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# A Multi-Choice Programming Approach to Zero-Sum Games with Multiple Goals

Arpita Panda[1], Subhrananda Goswami[2]†and Chandan Bikash Das[3]

[1]Department of Mathematics

Sonakhali Girls's High School, Sonakhali, Daspur, Paschim Midnapore, West Bengal, India

[2]Department of Computer Science

Haldia Institute of Technology, Haldia, Purba Midnapore, West Bengal, India

E-mail: subhrananda_usca@yahoo.co.in

[3]Department of Mathematics,Tamralipta Mahavidyalaya, Tamluk, Purba Midnapore-721636,

West Bengal, India, E-mail: cdas_bikash@yahoo.co.in

**Abstract**

This paper investigate a class of two person zero-sum multiple pay off games in which each component is deterministic. We consider a class of games in which multiple goals or objectives are present. The goal values are assume in ranges. Multi-Choice goal programming methodology is applied here. This model is a linear programming model where constrains are crisp. This model is illustrated through a numerical example.

**Keywords:** Zero-sum Game, Multi-Choice Programming, Goal.

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†Corresponding author

# 1 Introduction

The concept of multiple payoff games was introduced by blackwell [2], and in more depth later by Contini [3]. Contini examines two models of random games with vector payoffs namely the expected value model and the probability maximization model. Since the opponent strategy is assumed known, both models reduce to the usual vector valued linear optimization problem. Zeleny [4] examines vector valued two person zero-sum games and shows that by introducing by parameter vector $\lambda = (\lambda_1, \ldots, \lambda_m)$, the game reduces to a parametric linear programming problem. He then disscussed the concept of Pareto Optimal solutions and ideal points of vecter valued games. Aubin [1] describes a class of hierarchical games in which the objective of decision macker or central unit of an organization is to choose a strategy which minimizes the total squared deviation of the payoffs to subunits from the values of the games for these subunits. Aubin suggests the gradient algorithm as a solution method for such mimimax problems. The methodology Known as goal programming (GP) stems from the work of Charnes and Cooper [9], with further development by Lee [10], Ignizio [8],Tamiz et. al. [14] and Romero [11] among others. GP can also be though often extension of linear programming (LP) to treat multiple, normally conflicting objective problems. It allows decision maker (DMs) to set his/her aspiration levels for each goal. Unwanted deviations from this set of aspiration values are then minimized in an achievement functions. The large number of goal programming applications in many and diverse fields are accounted by Jones and Tamiz [12]. The oldest and still most widely used form of achivement function for GP is represented as follows.

# 2 Mathematical Formulation

## 2.1 Preliminaries

**Zero-Sum Game :**
The zero-sum property (if one gains, another loses) means that any result of a zero-sum situation is Pareto optimal (generally, any game where all strategies are Pareto optimal is called a conflict game). A game in which the gain of one player is a loss of another player is called a zero-sum game.

**Pure Strategy and Mixed Strategy:**
A pure strategy is a decision making rule in which one particular course of action is selected, while a mixed strategy is a decision making rule in which a player decides to choose his course of action with some definite probability distribution.
The mixed strategy of the matrix game (1.1.1) for player PI and PII are defined as follows:

$$Y = \{ \ y \in R^m; \ \sum_{i=1}^{m} y_i = 1; \ y_i \geq 0, \quad i = 1, 2, \ldots, m \ \} \tag{1}$$

$$Z = \{ \ z \in R^n; \ \sum_{j=1}^{n} z_j = 1; \ z_j \geq 0, \quad j = 1, 2, \ldots, n \ \} \tag{2}$$

We remark that the pure strategies for both players are the extreme points of $Y$ and $Z$.

## 2.2 Mathematical Model

The objective of player PI is then to select a strategy $x^*$ such that if PII select $y^*$, the expected payoff form the $k^{th}$ attribute is at least $g^k$. Let $A^k$ denote the matrix whose $(i, j)$ element is $a_{ij}^k$ , this implies that

$$x^* A^k y^* \geq g^k$$

Let $w^k, k = 1, \ldots, K$ denote the relative weights attached to $K$ goals, player PIs problem then becomes that of determining $x^* \in X$, which solves following model **M1**

**M1:**

$$\begin{aligned} \min_{x\in X}\min_{y\in Y}\min_{d^k} \quad &: \quad \sum_{k=1}^{K} w^k d^k \\ \text{subject to} \quad x^* A^k y^* + d^k &\geq g^k \\ d^k &\geq 0, \forall k \end{aligned} \tag{3}$$

The operator $\min_{d^k}$ is required here to ensure that $d^k$ chosen actually measure the under achievement of the $g^k$.

The problem **M1** is equivalent to following **M2** ***(see Appendix 1 )***

**M2:**

$$\min_{x\in X} \quad \beta \tag{4}$$

$$\begin{aligned} \text{subject to} \quad \sum_{i=1}^{m} x_i A_{ij}(r) &\leq 0, \forall j, r \\ x &\in X \end{aligned}$$

where,

$$A(r) = \sum_{k=1}^{K} C^k \alpha_r^k; \tag{5}$$

$$C^k = c_{ij} = g^k - a_{ij}^k;$$

$$A_{ij}(r) = (i, j)^{th}\text{component of } A(r);$$

$$\alpha^k\text{is the set of extreme point of} \psi = \{\alpha = (\alpha^1, \ldots, \alpha^k)\};$$

$$r = 1, \ldots, R\text{is the index set of extreme points of} \psi$$

Letting $a_{ij}^k$ represent the $k^th$ payoff to player PI, if player PI chooses strategy $i$ with probability $x_i, i = 1, \ldots, m$, and player PII chooses strategy $j$ with probability $y_j, j = 1, \ldots, n$, it follows that $\sum_{i=1}^{m} a_{ij}^k$ is the $k^{th}$ payoff of player PI.Then

$$d_j^k = \max\{0, g^k - \sum_{i=1}^{m} a_{ij}^k x_i\}$$

is the under achievement of goal $g^k$ corresponding to this strategy combination. Since $\sum_{i=1}^{m} x_i = 1$ and $g^k$ are constants so we have

$$d_j^k = \max\{0, \sum_{i=1}^{m} (g^k - a_{ij}^k) x_i\}$$

Since each of these $d^k$ values are weighted by corresponding $w^k$ value, player PI select $x \in X$ which will yield

$$\min_x \{\max\{\sum_{k=1}^{K} w^k d_1^k, \ldots, \sum_{k=1}^{K} w^k d_n^k\}\}$$

**M3:**

$$\begin{aligned} & min & & v \\ \text{s.t.} \quad & v & \geq & \sum_{k=1}^{K} w^k d_j^k, \forall j \\ & d_j^k & = & \max\{0, \sum_{i=1}^{m} (g^k - a_{ij}^k) x_i\} \quad \forall j, \forall k \\ & x & \in & X \end{aligned} \tag{6}$$

where $v$ is the value of the game. The model M3 is equivalent to the following model M4***(see Appendix 2 )***

**M4:**

$$\begin{aligned} min \quad & v \\ \text{s.t.} \quad v \geq & \sum_{k=1}^{K} w^k f_j^k, \forall j \\ f_j^k \geq & \sum_{i=1}^{m} (g^k - a_{ij}^k) x_i \quad \forall j, \forall k \\ x \in & X \\ f_j^k \geq & 0 \end{aligned} \tag{7}$$

The model **M4** is the goal programming model for zero-sum games.

**Theorem:** Model M1and Model **M4** are equivalent. (proof: **(see Appendix 3 )** Hence the problem can be expressed as following model **M3** as follows.

# 3 Solution Procedure

In order to solve the model M4 we apply modified multi-choice goal programming method. By a new concept of upper ($g_{max}^k$) and lower ($g_{min}^k$) bounds of the $k^{th}$ aspiration level, $z^k$, is introduced to the multi-choice goal programming-achievement. Where $z^k$ is the continuous variable, $g_{min}^k \leq z^k \leq g_{max}^k$. By the modified multi-choice goal programming-achievement, model **M4** can be reformatted as following two models **M5a** and **M5b**.

**M5a: Case I: for the case of more the better**

$$\begin{aligned} min \quad & v \\ \text{s.t.} \quad v \geq & \sum_{k=1}^{K} \{w^k f_j^k + r_k(e_k^+ + e_k^-)\}, \forall j \\ f_j^k \geq & \sum_{i=1}^{m} (z^k - a_{ij}^k) x_i \quad \forall j, \forall k \\ z^k - e_k^+ + e_k^- = & g_{max}^k, \forall k \\ g_{min}^k \leq & z^k \leq g_{max}^k \\ x \in & X \\ f_j^k \geq & 0 \forall k \end{aligned} \tag{8}$$

where $f_j^k$ is the positive deviation attached to the $j^{th}$ goal in (1); $e_k^+$ and $e_k^-$ positive and negative deviation attached to $|z^k - g_{max}^k|$ in (2);$r_k$ is the weight attached to $|z^k - g_{max}^k|$.

**M5b: Case I: for the case of less the better**

$$\begin{aligned} min \quad & v & (9)\\ \text{s.t.} \quad v \ &\geq \ \sum_{k=1}^{K}\{w^k f_j^k + r_k(h_k^+ + h_k^-)\}, \forall j \\ f_j^k \ &\geq \ \sum_{i=1}^{m}(z^k - a_{ij}^k)x_i \quad \forall j, \forall k \\ z^k - h_k^+ + h_k^- \ &= \ g_{min}^k, \forall k \\ g_{min}^k \ &\leq \ z^k \leq g_{max}^k \\ x \ &\in \ X \\ f_j^k \ &\geq \ 0 \forall k \end{aligned}$$

where $f_j^k$ is the positive deviation attached to the $j^{th}$ goal in (1); $h_k^+$ and $h_k^-$ positive and negative deviation attached to $|z^k - g_{max}^k|$ in (2);$r_k$ is the weight attached to $|z^k - g_{max}^k|$.

# 4 Numerical Example

**Example 1** : Consider the following multi-pay offs game

$$A^1 = \begin{bmatrix} 1 & -6 \\ 2 & 7 \end{bmatrix}, \qquad A^2 = \begin{bmatrix} 10 & 7 \\ 3 & -4 \end{bmatrix}, \qquad A^3 = \begin{bmatrix} 2 & 1 \\ -6 & -1 \end{bmatrix}$$

with goals $g^1 \in (0,1), g^2 \in (2,5), g^3 \in (1,4)$ with $w^1 = 2, w^2 = .5, w^3 = 1, \alpha^1 = .5\alpha^2 = .5, \alpha^3 = 1$

By model **M5a** for **Example 1** by **Lingo package** we get following results.

$v = 4.9, f_1^1 = 0.0, f_1^2 = 2.467, f_1^3 = 5.6, f_2^1 = 1.15, f_2^2 = 1.217, f_2^3 = 2.25, e_1^+ = 0.0, e_1^- = 0.0, e_2^+ = 0.0, e_2^- = 1.73, e_3^+ = 0.0, e_3^- = 0.0, z^1 = 1.0, z^2 = 3.267, z^3 = 4, x_1 = .55, x_2 = .45.$

By model M5b for Example 1 by Lingo package we get following results.
$v = 1.7857, f_1^1 = 0.0, f_1^2 = 1.43, f_1^3 = 2.14, f_2^1 = 0.89, f_2^2 = 0.0, f_2^3 = 0.0, h_1^+ = 0.0, h_1^- = 0.0, h_2^+ = 0.0, h_2^- = 0.0, h_3^+ = 0.0, h_3^- = 0.0, z^1 = 0.0, z^2 = 2.0, z^3 = 1.0, x_1 = .607, x_2 = .393$.

**Example 2** : Consider the following multi-pay offs game

$$A^1 = \begin{bmatrix} 2 & -4 & 3 & 5 \\ 3 & 2 & 7 & 4 \\ 5 & 2 & 6 & 3 \\ 2 & 4 & -3 & 7 \end{bmatrix}, \qquad A^2 = \begin{bmatrix} 10 & 5 & 6 & 7 \\ 3 & -4 & 6 & -2 \\ 2 & 7 & 8 & 3 \\ 5 & 6 & 2 & 4 \end{bmatrix}, \qquad A^3 = \begin{bmatrix} 5 & 6 & 3 & -4 \\ 4 & -2 & 7 & 3 \\ -5 & 2 & -6 & 5 \\ 4 & 3 & 2 & 1 \end{bmatrix}$$

with goals $g^1 \in (4, 8), g^2 \in (6, 15), g^3 \in (5, 14)$ with $w^1 = 2, w^2 = .5, w^3 = 1, \alpha^1 = .5\alpha^2 = .5, \alpha^3 = 1$
By model **M5a** for **Example 2** by **Lingo package** we get following results.

$v = 23.87446, f_1^1 = 0.6103896, f_1^2 = 1.731602, f_1^3 = 14.73160, f_2^1 = 3.424242, f_2^2 = 0.047619, f_2^3 = 10.78788, f_3^1 = 0.7532468, f_3^2 = 0.000000, f_3^3 = 16.17749, f_4^1 = 0.000000, f_4^2 = 2.134199, f_4^3 = 12.06061, e_1^+ = 0.0, e_1^- = 3.727273, e_2^+ = 0.0, e_2^- = 8.653680, e_3^+ = 0.0, e_3^- = 0.0, z^1 = 4.272727, z^2 = 6.346320, z^3 = 14.00000, x_1 = 0.2554113, x_2 = 0.000000, x_3 = 0.5541126, x_4 = 0.1904762$.

By model M5b for **Example 2** by Lingo package we get following results.
$v = 8.968009, f_1^1 = 0.8685364, f_1^2 = 1.717791, f_1^3 = 5.297108, f_2^1 = 3.033304, f_2^2 = 0.3865031, f_2^3 = 2.298861, f_3^1 = 1.151621, f_3^2 = 0.000000, f_3^3 = 6.448729, f_4^1 = 0.000000, f_4^2 = 2.368975, f_4^3 = 2.784400, e_1^+ = 0.4320746, e_1^- = 0.000000, e_2^+ = 0.0, e_2^- = 0.000000, e_3^+ = 0.0, e_3^- = 0.0, z^1 = 4.432075, z^2 = 6.000000, z^3 = 5.000000, x_1 = 0.1831727, x_2 = 0.0701139, x_3 = 0.4978089, x_4 = 0.2489045$.

## 5 Conclusion

In this paper a class of zero-sum games are developed with multiple goals. Then solve the model through modified multi-choice goal programming method. In this modified multi-choice goal program is provided in which does not involve multiplicative terms of binary variables to model the aspiration levels. This leads to it being more easily implemented and easily understood by readers.

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