AMO - Advanced Modeling and Optimization, Volume 15, Number 3, 2013

SOR-Like Methods for Non-Hermitian Positive Definite Linear Complementarity Problems

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Abstract. Many problems in various scientific computing, operations research, management science and engineering areas can lead to the solution of a linear complementarity problem (*LCP*). This paper provides the efficient iterative algorithm for the large sparse Non-Hermitian positive definite systems of *LCP*, based on the splitting of the coefficient matrix and fixed-point principle. Also, the global convergence properties of the proposed method have been analyzed. Numerical results show the applicability of our method.

Keywords: Linear complementarity problem, SOR method, Non-Hermitian positive definite matrices.

1 Introduction

For a given real vector $q \in R^n$ and a given matrix $M \in R^{n\times n}$ the linear complementarity problem Abbreviated as *LCP (M, q)*, consists in finding vectors $z \in R^n$ such that,

$$\begin{cases} w = Mz + q, \\ z \geq 0, w \geq 0 , \\ z^T w = 0. \end{cases} \tag{1.1}$$

Where, z^T denotes the transpose of the vector z.

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Many problems in various scientific computing and engineering areas can lead to the solution of an *LCP* of the form (1.1). For more details see [Murty, 1988; Cottle, Pang and Stone,1992; Billups and Murty, 2000; Bazaraa, Sherali and Shetty , 2006].

The early motivation for studying the *LCP* was because the *KKT* optimality conditions for linear and quadratic programs constitute an *LCP* of the form (1.1). However, the study of *LCP* really came into prominence since the 1960s and in several decades, many methods for solving the *LCP (M, q)* have been introduced.

Most of these methods originate from those for the system of the linear equations where may be classified into two principal kinds. The book by [Cottle, Pang and Stone, 1992] is a good reference for direct methods developed to solve *LCP*. Another important class of methods used to tackle *LCP* is the iterative methods. Much attention has recently been paid on a class of iterative methods called the matrix-splitting method, see [Cryer, 1971; Mangasarian,1977;1984; Pang, 1982; Machida, Fukushima and Ibaraki,1995; Yuan and Song, 2003; Li and Dai, 2007; SaberiNajafi and Edalatpanah, 2011, 2012, 2013; Li, Giang, Cheng and Liao 2011] . Matrix splitting method for *LCP* exploits particular features of matrices such as the sparsity and the block structure and in these methods, most convergence results have been established for the case that the system matrix *M* is symmetric positive definite, *M-matrix*, *H-matrix* or diagonally dominant. The case that *M* is Non-Hermitian is much more difficult and, as we have known, only a few results were reported about this case of matrix for *LCP (M, q)*.

Recently , [Zhang and Benzi , 2010], present a different approach to solve systems of linear equations, called the P-regular splitting iterative methods for Non-Hermitian positive definite Linear systems. In this paper, we extend this approach to *LCP* and set up an efficient iterative method for solving *LCP* (1.1). Existence, uniqueness and convergence results of our method are also presented when the coefficient matrix is Non-Hermitian positive definite. The method begins with an initial point chosen arbitrarily and converges to optimal solution of original problem after finite iterations. The effectiveness of the method is demonstrated by its ability to solve some standard test problems found in the literature.

2 Prerequisite

We begin with some basic notation and preliminary results which we refer to later.

Definition 2.1

The matrix $M=[m_{ij}]$ is Non-Hermitian positive definite if the Hermitian part $H=(M+M^*)/2$ is Hermitian positive definite, where M^* denotes the conjugate transpose of the matrix *M*. Furthermore, $|x|$ denotes the vector whose components is the absolute value of the corresponding components of x and $\rho(M)$ denotes the spectral radius of matrix *M*.

Definition 2.2 [Yuan and Song, 2003]. For $x \in R^n$, vector x_+ is defined such that $(x_+)_j = max\{0,x_j\}, j = 1,2,. . . ,n$. Then, for any $x, y \in R^n$, the following facts hold:
1. $(x + y)_+ \leq x_+ + y_+$,
2. $x_+ - y_+ \leq (x - y)_+$,
3. $|x| = x_+ + (-x)_+$,
4. $x \leq y$ implies $x_+ \leq y_+$.

Lemma 2.1 [Mangasarian,1977]. *LCP(M, q)* can be equivalently transformed to a fixed-point system of equations

$$z = (z - \alpha E(Mz + q))_+, \qquad (2.1)$$

where α is some positive constant and E is a diagonal matrix with positive diagonal elements.

3 SOR-Like method for Non-Hermitian Positive Definite *LCP(M, q)*

Let the matrix M is split as

$$M=D+L+U\,, \qquad (3.1)$$

where D diagonal, L and U are strictly lower and upper triangular matrices of M, respectively. Then by choice of $\alpha E = D^{-1}$ and Lemma 2.1 we have,

$$z = (z - D^{-1}(Mz + q))_+. \qquad (3.2)$$

Now, we propose our iterative method Successive Overrelaxation Like method (SOR-Like Method) for *LCP(M, q)* by the following;

Algorithm1. SOR-Like method for *LCP*

Step 1.choos an initial vector $z^0 \in R^n$, parameter w and set k=0.
Step 2. For k=0,1,2,… do,

$$z^{k+1} = (z^k - wD^{-1}[(L - U^*)z^{k+1} + (D + U + U^*)z^k + q])_+,$$

Step 3. if $z^{k+1} = z^k$, then stop; otherwise, set k=k+1 and go to step 2.

In the next section, we will establish the existence and uniqueness of the solution of our method when the coefficient matrix is Non-Hermitian positive definite.

Theorem3.1. Let $M \in C^{n\times n}$ be non-Hermitian positive definite with $H=(M+M^*)/2$ its Hermitian part, $\eta=\lambda_{\min}(B)$ be the smallest eigenvalue of $B=H-2D^{-1}(U+U^*)$ and

$$\begin{cases} w\in(0,1], & if \quad \eta\geq 0, \\ w\in(0,1), & if \quad \eta=0, \\ w\in(0,\dfrac{2}{2-\eta}), & f \quad \eta<0. \end{cases}$$

Then for any initial vector z^0, Algorithm1 convergence to the unique solution of *LCP (M, q).*

Proof. For existence, consider Algorithm 1 , then we have;

$$z^{k+1}=(z^k-wD^{-1}[(M-(L-U^*))z^k+q+(L-U^*)z^{k+1}])_+.$$

Then,

$$\begin{aligned} z^{k+1}-z^k &= (z^k-wD^{-1}[(M-(L-U^*))z^k+q+(L-U^*)z^{k+1}])_+ \\ &\quad -(z^{k-1}-wD^{-1}[(M-(L-U^*))z^{k-1}+q+(L-U^*)z^k])_+ \\ &\leq [(z^k-z^{k-1})-wD^{-1}[(M-(L-U^*))(z^k-z^{k-1})+(L-U^*)(z^{k+1}-z^k)]]_+. \end{aligned}$$

Thus,

$$(z^{k+1}-z^k)_+ \leq [(I-wD^{-1}(M-(L-U^*)))(z^k-z^{k-1})]_+ + [-wD^{-1}(L-U^*)(z^{k+1}-z^k)]_+.$$

Similarly,

$$(z^k-z^{k+1})_+ \leq [(I-wD^{-1}(M-(L-U^*)))(z^{k-1}-z^k)]_+ + [-wD^{-1}(L-U^*)(z^k-z^{k+1})]_+.$$

On the other hand, by Definition 2.2 , we have;

$$\left|z^{k+1}-z^k\right| \leq \left|I-wD^{-1}(M-(L-U^*))\right|.\left|z^k-z^{k-1}\right| + wD^{-1}\left|L-U^*\right|.\left|z^{k+1}-z^k\right|.$$

Since $(L-U^*)$ is a strictly lower triangular matrix , then $(I-wD^{-1}\left|L-U^*\right|)$ is invertible and its inverse is nonnegative. Then;

$$\left|z^{k+1}-z^k\right| \leq Q^{-1}R\left|z^k-z^{k-1}\right|,$$

Where;

$$\begin{cases} Q=I-wD^{-1}\left|L-U^*\right|, \\ R=\left|I-wD^{-1}(M-(L-U^*)\right|. \end{cases}$$

Note that the matrix $Q^{-1}R$ is nonnegative. We know if $\rho(Q^{-1}R) < 1$, then sequence $\{z^k\}$ of Algorithm1 converges to a solution $\bar{z}$ of *LCP*. Therefore by applying [Zhang and Benzi, 2010, Theorem3.1], we know that there exist a vector which solves *LCP*.

For uniqueness of the solution for *LCP*,

Let $\bar{\bar{z}}$ is another solution of *LCP*. Then by following the proof process of existence of the solution, we have,

$$\left|\bar{z} - \bar{\bar{z}}\right| = (\bar{z} - \bar{\bar{z}})_+ + (\bar{\bar{z}} - \bar{z})_+ \le Q^{-1}R\left|\bar{z} - \bar{\bar{z}}\right|.$$

And since $\rho(Q^{-1}R) < 1$ we have,

$$\left|\bar{z} - \bar{\bar{z}}\right| = 0.$$

Namely,

$$\bar{z} = \bar{\bar{z}}.$$

And the proof is completed.■

4 Numerical Results

In this section, we give examples to illustrate the results obtained in previous sections.
In these examples the initial approximation is $z^0 = (1,1,\ldots,1)^T \in R^n$ and as a stopping criterion we choose;

$$\begin{cases} \left\|\min(Mz^k + q, z^k)\right\|_\infty \le 10^{-6}, \\ \left\|\min(\bar{M}z^k + \bar{q}, z^k)\right\|_\infty \le 10^{-6}. \end{cases}$$

These examples computed with MATLAB7 on a personal computer Pentium 4**.**

Example4.1. Consider *LCP (M, q)* with following system $M \in R^{n\times n}$ and $q \in R^n$.

Matrix *M* is a banded matrix select from [Lam, 1999], which use in Quantum Chemistry, i.e.:

$$M=\begin{bmatrix} 1 & 0.21 & 1.2 & 0 & 0.13 & 1.42 & & & & \\ 0.11 & 2 & 0.21 & 1.2 & 0 & 0.13 & 1.42 & & & \\ 0.12 & 0.11 & 3 & 0.21 & 1.2 & 0 & \ddots & \ddots & & \\ 0 & 0.12 & 0.11 & 4 & \ddots & \ddots & \ddots & \ddots & \ddots & \\ 0.34 & 0 & 0.12 & \ddots & \ddots & \ddots & \ddots & \ddots & 0.13 & 1.42 \\ 0.45 & 0.34 & 0 & \ddots & \ddots & \ddots & 0.21 & 1.2 & 0 & 0.13 \\ & 0.45 & \ddots & \ddots & \ddots & 0.11 & 997 & 0.21 & 1.2 & 0 \\ & & \ddots & \ddots & \ddots & 0.12 & 0.11 & 998 & 0.21 & 1.2 \\ & & & \ddots & 0.34 & 0 & 0.12 & 0.11 & 999 & 0.21 \\ & & & & 0.45 & 0.34 & 0 & 0.12 & 0.11 & 1000 \end{bmatrix}_{1000\times 1000}$$

And,

$$q=(-1,1,\ldots,(-1)^n)^T \in R^n.$$

Evidently, M is Non-Hermitian positive definite. Then *LCP (M, q)* has a unique solution and we solved the 1000×1000 matrix yielded by the SOR-like iterative method. In this experiment, the sign of $\eta=\lambda_{\min}(B)$ is positive.

In Table 1, with several values we report the CPU time (CPU) and number of iterations (Iter) for the corresponding SOR-like method (when *n=1000*). From the table, we can see that the SOR-like iterative method is useful for solving the *LCP*.

Table1 Shows the results of example 4.1 for SOR-Like

Method	SOR-Like	
w	**Iter**	**CPU**
0.3	48	0.456272
0.4	35	0.323434
0.5	27	0.251154
0.6	21	0.195035
0.7	17	0.155504
0.8	13	0.119373
0.9	12	0.109370
1.0	29	0.160918
1.5	∞	------------

Example4.2. Consider some randomly generated *LCP* with Non-Hermitian positive definite *M* where the data (M, q) are generated by the Matlab scripts:

```
function randomforLCP(n)
rand('state',0);
R=rand(n,n);
M=R+n*eye(n);
q=rand(n,1);
```

In Table 2, we report the CPU time (CPU) and number of iterations (Iter) for the corresponding SOR-like method for different values of n and w.

Table2 Shows the results of example 4.2 for SOR-Like

Method		SOR-Like		
n	the sign of η	w	**Iter**	**CPU**
100	+	0.1	135	0.005278
		0.5	24	0.001008
		1.0	19	0.000720
1000	+	0.1	145	0.399574
		0.5	26	0.070849
		1.0	19	0.051053
1500	+	0.1	147	0.901282
		0.5	26	0.158273
		1.0	19	0.116798

5. Conclusions

In this paper, we have proposed a new iterative method for solving a class of linear Complementarity problems with Non-Hermitian positive definite matrices. We have also, established global convergence for this method. Both theoretical analyses and numerical experiments have shown that the proposed algorithm seems promising for solving the *LCP*.

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