$\begin{aligned}
\|x_{n+1} - x_n\| &\leq \alpha_n\rho\|x_n - x_{n-1}\| + (1-\alpha_n)\{\|x_n - x_{n-1}\| + |\lambda_n - \lambda_{n-1}|\|Ax_{n-1}\| \\
&\quad + |\beta_n - \beta_{n-1}|(\|Sx_{n-1}\| + \|z_{n-1}\|)\} + |\alpha_n - \alpha_{n-1}|(\|f(x_{n-1})\| + \|T_n y_{n-1}\|) \\
&\leq (1-(1-\rho)\alpha_n)\|x_n - x_{n-1}\| + |\lambda_n - \lambda_{n-1}|\|Ax_{n-1}\| + |\beta_n - \beta_{n-1}|(\|Sx_{n-1}\| \\
&\quad + \|z_{n-1}\|) + |\alpha_n - \alpha_{n-1}|(\|f(x_{n-1})\| + \|T_n y_{n-1}\|) \\
&\leq (1-(1-\rho)\alpha_n)\|x_n - x_{n-1}\| + M(|\lambda_n - \lambda_{n-1}| + |\beta_n - \beta_{n-1}| + |\alpha_n - \alpha_{n-1}|).
\end{aligned} \qquad (3.10)$$

Where

$$M = \max\{\sup_{n\geq 1}\|Ax_{n-1}\|, \sup_{n\geq 1}(\|Sx_{n-1}\| + \|z_{n-1}\|), \sup_{n\geq 1}(\|f(x_{n-1})\| + \|T_n y_{n-1}\|)\}.$$

It follows by conditions $(a)-(d)$ of Algorithm 3.1 and Lemma 2.4 that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0.$$

Next, we show that $\lim_{n\to\infty}\|u_n - x_n\| = 0$. Since $x^* \in \Omega^* \cap S^* \cap F(T)$ and $\alpha_n + \sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i) = 1$, by using (3.2) and (3.4), we obtain

$$\begin{aligned}
\|x_{n+1}-x^*\|^2 &\leq \|\alpha_n f(x_n) + \sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)T_i y_n - \alpha_n x^* - \sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)T_i x^*\|^2 \\
&\leq \alpha_n\|f(x_n)-x^*\|^2 + \sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\|T_i y_n - T_i x^*\|^2 \\
&\leq \alpha_n\|f(x_n)-x^*\|^2 + \sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\|y_n - x^*\|^2 \\
&\leq \alpha_n\|f(x_n)-x^*\|^2 + (1-\alpha_n)(\beta_n\|Sx_n - x^*\|^2 \\
&\quad +(1-\beta_n)\|z_n - x^*\|^2) \qquad (3.11) \\
&\leq \alpha_n\|f(x_n)-x^*\|^2 + (1-\alpha_n)\beta_n\|Sx_n - x^*\|^2 \\
&\quad +(1-\alpha_n)(1-\beta_n)\{\|u_n - x^*\|^2 - \mu_2(2\theta_2-\mu_2)\|B_2u_n - B_2x^*\|^2 \\
&\quad -\mu_1(2\theta_1-\mu_1)\|B_1v_n - B_1y^*\|^2\} \\
&\leq \alpha_n\|f(x_n)-x^*\|^2 + \beta_n\|Sx_n - x^*\|^2 + (1-\alpha_n)(1-\beta_n)\{\|x_n - x^*\|^2 \\
&\quad -\lambda_n(2\alpha-\lambda_n)\|Ax_n - Ax^*\|^2 - \mu_2(2\theta_2-\mu_2)\|B_2u_n - B_2x^*\|^2 \\
&\quad -\mu_1(2\theta_1-\mu_1)\|B_1v_n - B_1y^*\|^2\} \\
&\leq \alpha_n\|f(x_n)-x^*\|^2 + \beta_n\|Sx_n - x^*\|^2 + \|x_n - x^*\|^2 \\
&\quad -(1-\alpha_n)(1-\beta_n)\{\lambda_n(2\alpha-\lambda_n)\|Ax_n - Ax^*\|^2 + \mu_2(2\theta_2-\mu_2)\|B_2u_n - B_2x^*\|^2 \\
&\quad +\mu_1(2\theta_1-\mu_1)\|B_1v_n - B_1y^*\|^2\}
\end{aligned}$$

Then from the above inequality, we get

$$\begin{aligned}
&(1-\alpha_n)(1-\beta_n)\{\lambda_n(2\alpha-\lambda_n)\|Ax_n-Ax^*\|^2+\mu_2(2\theta_2-\mu_2)\|B_2u_n-B_2x^*\|^2\\
&+\mu_1(2\theta_1-\mu_1)\|B_1v_n-B_1y^*\|^2\}\\
\leq\ & \alpha_n\|f(x_n)-x^*\|^2+\beta_n\|Sx_n-x^*\|^2+\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2\\
\leq\ & \alpha_n\|f(x_n)-x^*\|^2+\beta_n\|Sx_n-x^*\|^2+(\|x_n-x^*\|+\|x_{n+1}-x^*\|)\|x_{n+1}-x_n\|.
\end{aligned}$$

Since $\liminf_{n\to\infty}\lambda_n\leq\limsup_{n\to\infty}\lambda_n<2\alpha, \lim_{n\to\infty}\|x_{n+1}-x_n\|=0, \alpha_n\to 0$ and $\beta_n\to 0$, we obtain $\lim_{n\to\infty}\|B_2u_n-B_2x^*\|=0, \lim_{n\to\infty}\|B_1v_n-B_1y^*\|=0$ and $\lim_{n\to\infty}\|Ax_n-Ax^*\|=0$. From (2.2), we have

$$\begin{aligned}
\|u_n-x^*\|^2 &= \|P_C[x_n-\lambda_nAx_n]-P_C[x^*-\lambda_nAx^*]\|^2\\
&\leq \langle u_n-x^*,(x_n-\lambda_nAx_n)-(x^*-\lambda_nAx^*)\rangle\\
&= \frac{1}{2}\{\|u_n-x^*\|^2+\|(x_n-\lambda_nAx_n)-(x^*-\lambda_nAx^*)\|^2\\
&\quad -\|u_n-x^*-[(x_n-\lambda_nAx_n)-(x^*-\lambda_nAx^*)]\|^2\}.
\end{aligned}$$

Hence

$$\begin{aligned}
\|u_n-x^*\|^2 &\leq \|(x_n-\lambda_nAx_n)-(x^*-\lambda_nAx^*)\|^2-\|u_n-x_n+\lambda_n(Ax_n-Ax^*)\|^2\\
&\leq \|x_n-x^*\|^2-\|u_n-x_n+\lambda_n(Ax_n-Ax^*)\|^2\\
&\leq \|x_n-x^*\|^2-\|u_n-x_n\|^2+2\lambda_n\|u_n-x_n\|\|Ax_n-Ax^*\|. \qquad (3.12)
\end{aligned}$$

From (3.11), (3.4) and the above inequality, we have

$$\begin{aligned}
\|x_{n+1}-x^*\|^2 &\leq \alpha_n\|f(x_n)-x^*\|^2+(1-\alpha_n)(\beta_n\|Sx_n-x^*\|^2+(1-\beta_n)\|z_n-x^*\|^2)\\
&\leq \alpha_n\|f(x_n)-x^*\|^2+(1-\alpha_n)(\beta_n\|Sx_n-x^*\|^2+(1-\beta_n)\|u_n-x^*\|^2)\\
&\leq \alpha_n\|f(x_n)-x^*\|^2+(1-\alpha_n)\{\beta_n\|Sx_n-x^*\|^2+(1-\beta_n)(\|x_n-x^*\|^2-\|u_n-x_n\|^2\\
&\quad +2\lambda_n\|u_n-x_n\|\|Ax_n-Ax^*\|)\}\\
&\leq \alpha_n\|f(x_n)-x^*\|^2+\beta_n\|Sx_n-x^*\|^2+\|x_n-x^*\|^2-(1-\alpha_n)(1-\beta_n)\|u_n-x_n\|^2\\
&\quad +2\lambda_n\|u_n-x_n\|\|Ax_n-Ax^*\|.
\end{aligned}$$

Hence

$$
\begin{aligned}
(1-\alpha_n)(1-\beta_n)\|u_n-x_n\|^2 &\leq \alpha_n\|f(x_n)-x^*\|^2+\beta_n\|Sx_n-x^*\|^2+\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2 \\
&\quad +2\lambda_n\|u_n-x_n\|\|Ax_n-Ax^*\| \\
&\leq \alpha_n\|f(x_n)-x^*\|^2+\beta_n\|Sx_n-x^*\|^2+(\|x_n-x^*\|+\|x_{n+1}-x^*\|)\|x_{n+1}-x_n\| \\
&\quad +2\lambda_n\|u_n-x_n\|\|Ax_n-Ax^*\||.
\end{aligned}
$$

Since $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0, \alpha_n\to 0, \beta_n\to 0$ and $\lim_{n\to\infty}\|Ax_n-Ax^*\|=0$, we obtain

$$
\lim_{n\to\infty}\|u_n-x_n\|=0. \tag{3.13}
$$

From (2.2), we get

$$
\begin{aligned}
\|v_n-y^*\|^2 &= \|P_C[u_n-\mu_2B_2u_n]-P_C[x^*-\mu_2B_2x^*]\|^2 \\
&\leq \langle v_n-y^*,(u_n-\mu_2B_2u_n)-(x^*-\mu_2B_2x^*)\rangle \\
&= \frac{1}{2}\{\|v_n-y^*\|^2+\|u_n-x^*-\mu_2(B_2u_n-B_2x^*)\|^2 \\
&\quad -\|u_n-x^*-\mu_2(B_2u_n-B_2x^*)-(v_n-y^*)\|^2\} \\
&\leq \frac{1}{2}\{\|v_n-y^*\|^2+\|u_n-x^*\|^2-\mu_2(2\theta_2-\mu_2)\|B_2u_n-B_2x^*\|^2 \\
&\quad -\|u_n-x^*-\mu_2(B_2u_n-B_2x^*)-(v_n-y^*)\|^2\} \\
&\leq \frac{1}{2}\{\|v_n-y^*\|^2+\|u_n-x^*\|^2-\|u_n-v_n-\mu_2(B_2u_n-B_2x^*)-(x^*-y^*)\|^2\} \\
&\leq \frac{1}{2}\{\|v_n-y^*\|^2+\|u_n-x^*\|^2-\|u_n-v_n-(x^*-y^*)\|^2 \\
&\quad +2\mu_2\langle u_n-v_n-(x^*-y^*),B_2u_n-B_2x^*\rangle-\mu_2^2\|B_2u_n-B_2x^*\|^2\} \\
&\leq \frac{1}{2}\{\|v_n-y^*\|^2+\|u_n-x^*\|^2-\|u_n-v_n-(x^*-y^*)\|^2 \\
&\quad +2\mu_2\|u_n-v_n-(x^*-y^*)\|\|B_2u_n-B_2x^*\|\}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
\|v_n-y^*\|^2 &\leq \|u_n-x^*\|^2-\|u_n-v_n-(x^*-y^*)\|^2+2\mu_2\|u_n-v_n-(x^*-y^*)\|\|B_2u_n-B_2x^*\| \\
&\leq \|x_n-x^*\|^2-\|u_n-x_n\|^2+2\lambda_n\|u_n-x_n\|\|Ax_n-Ax^*\| \\
&\quad -\|u_n-v_n-(x^*-y^*)\|^2+2\mu_2\|u_n-v_n-(x^*-y^*)\|\|B_2u_n-B_2x^*\| \qquad (3.14)
\end{aligned}
$$

where the last inequality follows from (3.12). On the other hand, from (3.1) and (2.2), we obtain

$$\begin{aligned}
\|z_n - x^*\|^2 &= \|P_C[v_n - \mu_1 B_1 v_n] - P_C[y^* - \mu_1 B_1 y^*]\|^2 \\
&\leq \langle z_n - x^*, (v_n - \mu_1 B_1 v_n) - (y^* - \mu_1 B_1 y^*)\rangle \\
&= \frac{1}{2}\{\|z_n - x^*\|^2 + \|v_n - y^* - \mu_1(B_1 v_n - B_1 y^*)\|^2 \\
&\quad -\|v_n - y^* - \mu_1(B_1 v_n - B_1 y^*) - (z_n - x^*)\|^2\} \\
&= \frac{1}{2}\{\|z_n - x^*\|^2 + \|v_n - y^*\|^2 - 2\mu_1\langle v_n - y^*, B_1 v_n - B_1 y^*\rangle + \mu_1^2\|B_1 v_n - B_1 y^*\|^2 \\
&\quad -\|v_n - y^* - \mu_1(B_1 v_n - B_1 y^*) - (z_n - x^*)\|^2\} \\
&\leq \frac{1}{2}\{\|z_n - x^*\|^2 + \|v_n - y^*\|^2 - \mu_1(2\theta_1 - \mu_1)\|B_1 v_n - B_1 y^*\|^2 \\
&\quad -\|v_n - y^* - \mu_1(B_1 v_n - B_1 y^*) - (z_n - x^*)\|^2\}
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{2}\{\|z_n - x^*\|^2 + \|v_n - y^*\|^2 - \|v_n - z_n - \mu_1(B_1 v_n - B_1 y^*) + (x^* - y^*)\|^2\} \\
&\leq \frac{1}{2}\{\|z_n - x^*\|^2 + \|v_n - y^*\|^2 - \|v_n - z_n + (x^* - y^*)\|^2 \\
&\quad +2\mu_1\langle v_n - z_n + (x^* - y^*), B_1 v_n - B_1 y^*\rangle\} \\
&\leq \frac{1}{2}\{\|z_n - x^*\|^2 + \|v_n - y^*\|^2 - \|v_n - z_n + (x^* - y^*)\|^2 \\
&\quad +2\mu_1\|v_n - z_n + (x^* - y^*)\|\|B_1 v_n - B_1 y^*\|\}
\end{aligned}$$

Hence

$$\begin{aligned}
\|z_n - x^*\|^2 &\leq \|v_n - y^*\|^2 - \|v_n - z_n + (x^* - y^*)\|^2 + 2\mu_1\|v_n - z_n + (x^* - y^*)\|\|B_1 v_n - B_1 y^*\| \\
&\leq \|x_n - x^*\|^2 - \|u_n - x_n\|^2 + 2\lambda_n\|u_n - x_n\|\|Ax_n - Ax^*\| \\
&\quad -\|u_n - v_n - (x^* - y^*)\|^2 + 2\mu_2\|u_n - v_n - (x^* - y^*)\|\|B_2 u_n - B_2 x^*\| \\
&\quad -\|v_n - z_n + (x^* - y^*)\|^2 + 2\mu_1\|v_n - z_n + (x^* - y^*)\|\|B_1 v_n - B_1 y^*\|
\end{aligned}$$

where the last inequality follows from (3.14). From (3.11) and the above inequality, we have

$$
\begin{aligned}
\|x_{n+1} - x^*\|^2 &\leq \alpha_n \|f(x_n) - x^*\|^2 + (1-\alpha_n)(\beta_n \|Sx_n - x^*\|^2 + (1-\beta_n)\|z_n - x^*\|^2) \\
&\leq \alpha_n \|f(x_n) - x^*\|^2 + (1-\alpha_n)\{\beta_n \|Sx_n - x^*\|^2 \\
&\quad +(1-\beta_n)(\|x_n - x^*\|^2 - \|u_n - x_n\|^2 + 2\lambda_n \|u_n - x_n\|\|Ax_n - Ax^*\|) \\
&\quad +(1-\beta_n)(-\|u_n - v_n - (x^* - y^*)\|^2 + 2\mu_2 \|u_n - v_n - (x^* - y^*)\|\|B_2u_n - B_2x^*\|) \\
&\quad +(1-\beta_n)(-\|v_n - z_n + (x^* - y^*)\|^2 + 2\mu_1 \|v_n - z_n + (x^* - y^*)\|\|B_1v_n - B_1y^*\|)\} \\
&\leq \alpha_n \|f(x_n) - x^*\|^2 + \beta_n \|Sx_n - x^*\|^2 + \|x_n - x^*\|^2 + 2\lambda_n \|u_n - x_n\|\|Ax_n - Ax^*\| \\
&\quad +2\mu_2 \|u_n - v_n - (x^* - y^*)\|\|B_2u_n - B_2x^*\| + 2\mu_1 \|v_n - z_n + (x^* - y^*)\|\|B_1v_n - B_1y^*\| \\
&\quad -(1-\alpha_n)(1-\beta_n)\{\|u_n - x_n\|^2 + \|u_n - v_n - (x^* - y^*)\|^2 + \|v_n - z_n + (x^* - y^*)\|^2\}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&(1-\alpha_n)(1-\beta_n)\{\|u_n - x_n\|^2 + \|u_n - v_n - (x^* - y^*)\|^2 + \|v_n - z_n + (x^* - y^*)\|^2\} \\
\leq\ & \alpha_n \|f(x_n) - x^*\|^2 + \beta_n \|Sx_n - x^*\|^2 + \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + 2\lambda_n \|u_n - x_n\|\|Ax_n - Ax^*\| \\
&+2\mu_2 \|u_n - v_n - (x^* - y^*)\|\|B_2u_n - B_2x^*\| + 2\mu_1 \|v_n - z_n + (x^* - y^*)\|\|B_1v_n - B_1y^*\| \\
\leq\ & \alpha_n \|f(x_n) - x^*\|^2 + \beta_n \|Sx_n - x^*\|^2 + (\|x_n - x^*\| + \|x_{n+1} - x^*\|)\|x_{n+1} - x_n\| \\
&+2\lambda_n \|u_n - x_n\|\|Ax_n - Ax^*\| + 2\mu_2 \|u_n - v_n - (x^* - y^*)\|\|B_2u_n - B_2x^*\| \\
&+2\mu_1 \|v_n - z_n + (x^* - y^*)\|\|B_1v_n - B_1y^*\|.
\end{aligned}
$$

Since $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0, \alpha_n \to 0, \beta_n \to 0$ and $\lim_{n\to\infty} \|Ax_n - Ax^*\| = 0, \lim_{n\to\infty} \|B_2u_n - B_2x^*\| = 0, \lim_{n\to\infty} \|B_1v_n - B_1y^*\| = 0$, we obtain

$$
\lim_{n\to\infty} \|u_n - v_n - (x^* - y^*)\| = 0 \quad \text{and} \quad \lim_{n\to\infty} \|v_n - z_n + (x^* - y^*)\| = 0.
$$

Since

$$
\|u_n - z_n\| \leq \|u_n - v_n - (x^* - y^*)\| + \|v_n - z_n + (x^* - y^*)\|
$$

we get

$$
\lim_{n\to\infty} \|u_n - z_n\| = 0. \tag{3.15}
$$

It folows from (3.13) and (3.15) that

$$
\lim_{n\to\infty} \|x_n - z_n\| = 0. \tag{3.16}
$$

Now, let $z \in \Omega^* \cap S^* \cap F(T)$, since for each $i \geq 1$, $T_i x_n \in C$ and $\alpha_n + \sum_{i=1}^n (\alpha_{i-1} - \alpha_i) = 1$, we have $\sum_{i=1}^n (\alpha_{i-1} - \alpha_i) T_i x_n + \alpha_n z \in C$. And

$$\begin{aligned}
\sum_{i=1}^n (\alpha_{i-1} - \alpha_i)(x_n - T_i x_n) &= P_C[\alpha_n f(x_n) + \sum_{i=1}^n (\alpha_{i-1} - \alpha_i) T_i y_n] + (1 - \alpha_n) x_n \\
&\quad -(\sum_{i=1}^n (\alpha_{i-1} - \alpha_i) T_i x_n + \alpha_n z) + \alpha_n z - x_{n+1} \\
&= P_C[\alpha_n f(x_n) + \sum_{i=1}^n (\alpha_{i-1} - \alpha_i) T_i y_n] + \alpha_n (z - x_{n+1}) \\
&\quad - P_C[\sum_{i=1}^n (\alpha_{i-1} - \alpha_i) T_i x_n + \alpha_n z] + (1 - \alpha_n)(x_n - x_{n+1}).
\end{aligned}$$

It follows that

$$\begin{aligned}
\sum_{i=1}^n (\alpha_{i-1} - \alpha_i) \langle x_n - T_i x_n, x_n - x^* \rangle &= \langle P_C[\alpha_n f(x_n) + \sum_{i=1}^n (\alpha_{i-1} - \alpha_i) T_i y_n] \\
&\quad - P_C[\sum_{i=1}^n (\alpha_{i-1} - \alpha_i) T_i x_n + \alpha_n z], x_n - x^* \rangle \\
&\quad + \alpha_n \langle z - x_{n+1}, x_n - x^* \rangle + (1 - \alpha_n) \langle x_n - x_{n+1}, x_n - x^* \rangle \\
&\leq \|\alpha_n (f(x_n) - z) + \sum_{i=1}^n (\alpha_{i-1} - \alpha_i)(T_i y_n - T_i x_n)\| \|x_n - x^*\| \\
&\quad + \alpha_n \|z - x_{n+1}\| \|x_n - x^*\| + (1 - \alpha_n) \|x_n - x_{n+1}\| \|x_n - x^*\| \\
&\leq \alpha_n \|f(x_n) - z\| \|x_n - x^*\| + \sum_{i=1}^n (\alpha_{i-1} - \alpha_i) \|y_n - x_n\| \|x_n - x^*\| \\
&\quad + \alpha_n \|z - x_{n+1}\| \|x_n - x^*\| + (1 - \alpha_n) \|x_n - x_{n+1}\| \|x_n - x^*\| \\
&= \alpha_n \|f(x_n) - z\| \|x_n - x^*\| + (1 - \alpha_n) \|y_n - x_n\| \|x_n - x^*\| \\
&\quad + \alpha_n \|z - x_{n+1}\| \|x_n - x^*\| + (1 - \alpha_n) \|x_n - x_{n+1}\| \|x_n - x^*\|
\end{aligned}$$

$$\begin{aligned}
&\leq \alpha_n \|f(x_n) - z\| \|x_n - x^*\| + (1 - \alpha_n) \|\beta_n S x_n + (1 - \beta_n) z_n - x_n\| \|x_n - x^*\| \\
&\quad + \alpha_n \|z - x_{n+1}\| \|x_n - x^*\| + (1 - \alpha_n) \|x_n - x_{n+1}\| \|x_n - x^*\| \\
&\leq \alpha_n \|f(x_n) - z\| \|x_n - x^*\| + (1 - \alpha_n) \beta_n \|S x_n - x_n\| \|x_n - x^*\| \\
&\quad + (1 - \alpha_n)(1 - \beta_n) \|z_n - x_n\| \|x_n - x^*\| + \alpha_n \|z - x_{n+1}\| \|x_n - x^*\| \\
&\quad + (1 - \alpha_n) \|x_n - x_{n+1}\| \|x_n - x^*\|.
\end{aligned}$$

From Lemma 2.6 and the above inequality, we get

$$\begin{aligned}\frac{1}{2}\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\|x_n-T_ix_n\|^2 &\leq \sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\langle x_n-T_ix_n, x_n-x^*\rangle \\ &\leq \alpha_n\|f(x_n)-z\|\|x_n-x^*\|+(1-\alpha_n)\beta_n\|Sx_n-x_n\|\|x_n-x^*\| \\ &\quad +(1-\alpha_n)(1-\beta_n)\|z_n-x_n\|\|x_n-x^*\|+\alpha_n\|z-x_{n+1}\|\|x_n-x^*\| \\ &\quad +(1-\alpha_n)\|x_n-x_{n+1}\|\|x_n-x^*\|.\end{aligned}$$

Since $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0, \alpha_n\to 0, \beta_n\to 0$ and $\lim_{n\to\infty}\|x_n-z_n\|=0$, we obtain

$$\lim_{n\to\infty}\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\|x_n-T_ix_n\|^2=0.$$

Since $(\alpha_{i-1}-\alpha_i)\|x_n-T_ix_n\|^2\leq\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\|x_n-T_ix_n\|^2$ and $\{\alpha_n\}$ is strictly decreasing, we have

$$\lim_{n\to\infty}\|x_n-T_ix_n\|=0.$$

Since $\{x_n\}$ is bounded, without loss of generality we can assume that $x_n \rightharpoonup w\in C$. It follows from Lemma 2.3 that $w\in F(T)$. Therfore $w_w(x_n)\subset F(T)$. □

Theorem 3.1 The sequence $\{x_n\}$ generated by Algorithm 3.1 converges strongly to $z = P_{\Omega^*\cap S^*\cap F(T)}f(z)$, which is the unique solution of the variational inequality

$$\langle (I-f)z, x-z\rangle\geq 0, \quad \forall x\in\Omega^*\cap S^*\cap F(T). \tag{3.17}$$

Proof. Since $\{x_n\}$ is bounded $x_n\rightharpoonup w$ and from Lemma 3.2, we have $w\in F(T)$. Next, we show that $w\in S^*$. Since $\lim_{n\to\infty}\|x_n-z_n\|=0$ and there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k}\to w$, it easy to observe that $z_{n_k}\to w$. For any $x,y\in C$, using (2.5), we have

$$\begin{aligned}\|Q(x)-Q(y)\|^2 &= \|P_C[P_C[x-\mu_2B_2x]-\mu_1B_1P_C[x-\mu_2B_2x]] \\ &\quad -P_C[P_C[y-\mu_2B_2y]-\mu_1B_1P_C[y-\mu_2B_2y]]\|^2 \\ &\leq \|(P_C[x-\mu_2B_2x]-P_C[y-\mu_2B_2y])-\mu_1(B_1P_C[x-\mu_2B_2x]-B_1P_C[y-\mu_2B_2y])\|^2 \\ &\leq \|P_C[x-\mu_2B_2x]-P_C[y-\mu_2B_2y]\|^2 \\ &\quad -\mu_1(2\theta_1-\mu_1)\|P_C[x-\mu_2B_2x]-P_C[y-\mu_2B_2y]\|^2\end{aligned}$$

$$\begin{aligned}
&\leq \|P_C[x-\mu_2 B_2 x] - P_C[y-\mu_2 B_2 y]\|^2 \\
&\leq \|(x-\mu_2 B_2 x) - (y-\mu_2 B_2 y)\|^2 \\
&\leq \|x-y\|^2 - \mu_2(2\theta_2-\mu_2)\|B_2 x - B_2 y\|^2 \\
&\leq \|x-y\|^2.
\end{aligned}$$

This implies that $Q : C \to C$ is nonexpansive. On the other hand

$$\begin{aligned}
\|z_n - Q(z_n)\|^2 &= \|P_C[P_C[u_n-\mu_2 B_2 u_n] - \mu_1 B_1 P_C[u_n-\mu_2 B_2 u_n]] - Q(z_n)\|^2 \\
&= \|Q(u_n) - Q(z_n)\|^2 \\
&\leq \|u_n - z_n\|^2.
\end{aligned}$$

Since $\lim_{n\to\infty}\|u_n - z_n\| = 0$ (see (3.15)), we have $\lim_{n\to\infty}\|z_n - Q(z_n)\| = 0$. It follows from Lemma 2.3 that $w = Q(w)$, which implies from Lemma 2.2 that $w \in S^*$.

Furthermore, we show that $w \in \Omega^*$. Let

$$Tv = \begin{cases} Av + N_C v, & \forall v \in C, \\ \emptyset, & \text{otherwise,} \end{cases}$$

where $N_C v := \{w \in H : \langle w, v-u\rangle \geq 0, \quad \forall u \in C\}$ is the normal cone to C at $v \in C$. Then T is maximal monotone and $0 \in Tv$ if and only if $v \in \Omega^*$ (see [21]). Let $G(T)$ denote the graph of T and let $(v,u) \in G(T)$, since $u - Av \in N_C v$ and $z_n \in C$, we have

$$\langle v - z_n, u - Av\rangle \geq 0. \tag{3.18}$$

On the other hand, it follows from $z_n = P_C[u_n - \lambda_n A u_n]$ and $v \in C$ that

$$\langle v - z_n, z_n - (u_n - \lambda_n A u_n)\rangle \geq 0$$

and

$$\langle v - z_n, \frac{z_n - u_n}{\lambda_n} + Au_n\rangle \geq 0.$$

Therefore, from (3.18) and inverse strongly monotonicity of A, we have

$$\begin{aligned}\langle v - z_{n_k}, u\rangle &\geq \langle v - z_{n_k}, Av\rangle \\ &\geq \langle v - z_{n_k}, Av\rangle - \langle v - z_{n_k}, \frac{z_{n_k} - u_{n_k}}{\lambda_{n_k}} + Au_{n_k}\rangle \\ &\geq \langle v - z_{n_k}, Av - Az_{n_k}\rangle + \langle v - z_{n_k}, Az_{n_k} - Au_{n_k}\rangle - \langle v - z_{n_k}, \frac{z_{n_k} - u_{n_k}}{\lambda_{n_k}}\rangle \\ &\geq \langle v - z_{n_k}, Az_{n_k} - Au_{n_k}\rangle - \langle v - z_{n_k}, \frac{z_{n_k} - u_{n_k}}{\lambda_{n_k}}\rangle\end{aligned}$$

Since $\lim_{n\to\infty} \|u_n - z_n\| = 0$ and $u_{n_k} \to w$, it easy to observe that $z_{n_k} \to w$. Hence, we obtain $\langle v - w, u\rangle \geq 0$. Since T is maximal monotone, we have $w \in T^{-1}0$ and hence $w \in \Omega^*$. Thus we have

$$w \in \Omega^* \cap S^* \cap F(T).$$

Next, we claim that $\limsup_{n\to\infty}\langle f(z) - z, x_n - z\rangle \leq 0$ where $z = P_{\Omega^*\cap S^*\cap F(T)} f(z)$.
Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle f(z) - z, x_n - z\rangle = \limsup_{k\to\infty}\langle f(z) - z, x_{n_k} - z\rangle = \langle f(z) - z, w - z\rangle \leq 0. \qquad .$$

Next, we show that $x_n \to z$.

$$\begin{aligned}\|x_{n+1} - z\|^2 &= \langle x_{n+1} - \alpha_n f(x_n) - \sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)T_i y_n, x_{n+1} - z\rangle \\ &\quad +\langle \alpha_n f(x_n) + \sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)T_i y_n - z, x_{n+1} - z\rangle \\ &\leq \langle \alpha_n f(x_n) + \sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)T_i y_n - z, x_{n+1} - z\rangle \\ &= \alpha_n\langle f(x_n) - f(z), x_{n+1} - z\rangle + \alpha_n\langle f(z) - z, x_{n+1} - z\rangle \\ &\quad +\sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)\langle T_i y_n - z, x_{n+1} - z\rangle \\ &\leq \alpha_n\|f(x_n) - f(z)\|\|x_{n+1} - z\| + \alpha_n\langle f(z) - z, x_{n+1} - z\rangle \\ &\quad +\sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)\|T_i y_n - z\|\|x_{n+1} - z\| \\ &\leq \alpha_n\rho\|x_n - z\|\|x_{n+1} - z\| + \alpha_n\langle f(z) - z, x_{n+1} - z\rangle \\ &\quad +\sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)\|y_n - z\|\|x_{n+1} - z\| \\ &\leq \alpha_n\rho\|x_n - z\|\|x_{n+1} - z\| + \alpha_n\langle f(z) - z, x_{n+1} - z\rangle \\ &\quad +(1 - \alpha_n)\{\beta_n\|Sx_n - Sz\| + \beta_n\|Sz - z\| + (1 - \beta_n)\|z_n - z\|\}\|x_{n+1} - z\|\end{aligned}$$

$$
\begin{aligned}
&\le \alpha_n\rho\|x_n - z\|\|x_{n+1} - z\| + \alpha_n\langle f(z) - z, x_{n+1} - z\rangle \\
&\quad +(1-\alpha_n)\{\beta_n\|x_n - z\| + \beta_n\|Sz - z\| + (1-\beta_n)\|x_n - z\|\}\|x_{n+1} - z\| \\
&\le (1-\alpha_n(1-\rho))\|x_n - z\|\|x_{n+1} - z\| + \alpha_n\langle f(z) - z, x_{n+1} - z\rangle \\
&\quad +(1-\alpha_n)\beta_n\|Sz - z\|\|x_{n+1} - z\| \\
&\le \frac{1-\alpha_n(1-\rho)}{2}(\|x_n - z\|^2 + \|x_{n+1} - z\|^2) + \alpha_n\langle f(z) - z, x_{n+1} - z\rangle \\
&\quad +(1-\alpha_n)\beta_n\|Sz - z\|\|x_{n+1} - z\|
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\|x_{n+1} - z\|^2 &\le \left(1 - \frac{2\alpha_n(1-\rho)}{1+\alpha_n(1-\rho)}\right)\|x_n - z\|^2 + \frac{2\alpha_n}{1+\alpha_n(1-\rho)}\langle f(z) - z, x_{n+1} - z\rangle \\
&\quad + \frac{2(1-\alpha_n)\beta_n}{1+\alpha_n(1-\rho)}\|Sz - z\|\|x_{n+1} - z\| \\
&\le \left(1 - \frac{2\alpha_n(1-\rho)}{1+\alpha_n(1-\rho)}\right)\|x_n - z\|^2 + \frac{2\alpha_n(1-\rho)}{1+\alpha_n(1-\rho)}\{\frac{1}{1-\rho}\langle f(z) - z, x_{n+1} - z\rangle \\
&\quad + \frac{(1-\alpha_n)\beta_n}{\alpha_n(1-\rho)}\|Sz - z\|\|x_{n+1} - z\|\}.
\end{aligned}
$$

Let $\gamma_n = \frac{2\alpha_n(1-\rho)}{1+\alpha_n(1-\rho)}$ and $\delta_n = \frac{2\alpha_n(1-\rho)}{1+\alpha_n(1-\rho)}\{\frac{1}{1-\rho}\langle f(z)-z, x_{n+1}-z\rangle + \frac{(1-\alpha_n)\beta_n}{\alpha_n(1-\rho)}\|Sz-z\|\|x_{n+1}-z\|\}$.
Since

$$
\sum_{n=1}^{\infty}\alpha_n = \infty, \quad 1+\alpha_n(1-\rho) \le 2 \quad \text{and} \quad \limsup_{n\to\infty}\{\frac{1}{1-\rho}\langle f(z)-z, x_{n+1}-z\rangle + \frac{(1-\alpha_n)\beta_n}{\alpha_n(1-\rho)}\|Sz-z\|\|x_{n+1}-z\|\} \le 0.
$$

It follows that

$$
\sum_{n=1}^{\infty}\gamma_n = \infty \quad \text{and} \quad \limsup_{n\to\infty}\frac{\delta_n}{\gamma_n} \le 0.
$$

Thus all the conditions of Lemma 2.4 are satisfied. Hence we deduce that $x_n \to z$.
$P_{\Omega^*\cap S^*\cap F(T)}f$ is a contraction, there exists a unique $z \in C$ such that $z = P_{\Omega^*\cap S^*\cap F(T)}f(z)$. From (2.1), it follows that z is the unique solution of the problem (3.17). This completes the proof. □

Theorem 3.2. Let C be a nonempty closed convex subset of a real Hilbert space H. Let $A, B_i : C \to H$ be α, θ_i-inverse strongly monotone mappings for each $i = 1, 2$, respectively. Let $S : C \to H$ be a nonexpansive mapping and $\{T_i\}_{i=1}^{\infty} : C \to C$ is a countable family of k_i-strict pseudo-contraction mappings such that $\Omega^* \cap S^* \cap F(T) \neq \emptyset$, where $F(T) = \bigcap_{i=1}^{\infty} F(T_i)$.

Let f be a ρ-contraction mapping. For a given $x_0 \in C$ arbitrarily, let the iterative sequences $\{u_n\}, \{x_n\}, \{y_n\}$ and $\{z_n\}$ be generated by

$$\begin{cases} u_n = P_C[x_n - \lambda_n A x_n]; \\ z_n = P_C[P_C[u_n - \mu_2 B_2 u_n] - \mu_1 B_1 P_C[u_n - \mu_2 B_2 u_n]]; \\ y_n = \beta_n S x_n + (1-\beta_n) z_n; \\ x_{n+1} = P_C[\alpha_n f(x_n) + \sum_{i=1}^{n} (\alpha_{i-1} - \alpha_i) T_i y_n], \quad \forall n \geq 0, \end{cases} \tag{3.19}$$

where $\mu_i \in (0, 2\theta_i)$ for each $i = 1, 2, \{\lambda_n\} \subset (0, 2\alpha)$, $\alpha_0 = 1$, $\{\alpha_n\}$ is a strictly decreasing sequence in $(0,1)$ and $\{\beta_n\}$ is a sequence in $(0,1)$ satisfying the following conditions:

(a) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(b) $\lim_{n\to\infty} \frac{\beta_n}{\alpha_n} = \tau \in (0, \infty)$,

(c) $\sum_{n=1}^{\infty} (\alpha_{n-1} - \alpha_n) < \infty$ and $\sum_{n=1}^{\infty} |\beta_{n-1} - \beta_n| < \infty$,

(d) $\lim_{n\to\infty} \frac{|\lambda_n - \lambda_{n-1}| + |\alpha_{n-1} - \alpha_n| + |\beta_{n-1} - \beta_n|}{\alpha_n \beta_n} = 0$,

(e) there exists a constant $K > 0$ such that $\frac{1}{\alpha_n} |\frac{1}{\beta_n} - \frac{1}{\beta_{n-1}}| \leq K$,

(f) $\liminf_{n\to\infty} \lambda_n < \limsup_{n\to\infty} \lambda_n < 2\alpha$ and $\sum_{n=1}^{\infty} |\lambda_{n-1} - \lambda_n| < \infty$.

Then sequence $\{x_n\}$ generated by Algorithm (3.19) converges strongly to $x^* \in \Omega^* \cap S^* \cap F(T)$, which is the unique solution of the variational inequality

$$\langle \frac{1}{\tau}(I - f)x^* + (I - S)x^*, x - x^* \rangle \geq 0, \quad \forall x \in \Omega^* \cap S^* \cap F(T). \tag{3.20}$$

Proof. From $\lim_{n\to\infty}(\beta_n/\alpha_n) = \tau \in (0, \infty)$, without loss of generality, we can assume that $\beta_n \leq (1+\tau)\alpha_n$ for all $n \geq 1$. Hence $\beta_n \to 0$. By similar argument as that lemmas 3.1 and 3.2, we can deduce that $\{x_n\}$ is bounded, $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$, $\lim_{n\to\infty} \|x_n - z_n\| = 0$ (see (3.16)) and $\|(I - T_i)x_n\| \to 0$. Then, we have

$$\|y_n - x_n\| \leq \beta_n \|x_n - Sx_n\| + (1-\beta_n)\|x_n - z_n\| \to 0 \quad \text{as} \quad n \to \infty. \tag{3.21}$$

It follows that, for all $i \geq 1$,

$$\|y_n - T_i x_n\| \leq \|y_n - x_n\| + \|x_n - T_i x_n\| \to 0 \quad \text{as} \quad n \to \infty. \tag{3.22}$$

From (3.21) and (3.22), we have

$$\|y_n - T_i y_n\| \leq \|y_n - T_i x_n\| + \|T_i x_n - T_i y_n\| \leq \|y_n - T_i x_n\| + \|y_n - x_n\| \to 0 \quad \text{as} \quad n \to \infty.$$

Setting $w_n := \alpha_n f(x_n) + \sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i) T_i y_n$. From (3.9) and (3.10), we obtain

$$\begin{aligned}
\frac{\|x_{n+1} - x_n\|}{\beta_n} &\leq \frac{\|w_n - w_{n-1}\|}{\beta_n} \\
&\leq (1-(1-\rho)\alpha_n)\frac{\|x_n - x_{n-1}\|}{\beta_n} + M\left(\frac{|\lambda_n - \lambda_{n-1}|}{\beta_n} + \frac{|\beta_n - \beta_{n-1}|}{\beta_n} + \frac{|\alpha_n - \alpha_{n-1}|}{\beta_n}\right) \\
&= (1-(1-\rho)\alpha_n)\frac{\|x_n - x_{n-1}\|}{\beta_{n-1}} + (1-(1-\rho)\alpha_n)\|x_n - x_{n-1}\|\left(\frac{1}{\beta_n} - \frac{1}{\beta_{n-1}}\right) \\
&\quad + M\left(\frac{|\lambda_n - \lambda_{n-1}|}{\beta_n} + \frac{|\beta_n - \beta_{n-1}|}{\beta_n} + \frac{|\alpha_n - \alpha_{n-1}|}{\beta_n}\right) \\
&\leq (1-(1-\rho)\alpha_n)\frac{\|x_n - x_{n-1}\|}{\beta_{n-1}} + \|x_n - x_{n-1}\|\left|\frac{1}{\beta_n} - \frac{1}{\beta_{n-1}}\right| \\
&\quad + M\left(\frac{|\lambda_n - \lambda_{n-1}|}{\beta_n} + \frac{|\beta_n - \beta_{n-1}|}{\beta_n} + \frac{|\alpha_n - \alpha_{n-1}|}{\beta_n}\right) \\
&\leq (1-(1-\rho)\alpha_n)\frac{\|x_n - x_{n-1}\|}{\beta_{n-1}} + \alpha_n K\|x_n - x_{n-1}\| \\
&\quad + M\left(\frac{|\lambda_n - \lambda_{n-1}|}{\beta_n} + \frac{|\beta_n - \beta_{n-1}|}{\beta_n} + \frac{|\alpha_n - \alpha_{n-1}|}{\beta_n}\right) \\
&\leq (1-(1-\rho)\alpha_n)\frac{\|w_{n-1} - w_{n-2}\|}{\beta_{n-1}} + \alpha_n K\|x_n - x_{n-1}\| \\
&\quad + M\left(\frac{|\lambda_n - \lambda_{n-1}|}{\beta_n} + \frac{|\beta_n - \beta_{n-1}|}{\beta_n} + \frac{|\alpha_n - \alpha_{n-1}|}{\beta_n}\right).
\end{aligned}$$

Let $\gamma_n = (1-\rho)\alpha_n$ and $\delta_n = \alpha_n K\|x_n - x_{n-1}\| + M\left(\frac{|\lambda_n - \lambda_{n-1}|}{\beta_n} + \frac{|\beta_n - \beta_{n-1}|}{\beta_n} + \frac{|\alpha_n - \alpha_{n-1}|}{\beta_n}\right)$. From conditions (a) and (d), we have

$$\sum_{n=1}^{\infty} \gamma_n = \infty \quad \text{and} \quad \lim_{n\to\infty} \frac{\delta_n}{\gamma_n} = 0.$$

By Lemma 2.4, we obtain

$$\lim_{n\to\infty} \frac{\|x_{n+1} - x_n\|}{\beta_n} = 0, \lim_{n\to\infty} \frac{\|w_{n+1} - w_n\|}{\beta_n} = \lim_{n\to\infty} \frac{\|w_{n+1} - w_n\|}{\alpha_n} = 0.$$

From (3.19), we have

$$x_{n+1} = P_C[w_n] - w_n + \alpha_n f(x_n) + \sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)(T_i y_n - y_n) + (1-\alpha_n) y_n.$$

Hence it follows that

$$\begin{aligned} x_n - x_{n+1} &= (1-\alpha_n)x_n + \alpha_n x_n \\ &\quad - \left(P_C[w_n] - w_n + \alpha_n f(x_n) + \sum_{i=1}^{n} (\alpha_{i-1} - \alpha_i)(T_i y_n - y_n) + (1-\alpha_n) y_n \right) \\ &= (1-\alpha_n)\left[\beta_n (x_n - Sx_n) + (1-\beta_n)(x_n - z_n)\right] + (w_n - P_C[w_n]) \\ &\quad + \sum_{i=1}^{n} (\alpha_{i-1} - \alpha_i)(y_n - T_i y_n) + \alpha_n (x_n - f(x_n)) \end{aligned}$$

and hence

$$\begin{aligned} \frac{x_n - x_{n+1}}{(1-\alpha_n)\beta_n} &= x_n - Sx_n + \frac{(1-\beta_n)}{\beta_n}(x_n - z_n) + \frac{1}{(1-\alpha_n)\beta_n}(w_n - P_C[w_n]) \\ &\quad + \frac{1}{(1-\alpha_n)\beta_n} \sum_{i=1}^{n} (\alpha_{i-1} - \alpha_i)(y_n - T_i y_n) + \frac{\alpha_n}{(1-\alpha_n)\beta_n}(x_n - f(x_n)) \end{aligned}$$

Let $v_n = \frac{x_n - x_{n+1}}{(1-\alpha_n)\beta_n}$. For any $z \in \Omega^* \cap S^* \cap F(T)$, we have

$$\begin{aligned} \langle v_n, x_n - z \rangle &= \frac{1}{(1-\alpha_n)\beta_n} \langle w_n - P_C[w_n], P_C[w_{n-1}] - z \rangle + \frac{\alpha_n}{(1-\alpha_n)\beta_n} \langle (I-f)x_n, x_n - z \rangle \\ &\quad + \langle x_n - Sx_n, x_n - z \rangle + \frac{(1-\beta_n)}{\beta_n} \langle x_n - z_n, x_n - z \rangle \\ &\quad + \frac{1}{(1-\alpha_n)\beta_n} \sum_{i=1}^{n} (\alpha_{i-1} - \alpha_i) \langle y_n - T_i y_n, x_n - z \rangle. \end{aligned} \tag{3.23}$$

Since S is nonexpansive mapping, f is ρ-contraction mapping and T_i is k_i-strict pseudo-contraction mapping. Then $(I-S)$ and $(I-T_i)$ are monotones, and f is strongly monotone with coefficient $(1-\rho)$. We can deduce that

$$\begin{aligned} \langle x_n - Sx_n, x_n - z \rangle &= \langle (I-S)x_n - (I-S)z, x_n - z \rangle + \langle (I-S)z, x_n - z \rangle \\ &\geq \langle (I-S)z, x_n - z \rangle, \end{aligned} \tag{3.24}$$

$$\begin{aligned} \langle (I-f)x_n, x_n - z \rangle &= \langle (I-f)x_n - (I-f)z, x_n - z \rangle + \langle (I-f)z, x_n - z \rangle \\ &\geq (1-\rho)\|x_n - z\|^2 + \langle (I-f)z, x_n - z \rangle, \end{aligned} \tag{3.25}$$

$$\begin{aligned} \langle (I-T_i)y_n, x_n - z \rangle &= \langle (I-T_i)y_n - (I-T_i)z, x_n - y_n \rangle + \langle (I-T_i)y_n - (I-T_i)z, y_n - z \rangle \\ &\geq \langle (I-T_i)y_n - (I-T_i)z, x_n - y_n \rangle \\ &= \langle (I-T_i)y_n, x_n - y_n \rangle \\ &= \langle (I-T_i)y_n, \beta_n (x_n - Sx_n) + (1-\beta_n)(x_n - z_n) \rangle. \end{aligned} \tag{3.26}$$

From (2.1), we get

$$\begin{aligned}\langle w_n - P_C[w_n], P_C[w_{n-1}] - z\rangle &= \langle w_n - P_C[w_n], P_C[w_{n-1}] - P_C[w_n]\rangle + \langle w_n - P_C[w_n], P_C[w_n] - z\rangle \\ &\geq \langle w_n - P_C[w_n], P_C[w_{n-1}] - P_C[w_n]\rangle. \qquad (3.27)\end{aligned}$$

Then, from (3.23)-(3.27), we have

$$\begin{aligned}\langle v_n, x_n - z\rangle \geq\ & \frac{1}{(1-\alpha_n)\beta_n}\langle w_n - P_C[w_n], P_C[w_{n-1}] - P_C[w_n]\rangle + \frac{\alpha_n}{(1-\alpha_n)\beta_n}\langle (I-f)z, x_n - z\rangle \\ & + \langle (I-S)z, x_n - z\rangle + \frac{(1-\beta_n)}{\beta_n}\langle x_n - z_n, x_n - z\rangle \\ & + \frac{(1-\beta_n)}{(1-\alpha_n)\beta_n}\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\langle (I-T_i)y_n, x_n - z_n\rangle \\ & + \frac{1}{(1-\alpha_n)}\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\langle (I-T_i)y_n, x_n - Sx_n\rangle + \frac{(1-\rho)\alpha_n}{(1-\alpha_n)\beta_n}\|x_n - z\|^2.\end{aligned}$$

Then, we obtain

$$\begin{aligned}\|x_n - z\|^2 \leq\ & \frac{1}{(1-\rho)\alpha_n}\|w_n - P_C[w_n]\|\|w_{n-1} - w_n\| - \frac{1}{(1-\rho)}\langle (I-f)z, x_n - z\rangle \\ & + \frac{(1-\alpha_n)\beta_n}{(1-\rho)\alpha_n}\left(\langle v_n, x_n - z\rangle - \langle (I-S)z, x_n - z\rangle\right) - \frac{(1-\beta_n)(1-\alpha_n)}{(1-\rho)\alpha_n}\langle x_n - z_n, x_n - z\rangle \\ & - \frac{(1-\beta_n)}{(1-\rho)\alpha_n}\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\langle (I-T_i)y_n, x_n - z_n\rangle \\ & - \frac{\beta_n}{(1-\rho)\alpha_n}\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\langle (I-T_i)y_n, x_n - Sx_n\rangle. \\ \leq\ & \frac{\|w_{n-1} - w_n\|}{(1-\rho)\alpha_n}\|w_n - P_C[w_n]\| - \frac{1}{(1-\rho)}\langle (I-f)z, x_n - z\rangle \\ & + \frac{(1-\alpha_n)\beta_n}{(1-\rho)\alpha_n}\left(\langle v_n, x_n - z\rangle - \langle (I-S)z, x_n - z\rangle\right) + \frac{1}{(1-\rho)}\frac{(1-\beta_n)}{\beta_n}\frac{\beta_n}{\alpha_n}\|x_n - z_n\|x_n - z\| \\ & + \frac{1}{(1-\rho)}\frac{(1-\beta_n)}{\beta_n}\frac{\beta_n}{\alpha_n}\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\|(I-T_i)y_n\|\|x_n - z_n\| \\ & - \frac{\beta_n}{(1-\rho)\alpha_n}\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\langle (I-T_i)y_n, x_n - Sx_n\rangle.\end{aligned}$$

By the condition (e) of Theorem 3.2, there exists a constant $N > 0$ such that $\frac{1-\beta_n}{\beta_n} \leq N$. Since $\lim_{n\to\infty}\|x_n - z_n\| = 0$, $v_n \to 0$, $(I-T_i)y_n \to 0$ and $\frac{\|w_{n-1}-w_n\|}{\alpha_n} \to 0$ as $n \to \infty$, then every weak cluster point of $\{x_n\}$ is also a strong cluster point. Since $\{x_n\}$ is bounded, by Lemma 3.2 there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converging to a point $x^* \in F(T)$, similar argument as that Theorem 3.1 we can show that $x^* \in \Omega^* \cap S^* \cap F(T)$.

From (3.23)-(3.27), it follows that, for any $z \in \Omega^* \cap S^* \cap F(T)$,

$$\begin{aligned}
\langle (I-f)x_{n_k}, x_{n_k}-z\rangle &= \frac{(1-\alpha_{n_k})\beta_{n_k}}{\alpha_{n_k}}\langle v_{n_k}, x_{n_k}-z\rangle - \frac{1}{\alpha_{n_k}}\langle w_{n_k}-P_C[w_{n_k}], P_C[w_{n_k-1}]-z\rangle \\
&\quad -\frac{(1-\alpha_{n_k})\beta_{n_k}}{\alpha_{n_k}}\langle x_{n_k}-Sx_{n_k}, x_{n_k}-z\rangle - \frac{(1-\alpha_{n_k})(1-\beta_{n_k})}{\alpha_{n_k}}\langle x_{n_k}-z_{n_k}, x_{n_k}-z\rangle \\
&\quad -\frac{1}{\alpha_{n_k}}\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\langle y_{n_k}-T_i y_{n_k}, x_{n_k}-z\rangle \\
&\le \frac{(1-\alpha_{n_k})\beta_{n_k}}{\alpha_{n_k}}\langle v_{n_k}, x_{n_k}-z\rangle + \frac{1}{\alpha_{n_k}}\|w_{n_k}-P_C[w_{n_k}]\|\|w_{n_k-1}-w_{n_k}\| \\
&\quad -\frac{(1-\alpha_{n_k})\beta_{n_k}}{\alpha_{n_k}}\langle x_{n_k}-Sx_{n_k}, x_{n_k}-z\rangle + \frac{(1-\beta_{n_k})}{\beta_{n_k}}\frac{\beta_{n_k}}{\alpha_{n_k}}\|\|x_{n_k}-z_{n_k}\|\|x_{n_k}-z\| \\
&\quad +\frac{(1-\beta_{n_k})}{\beta_{n_k}}\frac{\beta_{n_k}}{\alpha_{n_k}}\sum_{i=1}^{n_k}(\alpha_{i-1}-\alpha_i)\|(I-T_i)y_{n_k}\|\|x_{n_k}-z_{n_k}\| \\
&\quad -\frac{\beta_{n_k}}{\alpha_{n_k}}\sum_{i=1}^{n_k}(\alpha_{i-1}-\alpha_i)\langle (I-T_i)y_{n_k}, x_{n_k}-Sx_{n_k}\rangle. \qquad (3.28)
\end{aligned}$$

Since $\lim_{n\to\infty}\|x_n - z_n\| = 0$, $v_n \to 0$, $(I-T_i)y_n \to 0$ and $\frac{\|w_{n-1}-w_n\|}{\alpha_n} \to 0$, letting $k \to \infty$ in (3.28), we obtain

$$\langle (I-f)x^*, x^*-z\rangle \le -\tau\langle x^*-Sx^*, x^*-z\rangle$$

i.e.,

$$\langle \frac{1}{\tau}(I-f)x^* + (I-S)x^*, z-x^*\rangle \ge 0.$$

In the following, we show that (3.20) has unique solution. Assume that x' is another solution. Then, we have

$$\langle (I-f)x', x'-x^*\rangle \le -\tau\langle x'-Sx', x'-x^*\rangle \qquad (3.29)$$

and

$$\langle (I-f)x^*, x^*-x'\rangle \le -\tau\langle x^*-Sx^*, x^*-x'\rangle. \qquad (3.30)$$

Adding (3.29) and (3.30), we get

$$\begin{aligned}
(1-\rho)\|x'-x^*\|^2 &\le \langle (I-f)x'-(I-f)x^*, x'-x^*\rangle \\
&\le -\tau\langle (I-S)x'-(I-S)x^*, x'-x^*\rangle \\
&\le 0.
\end{aligned}$$

Then $x' = x^*$. Since (3.20) has unique solution, it follows that $w_w(x_n) = \{x^*\}$. Since every weak cluster point of $\{x_n\}$ is also a strong cluster point, we conclude that $\{x_n\} \to x^*$. This completes the proof. □

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