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On Some Divergence Measures between Fuzzy Sets and Aggregation Operations

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Abstract

In the present communication, three new divergence measures between fuzzy sets are proposed. The validity of these divergence measures is examined axiomatically. Relation of the proposed divergence measures with the cardinality of a universe of discourse is established. Some properties of these divergence measures are established. Relation of divergence between fuzzy sets with aggregation operations is investigated, and some properties are established. Application of divergence measures between fuzzy sets in strategic decision making is illustrated with numerical.

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1 Introduction

Proposed by [Zadeh, 1965], fuzzy set theory based upon the assumption that there are sets in which the degree of membership of an element lie between zero and one, gained vital interdisciplinary importance in many fields such as pattern recognition, image processing, fuzzy aircraft control, feature selection, Bio-informatics and many more. Uncertainty and fuzziness are basic elements of the human perspective and of many real-world objectives. The main use of information is to remove the uncertainty and fuzziness. The measure of uncertainty removed is the information measure while the measure of vagueness is a measure of fuzziness.

[Kullback and Leibler, 1951] introduced a divergence measure between two probability distributions (observed and proposed) of a random variable. [Bhandari and Pal, 1993] introduced a divergence measure between fuzzy sets (Fuzzy directed divergence) corresponding to probabilistic divergence measure introduced by [Kullback and Leibler, 1951]. [Verma and Sharma, 2011] proposed divergence measure between fuzzy sets corresponding to Inaccuracy measure introduced by [Kerrige, 1961]. [Bhatia and Singh, 2012] proposed generalized fuzzy directed divergence corresponding to generalized probabilistic divergence measures introduced by [Taneja, 2005a,2005b]. Many researchers studied the concept of fuzzy directed divergence from the point of view of the theory or from the point of view of applications. In this paper, both the aspects are studied.

In the present communication, some preliminaries related with the concept of fuzziness, probabilistic and fuzzy divergence measures are presented in section 2. In section 3, we introduce three new divergence measures between fuzzy sets, and some properties of these divergence measures are presented. In section 4, relation of divergence measures between fuzzy sets with aggregation operation is investigated, and some properties are established. In section 5, a model for strategic decision making is proposed, and same is explained by the help of a numerical. In section 6, conclusion is presented.

2 Prelimineries

2.1 Preliminaries On Fuzzy Set Theory

Definition 1 Let a universe of discourse $X = \{x_1, x_2, x_3...x_n\}$ then a fuzzy subset of universe X is defined as

$$A = \{(x, \mu_A(x)) \mid x \in X, \mu_A(x) \in [0, 1]\}$$

Where $\mu_A(x) : X \to [0, 1]$ is a membership function defined as follow:

$$\mu_A(x) = \begin{cases} 0 & \text{if x does not belong to A and there is no ambiguty} \\ 1 & \text{if x belong to A and there is no ambiguty} \\ 0.5 & \text{if there is maximum ambiguity whether x belongs to A or not} \end{cases}$$

In fact $\mu_A(x)$ associates with each $x \in X$ a grade of membership of the set A. Some notions related to fuzzy sets which we shall need in our discussion [18].

Containment ; $A \subset B \Leftrightarrow \mu_A(x) \leq \mu_B(x)$ for all $x \in X$

Equality ; $A = B \Leftrightarrow \mu_A(x) = \mu_B(x)$ for all $x \in X$

Compliment ; $\bar{A}$ = Compliment of A $\Leftrightarrow \mu_A(x) = 1 - \mu_A(x)$ for all $x \in X$

Union ; $A \cup B$ = Union of A and B $\Leftrightarrow \mu_{A\cup B}(x) = \max.\{\mu_A(x), \mu_B(x)\}$ for all $x \in X$

Intersection ; $A \cap B$ = Intersection of A and B $\Leftrightarrow \mu_{A\cap B}(x) = \min.\{\mu_A(x), \mu_B(x)\}$ for all $x \in X$

Product ; AB = Product of A and B $\Leftrightarrow \mu_{AB}(x) = \mu_A(x)\mu_B(x)$ for all $x \in X$

Sum; $A \oplus B$ = Sum of A and B $\Leftrightarrow \mu_{A\oplus B}(x) = \mu_A(x) + \mu_B(x) - \mu_A(x)\mu_B(x)$ for all $x \in X$

2.2 Probabilistic Divergence Measures

The relative entropy is a measure of the distance between two probability distributions. In statistics, it arises as the expected logarithm of the likelihood ratio. The relative entropy or the divergence measure between two probability distributions, $K(P, Q)$ is the measure of inefficiency of assuming that the distribution is Q when the true distribution is P. For example, if we knew the true distribution of the random variable, then we could construct a code with average description length $H(P)$. If, instead, we used the code for a distribution Q , we would need $H(P) + K(P, Q)$ bits on the average to describe the random variable[Kullback and Leibler, 1951].

The relative entropy or Kullback Leibler distance between two probability distributions is defined as

$$K(P,Q) = \sum_{i=1}^{n} p_i \log \frac{p_i}{q_i} \tag{1}$$

where $P, Q \in \Gamma_n$ and $\Gamma_n = \{P = (p_1, p_2, p_3...p_n)|p_i \geq \ 0, \sum_{i=1}^{n} p_i = 1, n \geq 2\}$ is the set of all complete finite discrete probability distributions.

With the development in literature, many parametric and non-parametric measures were suggested, based on applicability in different situations.

2.3 Divergence Measures Between Fuzzy sets

Definition 2. [Couso *et al.*, 2004] Let a universal set X and $F(X)$ be the set of all fuzzy subsets . A mapping $D : F(X) \times F(X) \to \mathbf{R}$ is called a divergence between fuzzy subsets if and only if the following axioms hold:

$d_1 :\ D(A,B) = D(B,A)$

$d_2 :\ D(A,A) = 0$

$d_3 :$ max. $\{D(A \cup C, B \cup C), D(A \cap C, B \cap C)\} \leq D(B,A)$ for any $A, B, C \in F(X)$

Instead of axiom d_3 if D(A,B) is convex in A and B even then it a valid measure of divergence.

[Bhandari and Pal, 1993] Introduced a measure of divergence between fuzzy sets corresponding to information theoretic divergence measure(1)as

$$D(A,B) = -\tfrac{1}{n} \textstyle\sum_i \{\mu_A(x_i) \log \frac{\mu_A(x_i)}{\mu_B(x_i)} + (1 - \mu_A(x_i)) \log \frac{(1-\mu_A(x_i))}{(1-\mu_B(x_i))}\} \tag{2}$$

Axioms d_1, d_2, d_3 are used to define a new divergence measure. Many divergence measures between fuzzy sets were defined in literature corresponding to existing probabilistic divergence measures.

3 Three New Symmetric Divergence Measures Between Fuzzy Sets

[Bhatia *et al.*, 2011] proposed the following symmetric divergence measures between discrete probability distributions P and Q .

$$M_{CA}(P,Q) = \sum_{i=1}^{n}[(\frac{p_i^2+q_i^2}{p_i+q_i}) - (\frac{p_i+q_i}{2})] \tag{3}$$

$$M_{CG}(P,Q) = \sum_{i=1}^{n}[(\frac{p_i^2+q_i^2}{p_i+q_i}) - \sqrt{p_i q_i}] \tag{4}$$

$$M_{CH}(P,Q) = \sum_{i=1}^{n}[(\frac{p_i^2+q_i^2}{p_i+q_i}) - (\frac{2p_i q_i}{p_i+q_i})] \tag{5}$$

where $P, Q \in \Gamma_n$ and $\Gamma_n = \{P = (p_1, p_2, p_3 ... p_n) | p_i \geq \ 0, \sum_{i=1}^{n} p_i = 1, n \geq 2\}$ is the set of all complete finite discrete probability distributions.

Corresponding to these divergence measures we propose the following divergence measure between fuzzy sets respectively.

$$M_1(A,B) = \sum_{i=1}^{n} \frac{(\mu_A(x_i) - \mu_B(x_i))^2}{2}[\frac{1}{(\mu_A(x_i)+\mu_B(x_i))} + \frac{1}{(2-\mu_A(x_i)-\mu_B(x_i))}] \tag{6}$$

$$M_2(A,B) = \sum_{i=1}^{n}[\frac{(\mu_A(x_i))^2+(\mu_B(x_i))^2}{\mu_A(x_i)+\mu_B(x_i)} - \sqrt{\mu_A(x_i)\mu_B(x_i)} + \sum_{i=1}^{n}[\frac{(1-\mu_A(x_i))^2+(1-\mu_B(x_i))^2}{2-\mu_A(x_i)-\mu_B(x_i)} - \sqrt{(1-\mu_A(x_i))(1-\mu_B(x_i))}] \tag{7}$$

$$M_3(A,B) = \sum_{i=1}^{n}(\mu_A(x_i) - \mu_B(x_i))^2[\frac{1}{(\mu_A(x_i)+\mu_B(x_i))} + \frac{1}{(2-\mu_A(x_i)-\mu_B(x_i))}] \tag{8}$$

Theorem 3.1 *$M_1(A,B), M_2(A,B)$ and $M_3(A,B)$ are valid measures of divergence between fuzzy sets.*

Proof Axiom d_1 and d_2 are obviously satisfied by $M_1(A,B)$, $M_2(A,B)$ and $M_3(A,B)$.

$$\frac{\partial^2 M_1(A,B)}{\partial^2 \mu_A(x_i)} = \frac{4(\mu_B(x_i))^2}{(\mu_A(x_i)+\mu_B(x_i))^4} + 2[\frac{4-(\mu_A(x_i))^2+(\mu_B(x_i))^2}{(2-\mu_A(x_i)-\mu_B(x_i))^2}] \geq 0$$

$$\frac{\partial^2 M_1(A,B)}{\partial^2 \mu_B(x_i)} = \frac{4(\mu_A(x_i))^2}{(\mu_A(x_i)+\mu_B(x_i))^4} + 2[\frac{4-(\mu_A(x_i))^2+(\mu_B(x_i))^2}{(2-\mu_A(x_i)-\mu_B(x_i))^2}] \geq 0$$

Therefore, $M_1(A,B)$ is convex function of fuzzy sets A and B. Hence in view of axioms d_1 and d_2 and the convexity property $M_1(A,B)$ is a valid measure of divergence between fuzzy sets A and B. Similarly $M_2(A,B)$ and $M_3(A,B)$ are also found to be valid measures of divergence between fuzzy sets A and B .

Theorem 3.2 *Let A and B be two fuzzy subsets of X then* $M_i(A\cup B, A\cap B)$ *=* $M_i(A,B)$ *, where i= 1,2,3 .*

Proof consider the sets

$$W_1 = \{x|x \in X,\ \mu_A(x_i) \geq \mu_B(x_i)\}$$

and

$$W_2 = \{x|x \in X,\ \mu_A(x_i) < \mu_B(x_i)\}$$

Using definitions in section (2.1), In set W_1,
$A\cup B$ = Union of A and B $\Leftrightarrow \mu_{A\cup B}(x) =$
$\max.\{\mu_A(x), \mu_B(x)\} = \mu_A(x)$ and $A\cap B$ = Intersection of A and B $\Leftrightarrow \mu_{A\cap B}(x)$
$= \min.\{\mu_A(x), \mu_B(x)\} = \mu_B(x)$
In set W_2,
$A\cup B$ = Union of A and B $\Leftrightarrow \mu_{A\cup B}(x) = \max.\{\mu_A(x), \mu_B(x)\} = \mu_B(x)$ and
$A\cap B$ = Intersection of A and B $\Leftrightarrow \mu_{A\cap B}(x) =$
$\min.\{\mu_A(x), \mu_B(x)\} = \mu_A(x)$
we have
$M_1(A\cup B, A\cap B) = \sum_{W_1}[\frac{(\mu_A(x_i)-\mu_B(x_i))^2}{2}[\frac{1}{(\mu_A(x_i)+\mu_B(x_i))} + \frac{1}{(2-\mu_A(x_i)-\mu_B(x_i))}]]$
$+ \sum_{W_2}[\frac{(\mu_A(x_i)-\mu_B(x_i))^2}{2}[\frac{1}{(\mu_A(x_i)+\mu_B(x_i))} + \frac{1}{(2-\mu_A(x_i)-\mu_B(x_i))}]]$
$= \sum_{X}[\frac{(\mu_A(x_i)-\mu_B(x_i))^2}{2}[\frac{1}{(\mu_A(x_i)+\mu_B(x_i))} + \frac{1}{(2-\mu_A(x_i)-\mu_B(x_i))}]]$
$=M_1(A,B)$
that is , $M_1(A\cup B, A\cap B) = M_1(A,B)$
similarly ,$M_2(A\cup B, A\cap B) = M_2(A,B)$ and $M_3(A\cup B, A\cap B) = M_3(A,B)$
Thus proof of theorem follows.
Theorem (3.2) shows that the divergence between $max.\{A,B\}$ and $min.\{A,B\}$ is same as divergence between A and B.

Theorem 3.3 *Let a universe of discourse* $X = \{x_1, x_2, x_3 ... x_n\}$ *,A be a crisp set and* A_F *be most fuzzy set then* $M_1(A, A_F) + M_3(A, A_F) = n$= *cardinality of X.*

Proof Since A_F is the most fuzzy set therefore $\mu_{A_F}(x_i) = 0.5$ therefore using equation (8)

$$M_1(A, A_F) = \sum_{i=1}^{n} \frac{(2\mu_A(x_i) - 1)^2}{4} [\frac{1}{(2\mu_A(x_i) + 1)} + \frac{1}{(3 - 2\mu_A(x_i))}] \quad (9)$$

Also , when A is a crisp set then $\mu_A(x_i) = 0 \ or \ 1$
consequently , $M_1(A, A_F) = \frac{n}{3}$
Similarly ,$M_3(A, A_F) = \frac{2n}{3}$
Thus proof of theorem follows.
Theorem(3.3) shows that for a given universe of discourse, the divergence of the most fuzzy set from an arbitrary fuzzy set remains constant.

Theorem 3.4 *Let a universe of discourse* $X = \{x_1, x_2, x_3 ... x_n\}$ *, A be fuzzy subset of A with complement* $\overline{A}$ *, then*
a. $Max.\ M_1(A, \overline{A}) = n$
b. $Max.\ M_2(A, \overline{A}) = 2n$
c. $Max.\ M_3(A, \overline{A}) = 2n$
This occurs when a is a non fuzzy (crisp)set.

Proof Using definitions in section (2.1), $\bar{A}$ = Compliment of A $\Leftrightarrow \mu_A(x) = 1 - \mu_A(x)$ from equations (8),(9),(10) the respective equations are

$$M_1(A, \overline{A}) = \sum_{i=1}^{n} (2\mu_A(x_i) - 1)^2 \quad (10)$$

$$M_2(A, \overline{A}) = \sum_{i=1}^{n} [(2\mu_A(x_i) - 1)^2 + (1 - \sqrt{\mu_A(x_i)(1 - \mu_A(x_i))}] \quad (11)$$

$$M_3(A, \overline{A}) = \sum_{i=1}^{n} 2(2\mu_A(x_i) - 1)^2 \quad (12)$$

when A is a crisp set then $\mu_A(x_i) = 0 \ or \ 1$
Thus for a crisp set A equations (10),(11) and (12) gives the desired result.
The divergence of an arbitrary fuzzy set is maximum from its compliment .

Theorem(3.4) shows that maximum value of divergence depends on the cardinality of the universe of discourse. Thus, on dividing the right-hand side of equations (6), (7) and (8) with n, 2n and 2n respectively we obtain three normalized divergence measures (value lie between 0 and 1) between fuzzy sets.

Theorem 3.5 *:Let* A *,*$B \in F(X)$ *then*
a. $M_i(A,B) = M_i(\overline{A},\overline{B})$
b. $M_i(A,\overline{A}) = M_i(\overline{A},A)$
c. $M_i(A,\overline{B}) = M_i(\overline{A},B)$
d.$M_i(A,B) + M_i(\overline{A},A) = M_i(\overline{A},\overline{B}) + M_i(A,\overline{B})$
where i= 1,2,3

Proof Proofs of (a) , (b) and (c) directly follow from the definition of $M_i(A,B)$ and (d) is a consequence of (b) and (c).

4 Relation of Divergence Measures with Aggregation Operations

4.1 Aggregation Operations

The aggregation operations for fuzzy sets are the operations by which several fuzzy sets are combined to produce a single set. e.g fuzzy union and fuzzy intersection are special cases of aggregation operations.

Definition 3 [Klir and Folger, 1988]: An aggregation operation is defined by the function $M : [0,1]^n \to [0,1]$ verifying
1.) $M(0,0,0...0) = 0$, $M(1,1,1...1) = 1$ (Boundary Conditions)
2.) M is Monotonic in each argument. (Monotonicity)

The use of monotone functions is justified in many decision making contexts, since it ensures consistency and reliability. The boundary conditions here are specified with the assumption that inputs are provided on the unit interval; however in certain cases, inputs naturally expressed on different intervals can be scaled appropriately. Aggregation functions are classed depending upon their behavior relative to the inputs. The most commonly used in application are averaging functions, which are usually interpreted as being representative of a given set of inputs or input vector.
An aggregation operation may be examined for some properties:

- Idempotent element
- Symmetry
- Neutral element
- Associativity
- Shift-Invariant
- Homogeneity
- Absorbing element
- Lipschitz's Continuous

4.2 Aggregation Operations and Divergence Measures between Fuzzy Sets

Theorem 4.1 *Let* $h_1, h_2, ..., h_n$ *be n mappings,* $h_i : [0,1] \times [0,1] \to [0,1]$ *for* $i = 1, 2, ..., n$ *verifying*
a. $h_i(x, x) = 0$ *for all* $x \in [0,1]$
b. $h_i(x, y) = h_i(y, x)$ *for all* $x, y \in [0,1]$
c. $h_i(1, 0) = h_i(0, 1) = 1$
d. $h_i(, y)$ *is non decreasing function over [0,y] and non decreasing over [y,1].*
Let Φ *be an aggregation operation. Then,*

$$D(A, B) = t\Phi(h_1(\mu_A(x_1), \mu_B(x_1)), h_2(\mu_A(x_2), \mu_B(x_2)), ..., h_n(\mu_A(x_n), \mu_B(x_n)))) \quad (13)$$

with $t > 0$ *is a divergence measure between fuzzy sets A and B.*

Proof Here Φ is an aggregation operation and let us see that D is a divergence measure.

- $D(A, B) = D(B, A)$ is trivial by given condition 2
- $D(A, A) = t\Phi(\overrightarrow{0}) = 0$ using condition 1 over h_i

- $D(A \cup C, B \cup C) = t\phi(h_1(\mu_{A\cup C}(x_1), \mu_{B\cup C}(x_1))),$
  $h_2(\mu_{A\cup C}(x_2), \mu_{B\cup C}(x_2)), ..., h_n(\mu_{A\cup C}(x_n), \mu_{B\cup C}(x_n)))$

Now it can be proved that $h_i(\mu_{A\cup C}(x_i), \mu_{B\cup C}(x_i))) \leq h_i(\mu_A(x_i), \mu_B(x_i))$ we have three cases:

1. If $\mu_C(x_i) \geq \mu_A(x_i), \mu_C(x_i) \geq \mu_B(x_i)$, then
   $h_i(\mu_{A\cup C}(x_i), \mu_{B\cup C}(x_i))) = h_i(\mu_C(x_i), \mu_C(x_i))) = 0$
   $\leq h_i(\mu_A(x_i), \mu_B(x_i)))$
2. If $\mu_A(x_i) \geq \mu_C(x_i), \geq \mu_B(x_i)$ or $\mu_B(x_i) \geq \mu_C(x_i), \geq \mu_A(x_i)$, then (considering the first possibility)
   $h_i(\mu_{A\cup C}(x_i), \mu_{B\cup C}(x_i))) = h_i(\mu_A(x_i), \mu_C(x_i)))$
   $\leq h_i(\mu_A(x_i), \mu_B(x_i)))$
   because $h_i(x, z) \geq h_i(x, y)$ if $x \geq y \geq z$ (using condition 4 on h_i).
3. If $\mu_A(x_i) \geq \mu_C(x_i), \mu_B(x_i) \geq \mu_C(x_i)$, then
   $h_i(\mu_{A\cup C}(x_i), \mu_{B\cup C}(x_i))) = h_i(\mu_A(x_i), \mu_B(x_i)))$

The same holds for the intersection. Hence $D(A, B)$ is a divergence measure between fuzzy sets A and B.

Remarks:

1. The constant $t > 0$ is needed to normalize the divergence measure $D(A, B)$
2. Divergence measure D obtained in theorem verify that: If $h_i(\mu_A(x_i), \mu_B(x_i))) = h_i(\mu_A(x_i), \mu_B(x_i)))$, for all i , then $D(A, B) = D(C, D)$ as Φ is applied over the same vectors.

Theorem 4.2 *Let D be a divergence measure. Then*

$$\Phi(\overrightarrow{p}) = \frac{D(A, \phi)}{D(X, \phi)} \tag{14}$$

with $\mu_A(x_i) = p_i$ *and* $\overrightarrow{p} = (p_1, p_2, ..., p_n)$ *is an aggregation operation.*

Proof We check the axioms of aggregation operation

- $\Phi(\overrightarrow{1}) = \frac{D(X,\phi)}{D(X,\phi)}$ and thus $\Phi(\overrightarrow{1}) = 1$.
  Now $\Phi(\overrightarrow{0}) = \frac{D(\phi,\phi)}{D(X,\phi)}$ and thus $\Phi(\overrightarrow{0}) = 0$.

- Let $\mu_A(x_i) = p_i$ and $\mu_B(x_i) = q_i$, $\overrightarrow{p} = (p_1, p_2, ..., p_n)$ and $\overrightarrow{q} = (q_1, q_2, ..., q_n)$ $\overrightarrow{p} \leq \overrightarrow{q} \Rightarrow p_i \leq q_i \Rightarrow \mu_A(x_i) \leq \mu_B(x_i))$
  Now using third condition of the divergence measure
$$D(B, \phi) \geq D(B \cap A, \phi \cap A) = D(A, \phi)$$
  we have
$$\Phi(\overrightarrow{p}) \leq \Phi(\overrightarrow{q})$$
  Hence Φ is an aggregation operation.

Example: Using the result of theorem(4.2) the divergence measures proposed in equations (8),(9) and (10) produces the aggregation functions Φ_1, Φ_2 and Φ_3 respectively as follows.

$$\Phi_1(\overrightarrow{p}) = \frac{1}{n}\sum_1^n \frac{p_i}{2 - p_i}$$

$$\Phi_2(\overrightarrow{p}) = \frac{1}{2n}\sum_1^n [p_i + \frac{1 + (1 - p_i)^2}{2 - p_i} - \sqrt{1 - p_i}]$$

$$\Phi_3(\overrightarrow{p}) = \frac{2}{n}\sum_1^n \frac{p_i}{2 - p_i}$$

5 Applications of Divergence Measures between Fuzzy Sets in Strategic Decision Making

Let the organization X want to apply m strategies $S_1, S_2, ..., S_m$ to meet a target. Let each strategy has varied degrees of effectiveness if cost associated with it is varied, let $\{C_1, C_2, ..., C_n\}$ be cost set . Let the fuzzy set X denotes the effectiveness of a particular strategy with uniform cost. Therefore

$$X = \{(X, \mu_X(S_i))|i = 1, 2, ...m\}$$

Further, let C_j be a fuzzy set denotes the degree of effectiveness of a strategy when it implemented with cost C_j.

$$C_j = \{(C_j, \mu_{C_j}(S_i))|i = 1, 2, ..., m\}$$

where j= 1,2...,n.
Taking $A = X$ and $B = C_j$ in the divergence measures defined in section 3,

and calculate $M_k(X, C_j)$ where k = 1,2,3.
Then $Min\{M_k(X, C_j)\}_{1 \le j \le n}$ determines the suitability of C_j. Let the minimum value is attained at C_t , $1 \le t \le n$ .With this C_t find $Max.\{\mu_{C_t}(S_i)\}_{1 \le i \le m}$, let it correspond to S_p ,$1 \le p \le m$.
Thus if the strategy S_p is implemented with budget of C_t the organization will meet its target in the most cost-effective manner.

Illustrative Example

Let m = n = 5 in the above model.
Table 1 shows the effectiveness of strategies at uniform cost and Table 2 shows the effectiveness of strategies at particular cost.

Table 1:

$\mu_X(S_1)$	$\mu_X(S_2)$	$\mu_X(S_3)$	$\mu_X(S_4)$	$\mu_X(S_5)$
0.8	0.9	0.6	0.5	0.7

Table 2:

C_j	$\mu_{C_j}(S_1)$	$\mu_{C_j}(S_2)$	$\mu_{C_j}(S_3)$	$\mu_{C_j}(S_4)$	$\mu_{C_j}(S_5)$
C_1	0.6	0.7	0.5	0.8	0.7
C_2	0.9	0.8	0.6	0.4	0.8
C_3	0.7	0.6	0.8	0.4	0.6
C_4	0.6	0.8	0.6	0.7	0.9
C_5	0.3	0.7	0.4	0.6	0.5

Table 3 shows the divergence between X and C_j , j = 1,2,3,4,5.

Table 3

C_j	$M_1(X, C_j)$	$M_2(X, C_j)$	$M_3(X, C_j)$
C_1	0.21912	0.33290	0.43824
C_2	0.06265	0.09453	0.12530
C_3	0.20204	0.30883	0.40408
C_4	0.17139	0.25994	0.34278
C_5	0.40679	0.62216	0.81358

According to the divergence measures presented in the Table 3 budget C_2 is more suitable and after examining the Table 2, it is observed that strategy S_1 is most effective. Therefore, the organization will achieve its target in the most cost-effective manner if the strategy S_1 is implemented with a budget C_2. Apart from application in decision making the divergence measures between fuzzy sets have applications in Bio-informatics [Poletti *et al.*, 2012], image thresholding [Fan *et al.*, 2011], [Bhatia and Singh, 2013].

6 Conclusion and Discussion

Three new divergence measures between fuzzy sets have been proposed in the present paper. Some properties of these divergence measures are established. Application of these measures has been studied in strategic decision making. Relation of divergence measures with aggregation operations established. Further, In order to provide more flexibility of applications in certain situations one or two parametric generalization of this type of divergence measures between fuzzy sets may be possible. The characterization of divergence measures between fuzzy sets is under study and will be reported separately.

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