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FRENET FRAME OF INVOLUTE CURVES OF BIHARMONIC CURVES IN THE HEISENBERG GROUP

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ABSTRACT. In this paper, we study involute curves of biharmonic curves in the Heisenberg group Heis^3. Finally, we find Frenet frame of invulute curves of biharmonic curves in the Heisenberg group Heis^3.

1. INTRODUCTION

Heisenberg group Heis^3 can be seen as the space $\mathbb{R}^3$ endowed with the following multipilcation:

$$(\overline{x},\overline{y},\overline{z})(x,y,z)=(\overline{x}+x,\overline{y}+y,\overline{z}+z-\frac{1}{2}\overline{x}y+\frac{1}{2}x\overline{y})$$

Heis^3 is a three-dimensional, connected, simply connected and 2-step nilpotent Lie group.

The Riemannian metric g is given by

$$g=dx^2+dy^2+(dz-xdy)^2.$$

The Lie algebra of Heis^3 has an orthonormal basis

$$\mathbf{e}_1=\frac{\partial}{\partial x},\quad \mathbf{e}_2=\frac{\partial}{\partial y}+x\frac{\partial}{\partial z},\quad \mathbf{e}_3=\frac{\partial}{\partial z}, \tag{1.1}$$

for which we have the Lie products

$$[\mathbf{e}_1,\mathbf{e}_2]=\mathbf{e}_3,\quad [\mathbf{e}_2,\mathbf{e}_3]=[\mathbf{e}_3,\mathbf{e}_1]=0$$

with

$$g(\mathbf{e}_1,\mathbf{e}_1)=g(\mathbf{e}_2,\mathbf{e}_2)=g(\mathbf{e}_3,\mathbf{e}_3)=1.$$

Let $\gamma:I\longrightarrow Heis^3$ be a non geodesic curve on the Heisenberg group Heis^3 parametrized by arc length. Let $\{\mathbf{T},\mathbf{N},\mathbf{B}\}$ be the Frenet frame fields tangent to the Heisenberg group Heis^3 along γ defined as follows:

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$\mathbf{T}$ is the unit vector field γ' tangent to γ, $\mathbf{N}$ is the unit vector field in the direction of $\nabla_{\mathbf{T}}\mathbf{T}$ (normal to γ), and $\mathbf{B}$ is chosen so that $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ is a positively oriented orthonormal basis. Then, we have the following Frenet formulas:

$$\begin{aligned} \nabla_{\mathbf{T}}\mathbf{T} &= \kappa\mathbf{N}, \\ \nabla_{\mathbf{T}}\mathbf{N} &= -\kappa\mathbf{T} + \tau\mathbf{B}, \\ \nabla_{\mathbf{T}}\mathbf{B} &= -\tau\mathbf{N}, \end{aligned}$$

where κ is the curvature of γ and τ is its torsion and

$$\begin{aligned} &g(\mathbf{T}, \mathbf{T}) = 1, \ g(\mathbf{N}, \mathbf{N}) = 1, \ g(\mathbf{B}, \mathbf{B}) = 1, \\ &g(\mathbf{T}, \mathbf{N}) = g(\mathbf{T}, \mathbf{B}) = g(\mathbf{N}, \mathbf{B}) = 0. \end{aligned}$$

Theorem 1.1. *Let $\gamma : I \longrightarrow Heis^3$ be a unit speed biharmonic curve with non-zero natural curvatures. Then, the parametric equations of γ are*

$$\begin{aligned} x(s) &= \cos\mathcal{C}s + \mathcal{B}_3, \\ y(s) &= \frac{1}{\mathcal{B}_1}\sin\mathcal{C}\sin[\mathcal{B}_1 s + \mathcal{B}_2] + \mathcal{B}_4, \\ z(s) &= \frac{1}{\mathcal{B}_1^2}\sin\mathcal{C}\cos\mathcal{C}\cos[\mathcal{B}_1 s + \mathcal{B}_2] + \frac{1}{\mathcal{B}_1}\sin\mathcal{C}\cos\mathcal{C}\sin[\mathcal{B}_1 s + \mathcal{B}_2] \\ &\quad + \frac{\mathcal{B}_3}{\mathcal{B}_1}\sin\mathcal{C}\sin[\mathcal{B}_1 s + \mathcal{B}_2] - \frac{1}{\mathcal{B}_1}\sin\mathcal{C}\cos[\mathcal{B}_1 s + \mathcal{B}_2] + \mathcal{B}_5, \end{aligned}$$

where $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \mathcal{B}_4, \mathcal{B}_5$ are constants of integration.

2. Involute Curves of Biharmonic Curves in the Lorentzian Heisenberg Group Heis^3

Definition 2.1. *Let unit speed curve $\gamma : I \longrightarrow Heis^3$ and the curve $\mathsf{C} : I \longrightarrow Heis^3$ be given. For $\forall s \in I$, then the curve C is called the involute of the curve γ, if the tangent at the point $\gamma(s)$ to the curve γ passes through the tangent at the point $\mathsf{C}(s)$ to the curve C and*

$$g(\mathbf{T}^*(s), \mathbf{T}(s)) = 0.$$

Let the Frenet-Serret frames of the curves γ and C be $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ and $\{\mathbf{T}^*, \mathbf{N}^*, \mathbf{B}^*\}$, respectively.

Theorem 2.2. *Let $\gamma : I \longrightarrow Heis^3$ be a unit speed biharmonic curve and C its involute curve on $Heis^3$. Then, the parametric equations of C are*

$$\begin{aligned} \mathsf{C}(s) &= [\Game\cos\mathcal{C} + \mathcal{B}_3]\mathbf{e}_1 + [(\Game - s)\sin\mathcal{C}\cos[\mathcal{B}_1 s + \mathcal{B}_2] + \frac{1}{\mathcal{B}_1}\sin\mathcal{C}\sin[\mathcal{B}_1 s + \mathcal{B}_2] + \mathcal{B}_4]\mathbf{e}_2 \\ &\quad + [\frac{1}{\mathcal{B}_1^2}\sin\mathcal{C}\cos\mathcal{C}\cos[\mathcal{B}_1 s + \mathcal{B}_2] + \frac{1}{\mathcal{B}_1}\sin\mathcal{C}\cos\mathcal{C}\sin[\mathcal{B}_1 s + \mathcal{B}_2] \\ &\quad + \frac{\mathcal{B}_3}{\mathcal{B}_1}\sin\mathcal{C}\sin[\mathcal{B}_1 s + \mathcal{B}_2] - \frac{1}{\mathcal{B}_1}\sin\mathcal{C}\cos[\mathcal{B}_1 s + \mathcal{B}_2] + \mathcal{B}_5 \\ &\quad - [\cos\mathcal{C}s + \mathcal{B}_3][\frac{1}{\mathcal{B}_1}\sin\mathcal{C}\sin[\mathcal{B}_1 s + \mathcal{B}_2] + \mathcal{B}_4] + (\Game - s)\sin\mathcal{C}\sin[\mathcal{B}_1 s + \mathcal{B}_2]]\mathbf{e}_3, \end{aligned} \tag{2.1}$$

where $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \mathcal{B}_4, \mathcal{B}_5, \eth$ *are constants of integration.*

Proof. The involute curve of γ curve may be given as

$$\mathsf{C}(s) = \gamma(s) + (\eth - s)\,\mathbf{T}(s), \tag{2.2}$$

where $\eth$ is constant of integration.

From Theorem 1.1, we get

$$\mathbf{T} = \cos\mathcal{C}\mathbf{e}_1 + \sin\mathcal{C}\cos[\mathcal{B}_1 s + \mathcal{B}_2]\,\mathbf{e}_2 + \sin\mathcal{C}\sin[\mathcal{B}_1 s + \mathcal{B}_2]\,\mathbf{e}_3.$$

Again by using Theorem 1.1, and (2.2) we get (2.1). Hence the proof is completed.

Theorem 2.3. *Let* $\gamma : I \longrightarrow Heis^3$ *be a unit speed biharmonic curve and* C *its involute curve on* $Heis^3$*. Then, the parametric equations of* C *are*

$$\begin{aligned}
x_{\mathsf{C}}(s) &= [\eth\cos\mathcal{C} + \mathcal{B}_3] \\
y_{\mathsf{C}}(s) &= [(\eth - s)\sin\mathcal{C}\cos[\mathcal{B}_1 s + \mathcal{B}_2] + \frac{1}{\mathcal{B}_1}\sin\mathcal{C}\sin[\mathcal{B}_1 s + \mathcal{B}_2] + \mathcal{B}_4], \\
z_{\mathsf{C}}(s) &= [\eth\cos\mathcal{C} + \mathcal{B}_3][(\eth - s)\sin\mathcal{C}\cos[\mathcal{B}_1 s + \mathcal{B}_2] + \frac{1}{\mathcal{B}_1}\sin\mathcal{C}\sin[\mathcal{B}_1 s + \mathcal{B}_2] + \mathcal{B}_4] \\
&\quad + [\frac{1}{\mathcal{B}_1^2}\sin\mathcal{C}\cos\mathcal{C}\cos[\mathcal{B}_1 s + \mathcal{B}_2] + \frac{1}{\mathcal{B}_1}\sin\mathcal{C}\cos\mathcal{C}\sin[\mathcal{B}_1 s + \mathcal{B}_2] \\
&\quad + \frac{\mathcal{B}_3}{\mathcal{B}_1}\sin\mathcal{C}\sin[\mathcal{B}_1 s + \mathcal{B}_2] - \frac{1}{\mathcal{B}_1}\sin\mathcal{C}\cos[\mathcal{B}_1 s + \mathcal{B}_2] + \mathcal{B}_5 \\
&\quad - [\cos\mathcal{C}s + \mathcal{B}_3][\frac{1}{\mathcal{B}_1}\sin\mathcal{C}\sin[\mathcal{B}_1 s + \mathcal{B}_2] + \mathcal{B}_4] + (\eth - s)\sin\mathcal{C}\sin[\mathcal{B}_1 s + \mathcal{B}_2]],
\end{aligned}$$

where $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \mathcal{B}_4, \mathcal{B}_5$ *are constants of integration.*

Proof. It is obvious from Theorem 2.2.

Theorem 2.4. *Let* $\gamma : I \longrightarrow Heis^3$ *be a unit speed biharmonic curve and* C *its involute curve on* $Heis^3$*. Then, Frenet frame of* C *are*

$$\begin{aligned}
\mathbf{T}^* &= \frac{1}{\kappa}\sin^2\mathcal{C}\cos[\mathcal{B}_1 s + \mathcal{B}_2]\sin[\mathcal{B}_1 s + \mathcal{B}_2]\,\mathbf{e}_1 - \frac{1}{\kappa}\sin\mathcal{C}\sin[\mathcal{B}_1 s + \mathcal{B}_2]\,(\mathcal{B}_1 + \cos\mathcal{C})\mathbf{e}_2 \\
&\quad + \frac{1}{\kappa}\mathcal{B}_1\sin\mathcal{C}\cos[\mathcal{B}_1 s + \mathcal{B}_2]\,\mathbf{e}_3,
\end{aligned}$$

$$\begin{aligned}
\mathbf{N}^* &= [-\wp\kappa\cos\mathcal{C} + \frac{\wp\tau}{\kappa}[\mathcal{B}_1\sin^2\mathcal{C}\cos^2[\mathcal{B}_1 s + \mathcal{B}_2] + \sin^2\mathcal{C}\sin^2[\mathcal{B}_1 s + \mathcal{B}_2]\,(\mathcal{B}_1 + \cos\mathcal{C})]]\mathbf{e}_1 \\
&\quad + [-\wp\kappa\sin\mathcal{C}\cos[\mathcal{B}_1 s + \mathcal{B}_2] - \frac{\wp\tau}{\kappa}[\mathcal{B}_1\cos\mathcal{C}\sin\mathcal{C}\cos[\mathcal{B}_1 s + \mathcal{B}_2] \\
&\quad - \sin^3\mathcal{C}\cos[\mathcal{B}_1 s + \mathcal{B}_2]\sin^2[\mathcal{B}_1 s + \mathcal{B}_2]]]\mathbf{e}_2 \\
&\quad + [-\wp\kappa\sin\mathcal{C}\sin[\mathcal{B}_1 s + \mathcal{B}_2] - \frac{\wp\tau}{\kappa}[\cos\mathcal{C}\sin\mathcal{C}\sin[\mathcal{B}_1 s + \mathcal{B}_2]\,(\mathcal{B}_1 + \cos\mathcal{C}) \\
&\quad + \sin^3\mathcal{C}\cos^2[\mathcal{B}_1 s + \mathcal{B}_2]\sin[\mathcal{B}_1 s + \mathcal{B}_2]]]\mathbf{e}_3,
\end{aligned}$$

$$\begin{aligned}\mathbf{B}^* &= [\wp\tau\cos\mathcal{C} + \wp[\mathcal{B}_1\sin^2\mathcal{C}\cos^2[\mathcal{B}_1 s + \mathcal{B}_2] + \sin^2\mathcal{C}\sin^2[\mathcal{B}_1 s + \mathcal{B}_2](\mathcal{B}_1 + \cos\mathcal{C})]]\mathbf{e}_1 \\ &+ [\wp\tau\sin\mathcal{C}\cos[\mathcal{B}_1 s + \mathcal{B}_2] - \wp[\mathcal{B}_1\cos\mathcal{C}\sin\mathcal{C}\cos[\mathcal{B}_1 s + \mathcal{B}_2] \\ &- \sin^3\mathcal{C}\cos[\mathcal{B}_1 s + \mathcal{B}_2]\sin^2[\mathcal{B}_1 s + \mathcal{B}_2]]]\mathbf{e}_2 \\ &+ [\wp\tau\sin\mathcal{C}\sin[\mathcal{B}_1 s + \mathcal{B}_2] - \wp[\cos\mathcal{C}\sin\mathcal{C}\sin[\mathcal{B}_1 s + \mathcal{B}_2](\mathcal{B}_1 + \cos\mathcal{C}) \\ &+ \sin^3\mathcal{C}\cos^2[\mathcal{B}_1 s + \mathcal{B}_2]\sin[\mathcal{B}_1 s + \mathcal{B}_2]]]\mathbf{e}_3,\end{aligned}$$

where $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \mathcal{B}_4, \mathcal{B}_5$ *are constants of integration and*

$$\wp = \frac{1}{\sqrt{\kappa^2 + \tau^2}}.$$

Proof. Assume that γ be a unit speed spacelike biharmonic curve and C its involute curve on Heis^3.Then,

$$\mathsf{C}'(s) = (\eth - s)\,\kappa(s)\,\mathbf{N}(s).$$

Also, we have

$$\mathbf{T}^* = \mathbf{N} \text{ and } \mathbf{T}^* = -\mathbf{N}.$$

Now, we suppose that

$$\mathbf{T}^* = \mathbf{N}. \tag{2.3}$$

Using Theorem 2.2 and (2.3) we get

$$\begin{aligned}\mathbf{T}^* &= \frac{1}{\kappa}\sin^2\mathcal{C}\cos[\mathcal{B}_1 s + \mathcal{B}_2]\sin[\mathcal{B}_1 s + \mathcal{B}_2]\,\mathbf{e}_1 \\ &- \frac{1}{\kappa}\sin\mathcal{C}\sin[\mathcal{B}_1 s + \mathcal{B}_2](\mathcal{B}_1 + \cos\mathcal{C})\mathbf{e}_2 \\ &+ \frac{1}{\kappa}\mathcal{B}_1\sin\mathcal{C}\cos[\mathcal{B}_1 s + \mathcal{B}_2]\,\mathbf{e}_3.\end{aligned} \tag{2.4}$$

On the other hand, using (2.4) we obtain

$$\begin{aligned}\mathbf{N}^* &= [-\wp\kappa\cos\mathcal{C} + \frac{\wp\tau}{\kappa}[\mathcal{B}_1\sin^2\mathcal{C}\cos^2[\mathcal{B}_1 s + \mathcal{B}_2] \\ &+ \sin^2\mathcal{C}\sin^2[\mathcal{B}_1 s + \mathcal{B}_2](\mathcal{B}_1 + \cos\mathcal{C})]]\mathbf{e}_1 \\ &+ [-\wp\kappa\sin\mathcal{C}\cos[\mathcal{B}_1 s + \mathcal{B}_2] - \frac{\wp\tau}{\kappa}[\mathcal{B}_1\cos\mathcal{C}\sin\mathcal{C}\cos[\mathcal{B}_1 s + \mathcal{B}_2] \\ &- \sin^3\mathcal{C}\cos[\mathcal{B}_1 s + \mathcal{B}_2]\sin^2[\mathcal{B}_1 s + \mathcal{B}_2]]]\mathbf{e}_2 \\ &+ [-\wp\kappa\sin\mathcal{C}\sin[\mathcal{B}_1 s + \mathcal{B}_2] - \frac{\wp\tau}{\kappa}[\cos\mathcal{C}\sin\mathcal{C}\sin[\mathcal{B}_1 s + \mathcal{B}_2](\mathcal{B}_1 + \cos\mathcal{C}) \\ &+ \sin^3\mathcal{C}\cos^2[\mathcal{B}_1 s + \mathcal{B}_2]\sin[\mathcal{B}_1 s + \mathcal{B}_2]]]\mathbf{e}_3.\end{aligned}$$

Also,

$$\begin{aligned}\mathbf{B}^* &= [\wp\tau\cos\mathcal{C} + \wp[\mathcal{B}_1\sin^2\mathcal{C}\cos^2[\mathcal{B}_1 s + \mathcal{B}_2] \\ &+ \sin^2\mathcal{C}\sin^2[\mathcal{B}_1 s + \mathcal{B}_2](\mathcal{B}_1 + \cos\mathcal{C})]]\mathbf{e}_1 \\ &+ [\wp\tau\sin\mathcal{C}\cos[\mathcal{B}_1 s + \mathcal{B}_2] - \wp[\mathcal{B}_1\cos\mathcal{C}\sin\mathcal{C}\cos[\mathcal{B}_1 s + \mathcal{B}_2] \\ &- \sin^3\mathcal{C}\cos[\mathcal{B}_1 s + \mathcal{B}_2]\sin^2[\mathcal{B}_1 s + \mathcal{B}_2]]]\mathbf{e}_2 \\ &+ [\wp\tau\sin\mathcal{C}\sin[\mathcal{B}_1 s + \mathcal{B}_2] - \wp[\cos\mathcal{C}\sin\mathcal{C}\sin[\mathcal{B}_1 s + \mathcal{B}_2](\mathcal{B}_1 + \cos\mathcal{C}) \\ &+ \sin^3\mathcal{C}\cos^2[\mathcal{B}_1 s + \mathcal{B}_2]\sin[\mathcal{B}_1 s + \mathcal{B}_2]]]\mathbf{e}_3.\end{aligned}$$

Thus, we have theorem and the proof is finished.

3. Some pictures

In this section we draw some pictures about γ and C:

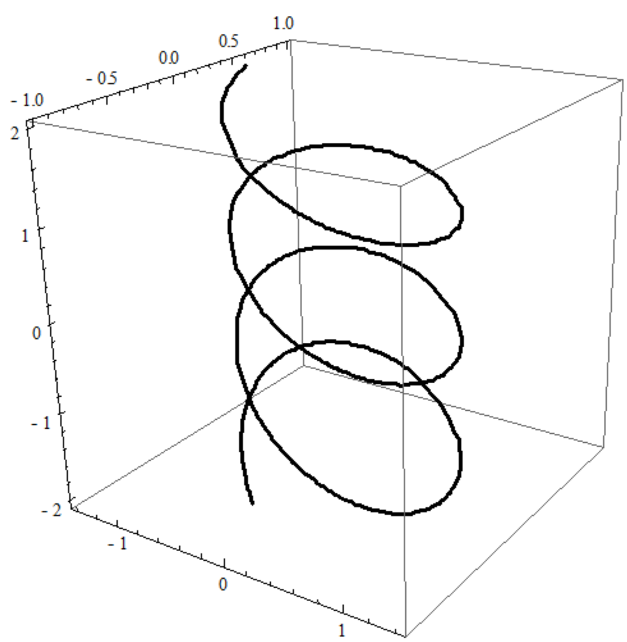

Fig.1

Fig.1: *A unit speed biharmonic curve.*

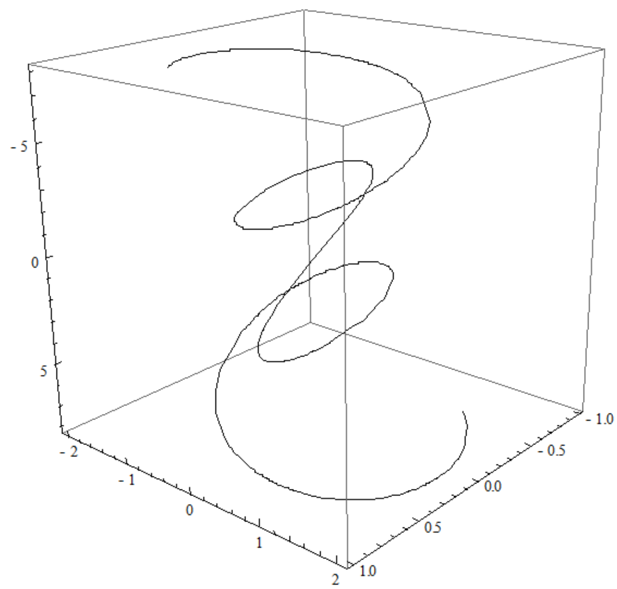

Fig.2

Fig.2: Involute curve of *a unit speed biharmonic curve.*

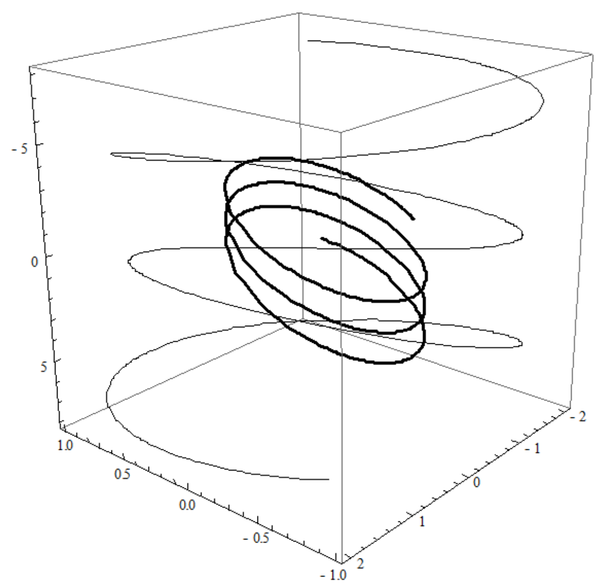

Fig.3

Fig.3: Using Mathematica both involute curve and its mate.

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