AMO - Advanced Modeling and Optimization, Volume 15, Number 3, 2013

SOLVING MULTI-OBJECTIVE OPTIMAL CONTROL PROBLEMS USING FUZZY AGGREGATION AND EMBEDDING METHOD

HASSAN ZAREI[1]

[1] *Department of Mathematics, Payame noor university, Iran.*
zarei2003@yahoo.com

ABSTRACT. In this paper we first convert the multi-objective optimal control problems to the multi-objective optimization problems in measure space. The fuzzy goals characterized by linear membership functions are assigned to objectives and the metamorphosed problem is replaced with an equivalent optimization problem whose solution is compromised Pareto optimal, using Zimmerman's fuzzy approach. The resulting optimization problem, which is linear with infinite dimensional in space of all pairs in Cartesian product of real numbers set and measure space, is approximated with a linear programming (LP) problem that its solution is used to construct a solution in piecewise constant level. Finally, some illustrative numerical examples are worked out to indicate the efficiency of the proposed method.
Keywords: Multi-objective optimal control, Measure theory, Fuzzy goal programming, Linear programming.

1. INTRODUCTION

Let $I = [0, \tau]$, $A = A_1 \times \ldots \times A_n \subseteq R^n$ and $U = U_1 \times \ldots \times U_m \subseteq R^m$ be compact sets and $\Omega = I \times A \times U$. Denote the space of all real valued continuous functions on Ω by $C(\Omega)$. Consider the following multi-objective optimal control problem (MOOCP),

$$Minimize(J_1(x,u), \ldots, J_k(x,u)) \tag{1.1}$$

Subject to

$$\dot{x} = g(t, x, u), \tag{1.2}$$

$$x(0) = x_0, x(\tau) = x_\tau, \tag{1.3}$$

[0]**AMO - Advanced Modeling and Optimization. ISSN: 1841-4311**

where $x : t \in I \longrightarrow x(t) \in A$ is the time-varying differentiable state variable and $u : t \in I \longrightarrow u(t) \in U$ is the time-varying measurable control variable. Moreover, $g : (t, x, u) \in \Omega \longrightarrow g(t, x, u) \in R^n$ is a continuous vector representing the dynamic system equations (on the interval I) with initial and terminal conditions given by the vectors x_0 and x_τ. Each individual objective function is:

$$J_i(x, u) = \int_0^\tau f_i(t, x, u)dt, \tag{1.4}$$

where $f_i \in C(\Omega), i = 1, \ldots, k$. The final time τ may be fixed or free. Contrary to single objective optimization, typically no single solution exists in multiple objective optimization. Because of this fact, the notion of Pareto optimality has been introduced. A solution is Pareto optimal if there exists no other solution that would improve some objective values without worsening at least one criterion, simultaneously. The MOOCPs have been studied by many authors. Some authors have used fuzzy set theory in conjunction with linear programming techniques as a very efficient tool for linear lumped and distributed parameter systems (See [1] and references therein). The Normal Boundary Intersection (NBI) [2] and the Normalized Normal Constraint (NNC) [3], which have been found to mitigate the disadvantages of the weighted sum (WS) method [4], have been successfully combined with direct optimal control approaches for the efficient solution of multi-objective optimal control problems. For example, in [5], a successful application of NBI and NNC for the multiple objective optimal control of (bio) chemical processes has been reported, and in [6] several scalarisation techniques for multi-objective optimization, e.g., WS, NNC and NBI have been integrated with fast deterministic direct optimal control approaches. Using the measure theory for solving optimal control problems based on the idea of Young [7], which was applied for the first time by Wilson and Rubio [8], has been theoretically established by Rubio in [9] . Considerable attention has been given to the optimal control problems by applying measure theoretical approach and this approach has proved to be a very efficient tool for nonlinear systems [10-13]. In this paper we follow this approach and propose a solution in piecewise constant level for MOOCPs.

In section 2, the MOOCP is transformed into a multi-objective optimization problem in measure space applying a measure theoretical approach. In section 3, a fuzzy goal is assigned for each objective, which is described by a linear membership function elicited through an interaction with the DM, and an infinite dimensional fuzzy goal programming is proposed with the final goal of finding a Pareto optimal solution which has the best satisfaction performance among other Pareto optimal solutions. Moreover, in section 4 the infinite dimensional fuzzy goal programming problem is approximated by an LP model, and a piecewise constant solution is achieved.

Some numerical experiments are provided in section 5. The last section is the conclusion.

2. MEASURE SPACE

Let B is an open ball containing $I \times A$, and $\acute{C}(B)$ is the space of all continuously differentiable real-valued functions on it. Denote the set of all continuously differentiable functions ψ on I that $\psi(0) = \psi(\tau) = 0$ by $D(I)$, and the set of functions $\upsilon \in C(\Omega)$ that depend only on time by $C^1(\Omega)$. As discussed in [9], (1.2)-(1.3) can be written in the integral forms as:

$$\int_0^\tau \varphi^g(t,x,u)dt = \varphi(\tau, x_\tau) - \varphi(0, x_0) = \Delta\varphi, \forall\varphi \in \acute{C}(B), \tag{2.1}$$

$$\int_0^\tau \psi^j(t,x,u)dt = 0, \forall\psi \in D(I), j = 1, \dots, n, \tag{2.2}$$

$$\int_0^\tau \upsilon(t)dt = a_\upsilon, \forall\upsilon \in C^1(\Omega), \tag{2.3}$$

where, $\varphi^g(t,x) = \varphi_x(t,x)g(t,x,u) + \varphi_t(t,x)$, $\psi^j = x_j\acute{\psi} + g_j(t,x,u)\psi$ and a_υ is the integral of υ over I. By the Riesz representation theorem [9], there exists a unique positive Radon measure μ on Ω that

$$\int_0^\tau F(t,x(t),u(t))dt = \int_\Omega F(t,x,u)d\mu = \mu(F), \forall F \in C(\Omega). \tag{2.4}$$

Therefore, the MOOCP (1.1)-(1.3) is converted into another optimization problem in measure space given by,

$$\underset{\mu \in M^+(\Omega)}{Minimize}(\mu(f_1), \dots, \mu(f_k)) \tag{2.5}$$

Subject to

$$\mu(\varphi^g) = \Delta\varphi, \varphi \in \acute{C}(B), \tag{2.6}$$

$$\mu(\psi^j) = 0, \psi \in D(I), j = 1, \dots, n, \tag{2.7}$$

$$\mu(\upsilon) = a_\upsilon, \upsilon \in C^1(\Omega), \tag{2.8}$$

where $M^+(\Omega)$ denotes the space of all positive Radon measures on Ω. According to the concept of Pareto optimality, the number of Pareto optimal solutions for optimization problems with multiply objectives can be infinite. There are several methods to find a Pareto optimal solution which, among other Pareto optimal solutions, has the best satisfaction of DM [14]. The next section deals with finding such solution.

3. FUZZY GOAL PROGRAMMING

One of the frequently used methods proposed by Zimmermann [15] in 1979 is based on incorporating the fuzzy goals for objectives and considering the equilibrium problem in terms of maximization of the degree of attainment for the aggregated fuzzy goals. The fuzzy goals are quantified by eliciting the corresponding membership functions, which usually are linear, through the interaction with the DM. For notational convenience, we express the multi-objective programming problem (2.5)-(2.8) as:

$$\underset{\mu\in Q}{Minimize}(\mu(f_1),\ldots,\mu(f_k)) \tag{3.1}$$

where Q is the set of all positive Radon measures on Ω , satisfying (2.6)-(2.8). Since we are interested in minimizing the objective functions, it is quite nature to define the linear membership function $\sigma(\mu(f_i)), i = 1\ldots,k$ for the fuzzy goal of the DM as

$$\sigma(\mu(f_i)) = \begin{cases} 1 & \mu(f_i) < \underline{\mu}_i \\ \frac{\overline{\mu}_i-\mu(f_i)}{\overline{\mu}_i-\underline{\mu}_i} & \underline{\mu}_i \le \mu(f_i) \le \overline{\mu}_i \\ 0 & \overline{\mu}_i < \mu(f_i), \end{cases} \tag{3.2}$$

where $\overline{\mu}_i$ and $\underline{\mu}_i$ are, respectively a minimum value and a maximum value of totally desirable levels for $\mu(f_i)$. Assume that $\mu_i, i = 1,\ldots,k$ minimizes $\mu(f_i)$ over Q . The existence of μ_i is guaranteed due to continuity of $\mu \in Q \to \mu(f_i)$ and compactness of Q with the weak*-topology [9]. As discussed in [14], the DM assesses suitable values for $\underline{\mu}_i$ and $\overline{\mu}_i$, within μ_i^{min} and μ_i^m, given by $\mu_i^{min} = \mu_i(f_i)$ and $\mu_i^m = \underset{j=1,\ldots,k}{max}\mu_j(f_i)$. Following the fuzzy decision of Bellman and Zadeh [15], the multi-objective problem (3.1) can be interpreted as:

$$\underset{\mu\in Q}{Maximize}\,\underset{j=1,\ldots,k}{Minimum}\,\sigma(\mu(f_j)). \tag{3.3}$$

By introducing the auxiliary variable λ, problem (3.3) can be equivalently transformed as

$$Maximize \qquad \lambda \tag{3.4}$$

Subject to

$$\mu(f_i) + \lambda(\overline{\mu}_i - \underline{\mu}_i) \le \overline{\mu}_i, i = 1,\ldots,k, \tag{3.5}$$

$$\mu \in Q. \tag{3.6}$$

If this problem has a unique optimal solution (μ^*, λ^*), then μ^* is a Pareto optimal solution of (3.1). If this sufficiently condition for Pareto optimality of μ^* doesn't satisfied, then we can test the Pareto optimality for μ^* by considering another auxiliary problem (for more details see [14]). Since Q is an infinite dimensional space, the problem (3.4)-(3.6) is an infinite-dimensional optimization problem and we are mainly interested in approximating it. The next is devoted to this subject.

4. Approximation

The maximization is considered not over the set Q, but over a subset of it denoted by requiring that only a finite number of constraints (2.6)-(2.8) be satisfied. Let, $\varphi_i, i = 1, \ldots, K_1$, $\psi_\ell, \ell = 1, \ldots, K_2$ and $\upsilon_s, s = 1, \ldots, S$ are taken from a total subset of $\acute{C}(B)$, $D(I)$ and $C^1(\Omega)$, respectively. Consider the following problem:

$$Maximize \qquad \lambda \tag{4.1}$$

Subject to

$$\mu(f_i) + \lambda(\overline{\mu}_i - \underline{\mu}_i) \leq \overline{\mu}_i, i = 1, \ldots, k, \tag{4.2}$$

$$\mu(\varphi_i^g) = \Delta\varphi_i, i = 1 \ldots K_1, \tag{4.3}$$

$$\mu(\psi_\ell^j) = 0, \ell = 1 \ldots K_2, j = 1, \ldots, n, \tag{4.4}$$

$$\mu(\upsilon_s) = a_{\upsilon_s}, s = 1 \ldots S. \tag{4.5}$$

It can be verified that the solution of problem (4.1)-(4.5) tends to the solution of problem (3.4)-(3.6) as K_1, K_2 and S tend to infinity [10]. Assume that $\Omega_N = \{y_1, \ldots, y_N\}$ is a countable dense subset of Ω. It has been proved that the problem has an optimal solution (μ^*, λ^*) with $\mu^* \approx \sum_{j=1}^N \alpha_j \delta(y_j)$, where $\alpha_j \geq 0$ and $\delta(y)$ is unitary atomic measure with the support being the singleton set $\{y\}$ characterized by $\delta(y)(F) = F(y), \forall F \in C(\Omega)$(for more details see [10]). Therefore, the problem (4.1)-(4.5) is approximated by an LP problem as follows:

$$\underset{\alpha \geq 0}{Maximize} \qquad \lambda \tag{4.6}$$

Subject to,

$$\sum_{j=1}^N \alpha_j f_i(y_j) + \lambda(\overline{\mu}_i - \underline{\mu}_i) \leq \overline{\mu}_i, i = 1, \ldots, k, \tag{4.7}$$

$$\sum_{j=1}^N \alpha_j \varphi_i^g(y_j) = \Delta\varphi_i, i = 1, \ldots, K_1, \tag{4.8}$$

$$\sum_{j=1}^N \alpha_j \psi_\ell^h(y_j) = 0, \ell = 1, \ldots, K_2, h = 1, \ldots, n, \tag{4.9}$$

$$\sum_{j=1}^N \alpha_j \upsilon_s(y_j) = a_{\upsilon_s}, s = 1, \ldots, S. \tag{4.10}$$

We note that $\mu_i, i = 1, \ldots, k$, is approximated by solving an LP problem which consists of minimizing the function $\sum_{j=1}^N \alpha_j f_i(y_j)$ over the constraints given by (4.8)-(4.10). For problems with the fixed or free final time τ, the dense set Ω_N and the total functions φ_is, ψ_ℓs and υ_ss are chosen according

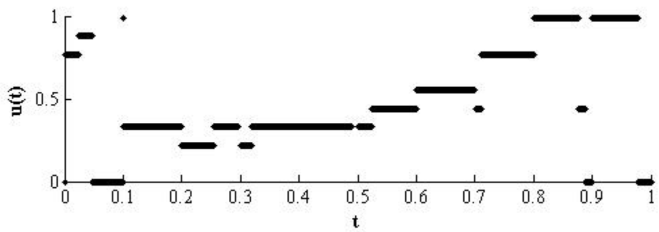

FIGURE 1. The approximate optimal control u.

to the discussions in [11, 12]. The procedure of constructing piecewise constant control functions from the solution of LP problem (4.6)-(4.10) which approximate the action of optimal measure is based on the analysis in [9]. Of course, we need only to construct the control function u, since x can be obtained by solving the ODEs (1.2).

5. NUMERICAL RESULTS

In this section, the approximate optimal solution for several MOOCPs in piecewise constant levels are achieved using the described method. We note that in each example the suitable total functions and the dense set Ω_N have been chosen through a series of numerical experiments, but the details are omitted here.

Example 1. consider the following example:

$$Minimize \qquad (J_1(x,u), J_2(x,u))$$

subject to,

$$\dot{x} = \frac{1}{2}x^2 sinx + u,$$
$$x(0) = 0, x(1) = 0.5,$$

where, $J_1(x,u) = \int_0^1 u^2(t)dt$ and $J_2(x,u) = \int_0^1 x^2(t)dt$. Calculating the individual minimum μ_i for these objective functions yields $\mu_1(f_1) = 0.2563$, $\mu_1(f_2) = 0.0993$, $\mu_2(f_1) = 0.4437$ and $\mu_2(f_2) = 0.0622$. Therefore, we set $\underline{\mu}_1 = \mu_1^{min} = \mu_1(f_1) = 0.2563$ and $\overline{\mu}_1 = \mu_1^m = \max_{j=1,2} \mu_j(f_1) = 0.4437$. Implementing the corresponding LP model (4.6)-(4.10), we achieve $\lambda^* = 0.6307$, and an optimal measure μ^* with $\mu^*(f_1) = 0.3254$ and $\mu^*(f_1) = 0.0759$. Therefore, $\sigma(\mu^*(f_1)) = 0.6313$ and $\sigma(\mu^*(f_2)) = 0.6303$. The resulting control function and the response of the system to this function are depicted in Fig. 1 and Fig. 2, respectively. Moreover, we found $x(1) = 0.4940$, which is close to the exact value.

Example2[6]. let $x_1(t)$, $x_2(t)$ and $u(t)$ denote the position, the velocity and the acceleration of a car at time t, respectively. The acceleration of the car is controlled by pushing the accelerator, or hitting the brakes. A simple dynamic model for the car is $\dot{x}_1 = x_2$ and $\dot{x}_2 = u$. The aim is to drive 400 m, starting and ending at rest, while minimizing on the one hand, the control

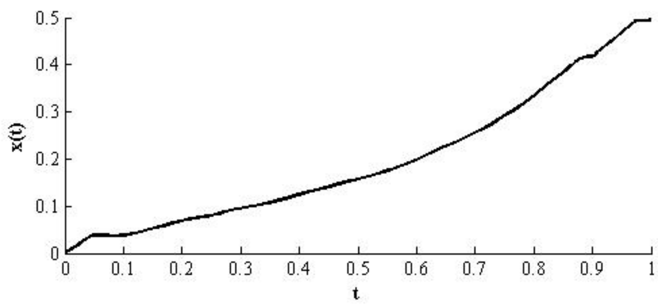

FIGURE 2. Dynamic behavior of the state variable $x(t)$ versus time .

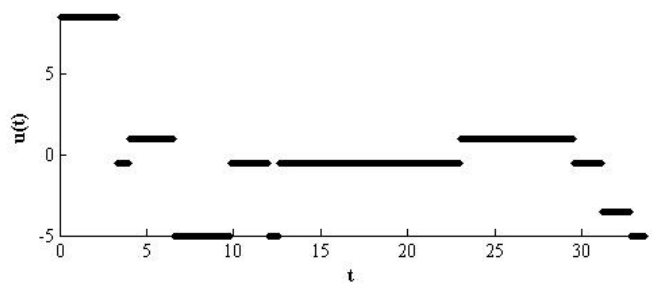

FIGURE 3. The approximate optimal control for the car example.

effort for accelerating (which can by interpreted as the fuel consumption) and on the other hand traveling time. Therefore, we have a MOOCP as:

$$Minimize \quad (J_1(x,u), J_2(x,u))$$

Subject to,

$$\dot{x}_1 = x_2,$$
$$\dot{x}_2 = u,$$
$$x_1(0) = 0, x_1(\tau) = 400,$$
$$x_2(0) = 0, x_2(\tau) = 0,$$

where, $J_1(x,u) = \int_0^\tau |u(t)|dt$ and $J_2(x,u) = \int_0^\tau dt$. Objectives J_1 and J_2 are obviously conflicting objectives since a small traveling time requires a high speed, and, hence, also a large consumption of fuel for reaching this velocity. Since infinitely fast accelerating and decelerating is impossible, the control is bounded between $-5 \le u(t) \le 8.5$. Additional constraint is speed limit given by $x_2(t) \le 40$. Calculating the individual minimum μ_i for these objective functions yields $\mu_1(f_1) = 46.9589$, $\mu_1(f_2) = 36.2393$, $\mu_2(f_1) = 155.5059$ and $\mu_2(f_2) = 32.8075$. Therefore, we set $\underline{\mu}_1 = 46.9589$, $\overline{\mu}_1 = 155.5056$, $\underline{\mu}_2 = 32.8075$ and $\overline{\mu}_2 = 36.2393$. Implementing the corresponding

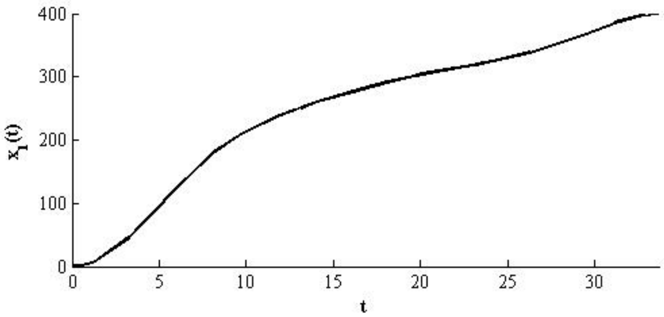

FIGURE 4. Optimal position of the car.

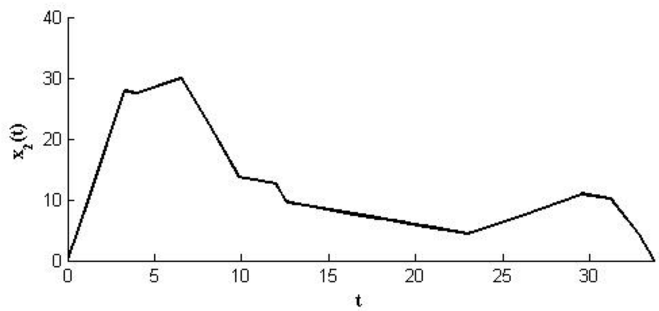

FIGURE 5. Optimal velocity of the car.

LP model (4.6)-(4.10), we achieve $\lambda^* = 0.7505$, and an optimal measure μ^* with $\mu^*(f_1) = 74.0425$ and $\mu^*(f_2) = 33.6638$. Therefore, $\sigma(\mu^*(f_2)) = \sigma(\mu^*(f_1)) = 0.7505$. The resulting suboptimal control and the response of the system to the obtained control function are depicted in Fig. 1 and Fig. 2, respectively. Moreover, we found $x_1(33.6638) = 399.6622$ and $x_2(33.6638) = 0.0008$, which are close to the exact values. The resulting control and the trajectories of this example are plotted in Figures 3-5.

Example3 (Reactor model) [5]. Consider the reactor model based on the 1D plug flow model with an irreversible first-order reaction which is given by a highly nonlinear model with independent variable the position $z \in [0, 1]$ along the reactor as :

$$\frac{d}{dz}x_1(z) = \frac{\alpha}{v}(1 - x_1)e^{\frac{\gamma x_2}{1+x_2}}$$

$$\frac{d}{dz}x_2(z) = \frac{\alpha}{v}\delta(1 - x_1)e^{\frac{\gamma x_2}{1+x_2}} + \frac{\beta}{v}(u - x_2)$$

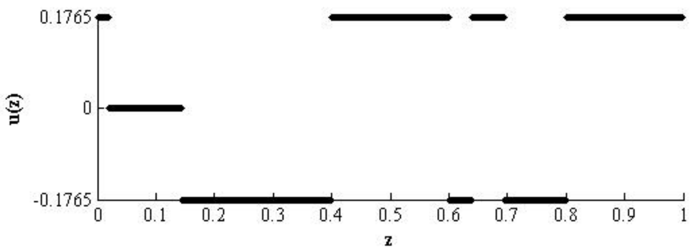

FIGURE 6. Optimal jacket temperature of the reactor.

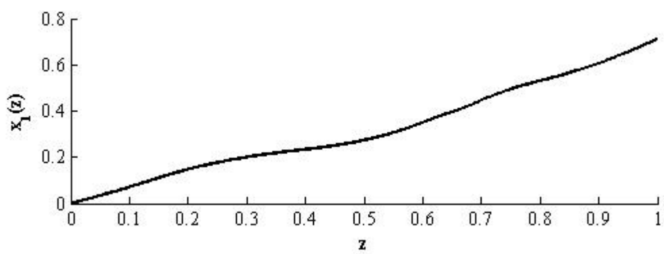

FIGURE 7. Optimal concentrations of the reactor.

with initial conditions $x(0) = (0,0)$, where, $x_1 = \frac{(C_F - C)}{C_F}$ is the dimensionless reactant concentration C, $x_2 = \frac{(T-T_F)}{T_F}$ the dimensionless reactor temperature T, and $u = \frac{(T_w - T_F)}{T_F}$, the dimensionless jacket temperature T_w. Bounds are imposed on the reactor and jacket temperatures for constructive reasons:

$$x_{2min} \le x_2(z) \le x_{2max}, u_{min} \le u(z) \le u_{max}.$$

The aim is to derive an optimal profile along the reactor for the jacket temperature profile $u(z)$. The two objectives considered are similar to the conflicting ones treated in [17], i.e., maximizing the conversion, which is related to minimizing the reactant concentration at the outlet:

$$J_1(x,u) = -x_1(\ell) = \int_0^{\ell} \frac{\alpha}{v}(1-x_1)e^{\frac{\gamma x_2}{1+x_2}}dz$$

and maximizing the net heat transfer between the reactor and its jacket, where heat transferred from the reactor to the jacket is assumed to a profit:

$$J_2(x,u) = \int_0^{\ell} (u(z) - x_2(z))dz.$$

Parameter values can be found in [5]. Calculating the individual minimum μ_i for these objective functions yields $\mu_2(f_1) = -0.4416$, $\mu_2(f_2) = -0.0054$, $\mu_1(f_1) = -0.9508$ and $\mu_1(f_2) = 0.0690$. Therefore, we set $\underline{\mu}_1 = -0.9508$,

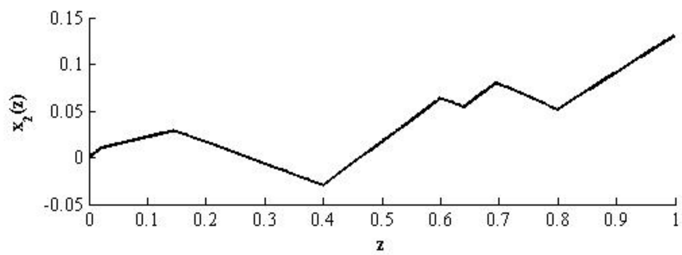

FIGURE 8. Optimal reactor temperature.

$\overline{\mu}_1 = -0.4416$, $\underline{\mu}_2 = -0.0054$ and $\overline{\mu}_2 = 0.0690$. Implementing the corresponding LP model (4.6)-(4.10), we achieve $\lambda^* = 0.5296$, and an optimal measure μ^* with $\mu^*(f_1) = -0.7113$ and $\mu^*(f_2) = 0.0296$; hence, $\sigma(\mu^*(f_1)) = 0.5297$ and $\sigma(\mu^*(f_2)) = 0.5296$. Figs 6-8 displays the optimal control and state profiles.

6. CONCLUSION

In this paper we used the measure theoretical approach and studied the optimal control problems with multiple objectives in measure space. A fuzzy goal programming is proposed by assigning fuzzy goals to objective functionals and then by using the Zimmermann's fuzzy approach we faced with an optimization problem whose solution is compromise Pareto optimal. The resulting optimization problem which is linear with infinite dimension is approximated by an LP problem which its solution obtained by simplex method is used to construct a solution in piecewise constant level. The numerical examples demonstrated the flexibility and efficiency of the proposed method.

REFERENCES

1. M. Sakawa, M. Inuiguchi, K. Kato, and T. Ikeda,An interactive fuzzy satisficing method for multiobjective optimal control problems in linear distributed-parameter systems, Fuzzy Sets and Systems 102 (1999) 237-246
2. I. Das, and J.E. Dennis, "Normal-Boundary Intersection: A new method for generating the Pareto surface in nonlinear multicriteria optimization problems," SIAM Journal on Optimization, vol. 8, pp. 631-657, 1998.
3. A. Messac, A. Ismail-Yahaya, and C.A. Mattson, "The normalized normal constraint method for generating the Pareto frontier," Structural and Multidisciplinary Optimization, vol. 25, pp. 86-98, 2003.
4. I. Das, and J.E. Dennis, "A closer look at drawbacks of minimizing weighted sums of objectives for Pareto set generation in multicriteria optimization problems," Structural Optimization, vol. 14, pp. 63-69, 1997.
5. F. Logist, P. M. M. Van Erdeghem, and J.F. Van Impe, "Efficient deterministic multiple objective optimal control of (bio) chemical processes," Chemical Engineering Science, vol. 64, pp. 2527-2538, 2009.

6. F. Logist, B. Houska, M. Diehl, and J. Van Impe, "Fast Pareto set generation for nonlinear optimal control problems with multiple objectives," Structural and Multidisciplinary Optimization, vol. 42, pp.591-603, 2010.
7. C. Young, Calculus of Variations and Optimal Control Theory, Sunders, Philadelphia, 1969.
8. D. A. Wilson, J. E. Rubio, "Existence of optimal controls for the diffusion equation" Journal of Optimization Theory and Applications., vol. 22, pp. 91-101, 1977 .
9. J. E. Rubio, Control and Optimization: The Linear Treatment of Non-linear Problems, Manchester University Press, Manchester, UK, 1986.
10. H. Zarei, A. V. Kamyad, S. Effati, Multiobjective optimal control of HIV dynamics, Mathematical Problems in Engineering 2010 - Article ID 568315. doi:10.1155/2010/568315.
11. H. H. Mehne, M. H. Farahi, and A. V. Kamyad, MILP modelling for the time optimal control problem in the case of multiple targets, Optimal Control Applications and Methods, vol. 27, no. 2, pp. 7791, 2006.
12. M. Gachpazan, A. H. Borzabadi, and A. V. Kamyad, A measure-theoretical approach for solving discrete optimal control problems, Applied Mathematics and Computation, vol. 173, no. 2, pp. 736752, 2006.
13. H. Zarei, A. V. Kamyad, and M. H. Farahi, Optimal Control of HIV Dynamic Using Embedding Method, Computational and Mathematical Methods in Medicine 2011, Article ID 674318. doi:10.1155/2011/674318
14. M. Sakawa, Fuzzy Sets and Interactive Multiobjective Optimization, Plenum Press, New York, 1993.
15. H.-J. Zimmerrnann, Fuzzy programming and linear programming with several objective functions, Fuzzy Sets and Systems 1 (1978)45-55.
16. R. E. Bellman, and L. A. Zadeh, "Decision making in a fuzzy environment," Management Science. Vol. 17, pp. 141-164, 1970.
17. F.Logist, P.Van Erdeghem, I. Smets, and J. Van Impe (2009). Optimal design of dispersive tubular reactors at steady-state using optimal control theory. Journal of Process Control, 19, 1191-1198.