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ASSOCIATED CURVES ACCORDING TO BISHOP FRAME IN EUCLIDEAN 3-SPACE

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Abstract. In this paper, we study associated curves in the Euclidean 3-space by utilizing the Bishop frame. Moreover, we define new curves called $\mathbf{M}_1-$*Direction Curve*, $\mathbf{M}_2-$*Direction Curve*, $\mathbf{M}_1-$*Donor Curve*, $\mathbf{M}_2$-*Donor Curve*. Finally, we charcterize new associated curves according to Bishop frame.

1. Introduction

The associated curves is called curves that found in a differential and mathematical relationship between two or more curves. Then, associated curves use a lot in the field of differential geometry such that Caustic curves, Inverse curves, Bertrand curves, etc. On these curves have been many studies, for example, Burke examined bertrand curves associated a pair of curves in [1], Niino also examined associated curves of a holomorphic curve in [10], Duquesne explored the infinite family of elliptic curves associated with simplest cubic fields in [4] and in [3], Choi introduce the notion of the principal (binormal)-direction curve and principal (binormal)-donor curve of a Frenet curve in $\mathbb{E}^3$.

Bishop frame, which is also called alternetive or parallel frame of the curves, was introduced by L.R. Bishop in 1975 by means of parallel vector fields. Recently, many research papers related to this concept have been treated in Euclidean space. For example, in [8] the outhors introduced a new version of Bishop frame and an application to spherical images and in [9] the outhors studied Minkowski space in $\mathbb{E}_1^3$.

In this paper, we obtain a new associated curves by using Bishop frame in $\mathbb{E}^{\mathbf{3}}$. Firstly, we summarize properties Bishop frame and Frenet frame which are parameterized by arc-length parameter s and the basic concepts on curves. The Finally, we give Bishop frame, curvature and torsion of associated curves according to Bishop frame in $\mathbb{E}^{\mathbf{3}}$.

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2. Preliminaries

Suppose M is a smooth manifold. Recall that a smooth curve in M is a smooth map $\gamma : I \to M$, where I is an interval in $\mathbb{R}$. For any $a \in I$, the tangent vector of γ at the point $\gamma(a)$ is

$$\gamma'(a) = \frac{d\gamma}{ds}(a) = d\gamma_a\left(\frac{d}{ds}\right),$$

where $\frac{d}{ds}$ is the standard coordinate tangent vector of $\mathbb{R}$.

Definition 2.1. *Let X be a smooth vector field on M. We say that a smooth curve $\gamma : I \to M$ is an integral curve of X if for any $t \in I$,*

$$\gamma'(t) = X_{\gamma(t)}. \tag{2.1}$$

Denote by $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ the moving Frenet–Serret frame along the curve γ in the space $\mathbb{E}^3$. For an arbitrary curve γ with first and second curvature, κ and τ in the space $\mathbb{E}^3$, the following Frenet–Serret formulae is given

$$\begin{aligned} \mathbf{T}' &= \kappa\mathbf{N}, \\ \mathbf{N}' &= -\kappa\mathbf{T} + \tau\mathbf{B}, \\ \mathbf{B}' &= -\tau\mathbf{N}, \end{aligned} \tag{2.2}$$

where curvature functions are defined by $\kappa = \kappa(s) = \|\mathbf{T}'(s)\|$ and $\tau(s) = -\langle \mathbf{B}', \mathbf{N}\rangle$. In the rest of the paper, we suppose everywhere $\kappa \neq 0$ and $\tau \neq 0$.

The Bishop frame or parallel transport frame is an alternative approach to defining a moving frame that is well defined even when the curve has vanishing second derivative. One can express parallel transport of an orthonormal frame along a curve simply by parallel transporting each component of the frame. The tangent vector and any convenient arbitrary basis for the remainder of the frame are used. The Bishop frame is expressed as

$$\begin{aligned} \mathbf{T}' &= k_1\mathbf{M}_1 + k_2\mathbf{M}_2, \\ \mathbf{M}_1' &= -k_1\mathbf{T}, \\ \mathbf{M}_2' &= -k_2\mathbf{T}, \end{aligned} \tag{2.3}$$

where we shall call the set $\{\mathbf{T}, \mathbf{M_1}, \mathbf{M_2}\}$ as Bishop trihedra and k_1 and k_2 as Bishop curvatures.

Now, we define some associated curves of a curve γ in $\mathbb{E}^3$ defined on an open interval I. For a Frenet curve $\gamma : I \to M$, consider a vector field V given by

$$V(s) = u(s)\mathbf{T}(s) + v(s)\mathbf{N}(s) + w(s)\mathbf{B}(s), \tag{2.4}$$

where u, v and w are functions on I satisfying $u^2(s) + v^2(s) + w^2(s) = 1$. Then, an integral curve $\bar{\gamma}(s)$ of V defined on I is a unit speed curve in $\mathbb{E}^3$, [3].

Remark 2.2. *A principal-direction (resp. the binormal) curve is an integral curve of $V(s)$ with $u(s) = w(s) = 0,\ v(s) = 1$ (resp. $u(s) = v(s) = 0,\ w(s) = 1$) for all s in (2.6),* [3].

Proposition 2.3. *Let γ be a Frenet curve in $\mathbb{E}^3$ and $\bar{\gamma}$ an integral curve of (2.4). Then, the principal-direction curve of $\bar{\gamma}$ equals to γ up to translation if and only if*

$$u(s) = 0,\ v(s) = -\cos(\int \tau(s)\,ds) \neq 0,\ w(s) = \sin(\int \tau(s)\,ds),$$

[3].

In rest of the article, we may assume that $\bar{s} = s$ without loss of generality.

3. $\mathbf{M}_1-$Direction Curve and $\mathbf{M}_1-$Donor Curve in $\mathbb{E}^3$

Definition 3.1. *Let β be a Frenet curve in $\mathbb{E}^3$. An integral curve of $\mathbf{M}_1$ is called $\mathbf{M}_1-$Direction Curve of β according to Bishop frame. Then,* a $\mathbf{M}_1-$*Direction Curve is an integral curve of*

$$V(s) = u(s)\,\mathbf{T}(s) + v(s)\,\mathbf{M}_1(s) + w(s)\,\mathbf{M}_2(s) \tag{3.1}$$

with $u(s) = w(s) = 0,\ v(s) = 1.$

Definition 3.2. *Let β be a Frenet curve in $\mathbb{E}^3$. An integral curve of $\mathbf{M}_2$ is called $\mathbf{M}_2-$Direction Curve of β according to Bishop frame. Then, a $\mathbf{M}_2-$Direction Curve is an integral curve of (3.1) with $u(s) = v(s) = 0,\ w(s) = 1.$*

Theorem 3.3. *Let β be a Frenet curve in $\mathbb{E}^3$ and $\bar{\beta}$ be an integral curve of (3.1). Then, the $\mathbf{M}_1-$Direction curve of $\bar{\beta}$ equals to β up to translation if and only if*

$$u(s) = 0,\ v(s) = -1,\ w(s) = 0. \tag{3.2}$$

Definition 3.4. *An integral curve of $-\mathbf{M}_1$ is called a $\mathbf{M}_1-$Donor Curve of β according to Bishop frame.*

Theorem 3.5. *Let β be a Frenet curve in $\mathbb{E}^3$ with the curvature κ and the torsion τ and $\bar{\beta}$ be the $\mathbf{M}_1-$Direction Curve of β with the curvature $\bar{\kappa}$ and the torsion $\bar{\tau}$. Then, the Frenet frame of $\bar{\beta}$ are*

$$\bar{\mathbf{T}}(s) = \mathbf{M}_1(s),\ \bar{\mathbf{N}}(s) = -\mathbf{T}(s),\ \bar{\mathbf{B}}(s) = \mathbf{M}_2(s)$$

and curvature $\bar{\kappa}$ and torsion $\bar{\tau}$ of $\bar{\beta}$ are given by

$$\bar{\kappa}(s) = k_1(s) \text{ and } \bar{\tau}(s) = -k_2(s).$$

Proof. From definition 3.1, we write

$$\bar{\beta}' = \bar{\mathbf{T}} = \mathbf{M}_1. \tag{3.3}$$

If we take the norm of the derivative of (3.3), then we get, for $k_1 > 0$,

$$\bar{\kappa}(s) = k_1(s). \tag{3.4}$$

On the other hand, we write by

$$\bar{\mathbf{N}}(s) = -\mathbf{T}(s),\ \bar{\mathbf{B}}(s) = \mathbf{M}_2(s).$$

Since $\bar{\mathbf{B}} = \mathbf{M}_2,\ \bar{\mathbf{B}}' = -k_2$ and

$$\bar{\tau} = -\left\langle \bar{\mathbf{B}}', \bar{\mathbf{N}} \right\rangle = -k_2. \tag{3.5}$$

Corollary 3.6. *Let β be a Frenet curve in $\mathbb{E}^3$ with the curvature κ and the torsion τ and $\bar{\beta}$ be the $\mathbf{M}_1-$Direction Curve of β with the curvature $\bar{\kappa}$ and the torsion $\bar{\tau}$. Then, the Bishop frame of $\bar{\beta}$ are given by*

$$\begin{aligned}
\bar{\mathbf{T}}(s) &= \mathbf{M}_1(s), \\
\bar{\mathbf{M}}_1(s) &= -\cos(\int k_2(s)ds)\mathbf{T}(s) + \sin(\int k_2(s)ds)\mathbf{M}_2(s), \\
\bar{\mathbf{M}}_2(s) &= \sin(\int k_2(s)ds)\mathbf{T}(s) + \cos(\int k_2(s)ds)\mathbf{M}_2(s).
\end{aligned}$$

Theorem 3.7. *If a curve β in $\mathbb{E}^3$ is a $\mathbf{M}_1-$Donor Curve of a curve $\bar{\beta}$ with the curvature $\bar{\kappa}$ and the torsion $\bar{\tau}$, then the curvature κ and the torsion τ of the curve β are given by*

$$\kappa(s) = \sqrt{\bar{\kappa}^2(s) + \bar{\tau}^2(s)} \text{ and } \tau(s) = -\frac{\bar{\kappa}^2(s)}{\bar{\kappa}^2(s) + \bar{\tau}^2(s)} \left(\frac{\bar{\tau}(s)}{\bar{\kappa}(s)}\right)'.$$

Proof. If we take the squares of (3.4), (3.5) and sum side by side this equalities, then we get

$$\kappa(s) = \sqrt{\bar{\kappa}^2(s) + \bar{\tau}^2(s)}.$$

Proof of $\tau(s)$ is clear from (3.5).

Corollary 3.8. *Let β be a Frenet curve in $\mathbb{E}^3$ with the curvature κ and the torsion τ and $\bar{\beta}$ be the $\mathbf{M}_1-$Direction Curve of β with the curvature $\bar{\kappa}$ and the torsion $\bar{\tau}$. Then, it satisfies*

$$\begin{aligned}
\frac{\bar{\tau}(s)}{\bar{\kappa}(s)} &= -\tan\theta(s), \\
\frac{\tau(s)}{\kappa(s)} &= -\frac{\bar{\kappa}^2(s)}{(\bar{\kappa}^2(s) + \bar{\tau}^2(s))^{\frac{3}{2}}} \left(\frac{\bar{\tau}(s)}{\bar{\kappa}(s)}\right)'.
\end{aligned}$$

Theorem 3.9. *Let β be a Frenet curve in $\mathbb{E}^3$ and $\bar{\beta}$ be an integral curve of (3.1). Then, the $\mathbf{M}_2-$Direction curve of $\bar{\beta}$ equals to β up to translation if and only if*

$$u(s) = a, \ v(s) = -\sqrt{1-a^2}\sin\theta(s), \ w(s) = \sqrt{1-a^2}\cos\theta(s).$$

Definition 3.10. *An integral curve of*

$$a\mathbf{T}(s) - \sqrt{1-a^2}\sin\theta(s)\mathbf{M}_1(s) + \sqrt{1-a^2}\cos\theta(s)\mathbf{M}_2(s)$$

is called a $\mathbf{M}_2-$Donor Curve of β according to Bishop frame.

Theorem 3.11. *Let β be a Frenet curve in $\mathbb{E}^3$ with the curvature κ and the torsion τ and $\bar{\beta}$ be the $\mathbf{M}_2-$Direction Curve of β with the curvature $\bar{\kappa}$ and the torsion $\bar{\tau}$. Then, curvature $\bar{\kappa}$ and torsion $\bar{\tau}$ of $\bar{\beta}$ are given by*

$$\bar{\kappa}(s) = k_2(s) \text{ and } \bar{\tau}(\mathbf{s}) = k_1(s).$$

Corollary 3.12. *Let β be a Frenet curve in $\mathbb{E}^3$ with the curvature κ and the torsion τ and $\bar{\beta}$ be the $\mathbf{M}_2-$Direction Curve of β with the curvature $\bar{\kappa}$ and the torsion $\bar{\tau}$. Then, the Bishop frame of $\bar{\beta}$ are given by*

$$\begin{aligned}
\bar{\mathbf{T}}(s) &= \mathbf{M}_2(s),\\
\bar{\mathbf{M}}_1(s) &= -\cos(\int k_1(s)ds)\mathbf{T}(s)+\sin(\int k_1(s)ds)\mathbf{M}_1(s),\\
\bar{\mathbf{M}}_2(s) &= -\sin(\int k_1(s)ds)\mathbf{T}(s)-\cos(\int k_1(s)ds)\mathbf{M}_1(s).
\end{aligned}$$

Corollary 3.13. *If a curve β in $\mathbb{E}^3$ is a $\mathbf{M}_2-$Donor Curve of a curve $\bar{\beta}$ with the curvature $\bar{\kappa}$ and the torsion $\bar{\tau}$, then the curvature κ and the torsion τ of the curve β are given by*

$$\kappa(s) = \sqrt{\bar{\kappa}^2(s) + \bar{\tau}^2(s)} \text{ and } \tau(s) = \frac{\bar{\tau}^2(s)}{\bar{\kappa}^2(s) + \bar{\tau}^2(s)} \left(\frac{\bar{\kappa}(s)}{\bar{\tau}(s)}\right)'.$$

Corollary 3.14. *Let β be a Frenet curve in $\mathbb{E}^3$ with the curvature κ and the torsion τ and $\bar{\beta}$ be the $\mathbf{M}_1-$Direction Curve of β with the curvature $\bar{\kappa}$ and the torsion $\bar{\tau}$. Then, it satisfies*

$$\begin{aligned}
\frac{\bar{\tau}(s)}{\bar{\kappa}(s)} &= \cot\theta(s),\\
\frac{\tau(s)}{\kappa(s)} &= \frac{\bar{\tau}^2(s)}{(\bar{\kappa}^2(s) + \bar{\tau}^2(s))^{\frac{3}{2}}} \left(\frac{\bar{\kappa}(s)}{\bar{\tau}(s)}\right)'.
\end{aligned}$$

References

[1] J. F. Burke, *Bertrand Curves Associated with a Pair of Curves,* Mathematics Magazine, 1 (34) (1960), 60-62.

[2] G. Canuto, *Associated Curves and Plücker Formulas in Grassmannians,* Inventiones Mathematicae, 53 (1979), 77-90.

[3] J. H. Choi, Y. H. Kim, *Associated Curves of a Frenet Curve and Their Applications,* Applied Mathematics and Computation, 18 (208) (2012), 9116–9124.

[4] S. Duquesne, *Elliptic Curves Associated with Simplest Quartic Fields,* Journal de Th´eorie des Nombres de Bordeaux, 19 (2010), 81–100.

[5] T. Körpınar, V. Asil, M.T. Sarıaydın and M. İncesu, *A Characterization for Bishop Equations of Parallel Curves According to Bishop Frame in E^3*, Bol. Soc. Paran. Mat, (in press).

[6] T. Körpınar, V. Asil, S. Baş, *Characterizing Inextensible Flows of Timelike Curves According to Bishop Frame in Minkowski Space*, Journal of Vectorial Relativity, 4 (5) (2010), 18-25.

[7] T. Körpınar, M. T. Sarıaydın, V. Asil, *On Characterization of Parallel Curves according to Bishop Frame in E^3*, (submitted).

[8] S. Yılmaz, *Poition Vectors of Some Special Space-like Curves According to Bishop Frame in Minkowski Space E_1^3*, Sci Magna, 5 (1) (2010), 48-50.

[9] S. Yılmaz, M. Turgut, *A New Version of Bishop Frame and An Aplication to Spherical Images*, J. Math. Anal. Appl., 371 (2010), 764-776.

[10] K. Niino, *Deficiencies of the Associated Curves of a Holomorphic Curve in the Projective Space,* Proceedings of the American Mathematical Society, 1 (59) (1976), 81-88.

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