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Some entropy measures of interval-valued intuitionistic fuzzy sets and their applications

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Abstract. The entropy of intuitionistic fuzzy sets is used to indicate the degree of fuzziness of an interval-valued intuitionistic fuzzy set(IvIFS). In this paper, we deal with the entropies of IvIFSs. Firstly, we propose a set of entropies on IvIFSs with a parameter $\lambda \in [0, +\infty)$, which generalizes the entropy measure defined by Gao, for IvIFSs, and then we prove that the new entropy is an increasing function with respect to the parameter λ. Finally, some numerical examples are given to illustrate the applications of the proposed entropy measures to pattern recognition.

Keywords. Interval-valued intuitionistic fuzzy sets; Interval-valued intuitionistic fuzzy entropy; Entropy measure

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1 Introduction

The intuitionistic fuzzy set(IFS) has been studied and applied in a variety of fields([2]-[5]). The character of IFS is that the values of its membership function and non-membership function are real numbers. However, in many applications, due to the increasing complexity of the social-economic environment and a lack of knowledge or data about the problem domains, the decision information may be provided with intervals, instead of real numbers. Thus, interval-valued intuitionistic fuzzy set(IvIFS), as a useful generation of IFS, was introduced by Atanassov[6], which is characterized by a membership function and a non-membership function whose values are intervals rather than real numbers. Thus, IvIFS is more flexible and practical than IFS in coping with fuzziness and uncertainty.

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Entropy measures is an important topic in the fuzzy set theory, and has been investigated by many researchers. Entropy as a measure of fuzziness was first mentioned by Zadeh[7]. Later, Deluca and Termini[8] presented some axioms to describe the fuzziness degree of fuzzy set, with which a fuzzy entropy based on Shannon's function was proposed. Szimidt and Kacprzyk [9] proposed a nonprobabilistic-type entropy measure with a geometric interpretation of IFSs, and many other entropy formulas can be found in [10,11,15,16].

As an important extension of IFS, IvIFS has many applications in real life. However, there is little investigation on the information measures of IvIFS, although some entropy measures have been presented in [15,16,21,22] recently. In this paper, motivated by the entropy formula given by Gao[21], we propose a set of entropies for IvIFSs, which extends Gao's entropy formula by introducing a parameter. We point out that the two entropies given by Ye[14] are equal and show some drawbacks of Ye's entropy.

The rest of the paper is structured as follows. In the next section, we introduce some basic concepts related to IvIFSs. Then we develop a set of entropies of IvIFSs, and some comparisons between the entropies are given. Some numerical examples are presented in Section 3 to illustrate the effective of the new entropy measures. Some conclusions are presented in the finial section.

2 A set of entropies on IvIFSs

In this paper, let $[I]$ denote the set of all the closed subintervals of $[0, 1]$.

Definition 2.1.[15] Let $[a_1, b_1], [a_2, b_2] \in [I]$, we define

$$[a_1, b_1] \leq [a_2, b_2], \text{ iff } a_1 \leq a_2 \text{ and } b_1 \leq b_2; [a_1, b_1] = [a_2, b_2], \text{ iff } a_1 = a_2 \text{ and } b_1 = b_2.$$

Definition 2.2.[6] An interval-valued intuitionistic fuzzy set(IvIFS) A in the finite universe X is expressed by the form

$$A = \{\langle x, \mu_A(x), v_A(x)\rangle | x \in X\},$$

where $\mu_A(x) = [\mu_A^-(x), \mu_A^+(x)] \in [I]$ is called membership interval of element x to IvIFS A, while $v_A(x) = [v_A^-(x), v_A^+(x)] \in [I]$ is the non-membership interval of that element to the set A, and the condition $0 \leq \mu_A^+(x) + v_A^+(x) \leq 1$ must hold for any $x \in X$.

For convenience of notations, we denote by IvIFS(X) the set of all the IvIFS in X. We call the interval $[1 - \mu_A^+(x) - v_A^+(x), 1 - \mu_A^-(x) - v_A^-(x)]$, abbreviated by $[\pi_A^-(x), \pi_A^+(x)]$ and denoted by $\pi_A(x)$, the interval-valued intuitionistic index of x in A, which is a hesitancy degree of x to A.

Definition 2.3.[15] Let $A, B \in$IvIFS(X), where $X = \{x_1, x_2, \ldots, x_n\}$, then some operations can be defined as follows:

$$A^C = \{\langle x_i, [v_A^-(x_i), v_A^+(x_i)], [\mu_A^-(x_i), \mu_A^+(x_i)]\rangle | x_i \in X\};$$

$$A \subseteq B \text{ iff } [\mu_A^-(x_i), \mu_A^+(x_i)] \le [\mu_B^-(x_i), \mu_B^+(x_i)] \text{ and } [v_A^-(x_i), v_A^+(x_i)] \ge [v_B^-(x_i) v_B^+(x_i)] \quad x_i \in X.$$

Now we give the entropy concept of IvIFS which is similar to the work of Zhang et al.[15].

Definition 2.4. A real function $E : \text{IvIFS}(X) \to [0,1]$ is named an entropy on IvIFSs, if E satisfies all the following properties:

(E1) $E(A) = 0$ iff A is a crisp set; (E2) $E(A) = 1$ iff $\mu_A(x_i) = v_A(x_i)$, $\forall x_i \in X$;

(E3) $E(A) = E(A^C)$; (E4) $E(A) \le E(B)$ if A is less fuzzy than B, which is defined as

$$\mu_A(x_i) \le \mu_B(x_i), v_A(x_i) \ge v_B(x_i), \text{ for } \mu_B(x_i) \le v_B(x_i);$$

$$\mu_A(x_i) \ge \mu_B(x_i), v_A(x_i) \le v_B(x_i), \text{ for } \mu_B(x_i) \ge v_B(x_i).$$

In [14], Ye introduced two fuzzy entropies formula for IvIFS by

$$E_{\text{Y}1}(A) = \frac{1}{n}\sum_{i=1}^{n}\{[\sin\frac{1+\mu_A^-(x_i)+pW_\mu(x_i)-v_A^-(x_i)-qW_v(x_i)}{4}\pi$$
$$+\sin\frac{1-\mu_A^-(x_i)-pW_\mu(x_i)+v_A^-(x_i)+qW_v(x_i)}{4}\pi - 1] \times \frac{1}{\sqrt{2}-1}\} \tag{1}$$

and

$$E_{\text{Y}2}(A) = \frac{1}{n}\sum_{i=1}^{n}\{[\cos\frac{1+\mu_A^-(x_i)+pW_\mu(x_i)-v_A^-(x_i)-qW_v(x_i)}{4}\pi$$
$$+\cos\frac{1-\mu_A^-(x_i)-pW_\mu(x_i)+v_A^-(x_i)+qW_v(x_i)}{4}\pi - 1] \times \frac{1}{\sqrt{2}-1}\}, \tag{2}$$

where $W_\mu(x_i) = \mu_A^+(x_i) - \mu_A^-(x_i), W_v(x_i) = v_A^+(x_i) - v_A^-(x_i)$, and $p, q \in [0,1]$ are two fixed numbers.

The following result can be found in [21], and we give a proof in detail for completeness.

Theorem 2.1. The entropies $E_{\text{Y}1}(A)$ and $E_{\text{Y}2}(A)$ are equal. That is, for each IvIFS A, $E_{\text{Y}1}(A) = E_{\text{Y}2}(A)$, and they are equivalent to the following formula:

$$E_{\text{Y}}(A) = \frac{1}{n}\sum_{i=1}^{n}\{[\sqrt{2}\cos\frac{\mu_A^-(x_i)+pW_\mu(x_i)-v_A^-(x_i)-qW_v(x_i)}{4}\pi - 1] \times \frac{1}{\sqrt{2}-1}\}. \tag{3}$$

Proof. Set $\Delta_1 = \frac{1+\mu_A^-(x_i)+pW_\mu(x_i)-v_A^-(x_i)-qW_v(x_i)}{4}\pi, \Delta_2 = \frac{1-\mu_A^-(x_i)-pW_\mu(x_i)+v_A^-(x_i)+qW_v(x_i)}{4}\pi$. Then, from the property of the trigonometric function, we get

$$E_{\text{Y}1}(A) = \frac{1}{n}\sum_{i=1}^{n}\{[\sin\Delta_1 + \cos(\frac{\pi}{2} - \Delta_2) - 1] \times \frac{1}{\sqrt{2}-1}\} = \frac{1}{n}\sum_{i=1}^{n}\{[\sin\Delta_1 + \cos\Delta_1 - 1] \times \frac{1}{\sqrt{2}-1}\}.$$

Similarly, we have

$$E_{\text{Y}2}(A) = \frac{1}{n}\sum_{i=1}^{n}\{[\cos\Delta_1 + \sin(\frac{\pi}{2} - \Delta_2) - 1] \times \frac{1}{\sqrt{2}-1}\} = \frac{1}{n}\sum_{i=1}^{n}\{[\cos\Delta_1 + \sin\Delta_1 - 1] \times \frac{1}{\sqrt{2}-1}\}.$$

Thus, $E_{\mathrm{Y1}}(A) = E_{\mathrm{Y2}}(A)$. Furthermore, we can simplify $E_{\mathrm{Y1}}(A)$ as follows:

$$\begin{aligned} E_{\mathrm{Y1}}(A) &= \frac{1}{n}\sum_{i=1}^{n}\{[\sqrt{2}(\frac{\sqrt{2}}{2}\sin\Delta_1 + \frac{\sqrt{2}}{2}\cos\Delta_1) - 1] \times \frac{1}{\sqrt{2}-1}\} \\ &= \frac{1}{n}\sum_{i=1}^{n}\{[\sqrt{2}\sin(\Delta_1 + \frac{\pi}{4}) - 1] \times \frac{1}{\sqrt{2}-1}\} = \frac{1}{n}\sum_{i=1}^{n}\{[\sqrt{2}\cos(\Delta_1 - \frac{\pi}{4}) - 1] \times \frac{1}{\sqrt{2}-1}\}. \end{aligned}$$

The completes the proof.

Example 2.1. Assume that $X = \{x\}$ and two IvIFS $A_1 = \{\langle x, [0.1, 0.2], [0.3, 0.4]\rangle\}$, and $A_2 = \{\langle x, [0.2, 0.3], [0.4, 0.5]\rangle\}$. We adopt $E_{\mathrm{Y}}(A)$ to calculate the entropies of A_1, A_2.

Intuitively, we can see that A_1 is more fuzzy than A_2. Set $p = q = 0.5$(medium value) in (3), we have

$$E_{\mathrm{Y}}(A) = \frac{1}{n}\sum_{i=1}^{n}\{[\sqrt{2}\cos\frac{\mu_A^-(x_i) + \mu_A^+(x_i) - v_A^-(x_i) - v_A^+(x_i)}{8}\pi - 1] \times \frac{1}{\sqrt{2}-1}\}. \tag{4}$$

From the entropy formula (3), we can get $E_Y(A_1) = E_Y(A_2) = 0.9580$, which is inconsistent with our intuition. The reason is that the entropy $E_{\mathrm{Y}}(A)$ only contains the difference between the membership degree and the nonmembership degree, and it does not contain the hesitancy degree.

Certainly, Gao[21] also noted these drawbacks of $E_{\mathrm{Y}}(A)$. In order to overcome them, Gao[21] proposed the following entropy for IvIFS by incorporating the hesitancy degree into the entropy formula:

$$E_{\mathrm{G}}(A) = \frac{1}{n}\sum_{i=1}^{n}\cos\frac{|\mu_A^-(x_i) - v_A^-(x_i)| + |\mu_A^+(x_i) - v_A^+(x_i)|}{2(2 + \pi_A^-(x_i) + \pi_A^+(x_i))}\pi. \tag{5}$$

The entropy measure $E_{\mathrm{G}}(A)$ reflects not only the difference between the membership degree and the nonmembership degree, but also the hesitancy degree. Thus it can measure the fuzziness and intuitionism of IvIFSs more comprehensively. However, the following example shows that the above entropy cannot distinguish the fuzzyness of some IvIFSs in some cases.

Example 2.2. Assume that $X = \{x\}$ and two IvIFS $A_3 = \{\langle x, [0.1, 0.2], [0.2, 0.5]\rangle\}$, and $A_4 = \{\langle x, [0.1, 0.2], [0.1, 0.6]\rangle\}$. We adopt $E_{\mathrm{G}}(A)$ to calculate the entropies of A_3, A_4.

From the entropy formula $E_{\mathrm{G}}(A)$, we have:

$$E_{\mathrm{G}}(A_3) = \frac{1}{n}\sum_{i=1}^{n}\cos\frac{0.1 + 0.3}{2(2+1)}\pi = \cos\frac{\pi}{15}, E_{\mathrm{G}}(A_4) = \frac{1}{n}\sum_{i=1}^{n}\cos\frac{0 + 0.4}{2(2+1)}\pi = \cos\frac{\pi}{15}.$$

Therefore $E_{\mathrm{G}}(A_3) = E_{\mathrm{G}}(A_4)$, then $E_{\mathrm{G}}(A)$ can not distinguish the fuzzyness of A_3 and A_4.

To enhance the distinguishing ability of the $E_{\mathrm{G}}(A)$, we propose the following entropy formula:

$$E(A, \lambda) = \frac{1}{n}\sum_{i=1}^{n}\cos\frac{|\mu_A^-(x_i) - v_A^-(x_i)| + |\mu_A^+(x_i) - v_A^+(x_i)|}{2(2 + \pi_A^-(x_i) + \pi_A^+(x_i) + \lambda\min\{\mu_A^-(x_i) + \mu_A^+(x_i), v_A^-(x_i) + v_A^+(x_i)\})}\pi, \tag{6}$$

where $\lambda \in [0, +\infty)$ is a parameter. Obviously, if $\lambda = 0$, then $E(A, \lambda)$ is reduced to the entropy $E_{\mathrm{G}}(A)$.

Remark 2.1 In fact, the different entropies may affect some specific application problems, such as pattern recognition and MADM[24]. For a MADM problem with alternatives expressed by IvIFSs, the optimists may choose the greater λ compared with the pessimists. On the contrary, the pessimists may choose the smaller λ. Therefore, the chosen λ of Equation (6) depends on the specific application.

Theorem 2.2. The mapping $E(A,\lambda)$, defined by (6), is an entropy measure for IvIFS, i.e., it satisfies all the properties in Definition 2.4.

Proof. We only need to prove that all the properties in Definition 2.4 hold. Firstly, we set

$$\tilde{E}_i(A,\lambda) = \cos(E_i(A,\lambda)\pi), \tag{7}$$

where

$$E_i(A,\lambda) = \frac{|\mu_A^-(x_i) - v_A^-(x_i)| + |\mu_A^+(x_i) - v_A^+(x_i)|}{2(2 + \pi_A^-(x_i) + \pi_A^+(x_i) + \lambda \min\{\mu_A^-(x_i) + \mu_A^+(x_i), v_A^-(x_i) + v_A^+(x_i)\})},$$

then

$$E(A,\lambda) = \frac{1}{n}\sum_{i=1}^{n} \tilde{E}_i(A,\lambda). \tag{8}$$

Obviously, $E_i(A,\lambda) \geq 0$. In the following, we prove that $E_i(A,\lambda) \leq 1/2$. Since

$$E_i(A,\lambda) \leq \frac{|\mu_A^-(x_i) - v_A^-(x_i)| + |\mu_A^+(x_i) - v_A^+(x_i)|}{2(2 + \pi_A^-(x_i) + \pi_A^+(x_i))} \doteq F_i(A,\lambda).$$

Therefore, we only need to prove that $F_i(A,\lambda) \leq 1/2$. If $\mu_A^-(x_i) \geq v_A^-(x_i), \mu_A^+(x_i) \geq v_A^+(x_i)$, then

$$F_i(A,\lambda) = \frac{\mu_A^-(x_i) - v_A^-(x_i) + \mu_A^+(x_i) - v_A^+(x_i)}{2(4 - \mu_A^-(x_i) - \mu_A^+(x_i) - v_A^-(x_i) - v_A^+(x_i))},$$

It is easy to get that $F_i(A,\lambda)$ is increasing with respect to $\mu_A^-(x_i)$ and $\mu_A^+(x_i)$. Next we prove that $F_i(A,\lambda)$ is decreasing with respect to $v_A^-(x_i)$ and $v_A^+(x_i)$. Taking the partial derivation of $F_i(A,\lambda)$ with respect to $v_A^-(x_i)$ and $v_A^+(x_i)$ respectively, yields

$$\frac{\partial F_i(A,\lambda)}{\partial \mu_A^-(x_i)} = \frac{-4 + 2(\mu_A^-(x_i) + \mu_A^+(x_i))}{2(4 - \mu_A^-(x_i) - \mu_A^+(x_i) - v_A^-(x_i) - v_A^+(x_i))^2} \leq 0.$$

$$\frac{\partial F_i(A,\lambda)}{\partial \mu_A^+(x_i)} = \frac{-4 + 2(\mu_A^-(x_i) + \mu_A^+(x_i))}{2(4 - \mu_A^-(x_i) - \mu_A^+(x_i) - v_A^-(x_i) - v_A^+(x_i))^2} \leq 0.$$

Then, in its definition region, the function $F_i(A,\lambda)$ reaches maximum when $\mu_A(x_i) = [1,1], v_A(x_i) = [0,0]$. That is: $E_i(A,\lambda) \leq F_i(A,\lambda) \leq \frac{1-0+1-0}{2(4-1-1-0-0)} = \frac{1}{2}$ for all $x_i \in X$. Similarly, we can get the same result when $\mu_A^-(x_i) \geq v_A^-(x_i), \mu_A^+(x_i) \leq v_A^+(x_i)$, or $\mu_A^-(x_i) \leq v_A^-(x_i), \mu_A^+(x_i) \geq v_A^+(x_i)$, or $\mu_A^-(x_i) \leq v_A^-(x_i), \mu_A^+(x_i) \leq v_A^+(x_i)$.

(E1) If A is a crisp set, then for any $x_i \in X$, we know

$$\mu_A(x_i) = [1,1], v_A(x_i) = [0,0] \quad \text{or} \quad \mu_A(x_i) = [0,0], v_A(x_i) = [1,1],$$

then $\pi_A(x_i) = [0,0]$ for each $x_i \in X$. From (6), we obtain that $E(A,\lambda) = 0$.

On the other hand, now suppose that $E(A,\lambda) = 0$. Since every term in the summation of (8) is non-negative, we deduce that every term should be zero, i.e., $\tilde{E}_i(A,\lambda) = 0$ for each $x_i \in X$. Then from (7) and $0 \le E_i(A,\lambda) \le 1/2$, we get

$$E_i(A,\lambda) = \frac{|\mu_A^-(x_i) - v_A^-(x_i)| + |\mu_A^+(x_i) - v_A^+(x_i)|}{2(2 + \pi_A^-(x_i) + \pi_A^+(x_i) + \lambda \min\{\mu_A^-(x_i) + \mu_A^+(x_i), v_A^-(x_i) + v_A^+(x_i)\})} = \frac{1}{2}. \quad (9)$$

That is

$$\begin{aligned} & |\mu_A^-(x_i) - v_A^-(x_i)| + |\mu_A^+(x_i) - v_A^+(x_i)| \\ = \; & 4 - (\mu_A^-(x_i) + v_A^-(x_i) + \mu_A^+(x_i) + v_A^+(x_i)) + \lambda \min\{\mu_A^-(x_i) + \mu_A^+(x_i), v_A^-(x_i) + v_A^+(x_i)\} \end{aligned}$$

which combines the equation $|a-b| + (a+b) = 2\max\{a,b\}$ for any $a, b \in R$ leads to

$$2\max\{\mu_A^-(x_i), v_A^-(x_i)\} + 2\max\{\mu_A^+(x_i), v_A^+(x_i)\} = 4 + \lambda \min\{\mu_A^-(x_i) + \mu_A^+(x_i), v_A^-(x_i) + v_A^+(x_i)\}.$$

Then from $0 \le \mu_A^-(x_i) \le \mu_A^+(x_i) \le 1, 0 \le v_A^-(x_i) \le v_A^+(x_i) \le 1$ and $\lambda \ge 0$, we can obtain

$$\max\{\mu_A^-(x_i), v_A^-(x_i)\} = 1, \max\{\mu_A^+(x_i), v_A^+(x_i)\} = 1 \quad (10)$$

and

$$\lambda \min\{\mu_A^-(x_i) + \mu_A^+(x_i), v_A^-(x_i) + v_A^+(x_i)\} = 0. \quad (11)$$

The first equation in (10) means that

$$\mu_A^-(x_i) = 1, \quad \text{or} \quad v_A^-(x_i) = 1.$$

If $\mu_A^-(x_i) = 1$, then from $0 \le \mu_A^-(x_i) \le \mu_A^+(x_i) \le 1, 0 \le v_A^-(x_i) \le v_A^+(x_i) \le 1$ and $\mu_A^+(x_i) + v_A^+(x_i) \le 1$, we have $\mu_A^-(x_i) = \mu_A^+(x_i) = 1, v_A^-(x_i) = v_A^+(x_i) = 0$. Similarly, if $v_A^-(x_i) = 1$, then from $0 \le \mu_A^-(x_i) \le \mu_A^+(x_i) \le 1, 0 \le v_A^-(x_i) \le v_A^+(x_i) \le 1$ and $\mu_A^+(x_i) + v_A^+(x_i) \le 1$, we have $\mu_A^-(x_i) = \mu_A^+(x_i) = 0, v_A^-(x_i) = v_A^+(x_i) = 1$. Thus,

$$\mu_A(x_i) = [1,1], v_A(x_i) = [0,0] \quad \text{or} \quad \mu_A(x_i) = [0,0], v_A(x_i) = [1,1]$$

any $x_i \in X$. This indicates that A is a crisp set.

(E2) Let $\mu_A(x_i) = v_A(x_i)$ for $x_i \in X$, i.e., $\mu_A^-(x_i) = v_A^-(x_i)$ and $\mu_A^+(x_i) = v_A^+(x_i)$. From Equation(6), we obtain $E(A,\lambda) = 1$.

Now suppose that $E(A,\lambda) = 1$. From (8) and $0 \le \tilde{E}_i(A,\lambda) \le 1$, we obtain that $\tilde{E}_i(A,\lambda) = 1$ for all $x_i \in X$. Then from (7) and $0 \le E_i(A,\lambda) \le 1/2$, we have

$$E_i(A,\lambda) = \frac{|\mu_A^-(x_i) - v_A^-(x_i)| + |\mu_A^+(x_i) - v_A^+(x_i)|}{2(2 + \pi_A^-(x_i) + \pi_A^+(x_i) + \lambda \min\{\mu_A^-(x_i) + \mu_A^+(x_i), v_A^-(x_i) + v_A^+(x_i)\})} = 0$$

for all $x_i \in X$. Therefore, $\mu_A^-(x_i) = v_A^-(x_i)$ and $\mu_A^+(x_i) = v_A^+(x_i)$ for each $x_i \in X$, which implies that $\mu_A(x_i) = v_A(x_i)$ for $x_i \in X$.

(E3) It is clear that $A^C = \{\langle x_i, [v_A^-(x_i), v_A^+(x_i)], [\mu_A^-(x_i), \mu_A^+(x_i)]\rangle | x_i \in X\}$, i.e., $\mu_{A^C}(x_i) = v_A(x_i) = [v_A^-(x_i), v_A^+(x_i)]$, and $v_{A^C}(x_i) = \mu_A(x_i) = [\mu_A^-(x_i), \mu_A^+(x_i)]$. By applying (6), we have $E(A^C, \lambda) = E(A, \lambda)$.

(E4) Suppose that $\mu_B(x_i) \leq v_B(x_i)$, i.e., $\mu_B^-(x_i) \leq v_B^-(x_i), \mu_B^+(x_i) \leq v_B^+(x_i)$ and $\mu_A(x_i) \leq \mu_B(x_i), v_A(x_i) \geq v_B(x_i)$, i.e.,

$$\mu_A^-(x_i) \leq \mu_B^-(x_i), \mu_A^+(x_i) \leq \mu_B^+(x_i), v_A^-(x_i) \geq v_B^-(x_i), v_A^+(x_i) \geq v_B^+(x_i) \tag{12}$$

for each $x_i \in X$. Then it follows that $\mu_A^-(x_i) \leq v_A^-(x_i)$ and $\mu_A^+(x_i) \leq v_A^+(x_i)$. Firstly, we prove that

$$E_i(A, \lambda) \geq E_i(B, \lambda).$$

Assume that the above inequality does not right. Then we have

$$\frac{v_A^-(x_i) - \mu_A^-(x_i) + v_A^+(x_i) - \mu_A^+(x_i)}{2(2 + \pi_A^-(x_i) + \pi_A^+(x_i) + \lambda\mu_A^-(x_i) + \lambda\mu_A^+(x_i))} < \frac{v_B^-(x_i) - \mu_B^-(x_i) + v_B^+(x_i) - \mu_B^+(x_i)}{2(2 + \pi_B^-(x_i) + \pi_B^+(x_i) + \lambda\mu_B^-(x_i) + \lambda\mu_B^+(x_i))},$$

That is

$$\begin{aligned} & (v_A^-(x_i) - \mu_A^-(x_i) + v_A^+(x_i) - \mu_A^+(x_i))(2 + \pi_B^-(x_i) + \pi_B^+(x_i) + \lambda\mu_B^-(x_i) + \lambda\mu_B^+(x_i)) \\ < \; & (v_B^-(x_i) - \mu_B^-(x_i) + v_B^+(x_i) - \mu_B^+(x_i))(2 + \pi_A^-(x_i) + \pi_A^+(x_i) + \lambda\mu_A^-(x_i) + \lambda\mu_A^+(x_i)). \end{aligned}$$

From the above inequality, we can deduce that

$$\begin{aligned} & (\mu_A^-(x_i) + \mu_A^+(x_i))(2v_B^-(x_i) + 2v_B^+(x_i) - 4) + (v_A^-(x_i) + v_A^+(x_i))\times \\ & (4 - 2\mu_B^-(x_i) - 2\mu_B^+(x_i)) + 4(\mu_B^-(x_i) + \mu_B^+(x_i)) - 4(v_B^-(x_i) + v_B^+(x_i)) \\ & +\lambda[(v_A^-(x_i) + v_A^+(x_i))(\mu_B^-(x_i) + \mu_B^+(x_i)) - (v_B^-(x_i) + v_B^+(x_i))(\mu_A^-(x_i) + \mu_A^+(x_i))] < 0. \end{aligned} \tag{13}$$

From inequality (12), we have $\mu_A^-(x_i) + \mu_A^+(x_i) \leq \mu_B^-(x_i) + \mu_B^+(x_i)$ and $v_A^-(x_i) + v_A^+(x_i) \geq v_B^-(x_i) + v_B^+(x_i)$. Thus

$$(\mu_A^-(x_i) + \mu_A^+(x_i))(2v_B^-(x_i) + 2v_B^+(x_i) - 4) \geq (\mu_B^-(x_i) + \mu_B^+(x_i))(2v_B^-(x_i) + 2v_B^+(x_i) - 4)$$

and

$$(v_A^-(x_i) + v_A^+(x_i))(4 - 2\mu_B^-(x_i) - 2\mu_B^+(x_i)) \geq (v_B^-(x_i) + v_B^+(x_i))(4 - 2\mu_B^-(x_i) - 2\mu_B^+(x_i)).$$

From inequality (12) again, we have $v_A^-(x_i) + v_A^+(x_i) \geq v_B^-(x_i) + v_B^+(x_i)$ and $\mu_B^-(x_i) + \mu_B^+(x_i) \geq \mu_A^-(x_i) + \mu_A^+(x_i)$. Thus

$$(v_A^-(x_i) + v_A^+(x_i))(\mu_B^-(x_i) + \mu_B^+(x_i)) - (v_B^-(x_i) + v_B^+(x_i))(\mu_A^-(x_i) + \mu_A^+(x_i)) \geq 0.$$

Substitute the above three inequalities into the left side of (10), we obtain that

$$\begin{aligned}
& (\mu_A^-(x_i)+\mu_A^+(x_i))(2v_B^-(x_i)+2v_B^+(x_i)-4)+(v_A^-(x_i)+v_A^+(x_i))\times \\
& (4-2\mu_B^-(x_i)-2\mu_B^+(x_i))+4(\mu_B^-(x_i)+\mu_B^+(x_i))-4(v_B^-(x_i)+v_B^+(x_i)) \\
& +\lambda[(v_A^-(x_i)+v_A^+(x_i))(\mu_B^-(x_i)+\mu_B^+(x_i))-(v_B^-(x_i)+v_B^+(x_i))(\mu_A^-(x_i)+\mu_A^+(x_i))] \\
\geq\ & (\mu_B^-(x_i)+\mu_B^+(x_i))(2v_B^-(x_i)+2v_B^+(x_i)-4)+(v_B^-(x_i)+v_B^+(x_i))(4-2\mu_B^-(x_i)-2\mu_B^+(x_i)) \\
& +4(\mu_B^-(x_i)+\mu_B^+(x_i))-4(v_B^-(x_i)+v_B^+(x_i)) \\
=\ & 0,
\end{aligned}$$

which contradicts with the inequality (13). This contradiction implies that $E_i(A,\lambda)\geq E_i(B,\lambda)$, for all $x_i\in X$. Since the function $\cos(\cdot)$ is decreasing on the interval $[0,\pi/2]$, from (7), we have $\tilde{E}_i(A,\lambda)\leq \tilde{E}_i(B,\lambda)$, for all $x_i\in X$. Then from (8), we have $E(A,\lambda)\leq E(B,\lambda)$.

Similarly, when $\mu_B(x_i)\leq v_B(x_i)$, and $\mu_A(x_i)\leq \mu_B(x_i), v_A(x_i)\geq v_B(x_i)$ for each $x_i\in X$, we can also prove that $E(A,\lambda)\leq E(B,\lambda)$. This completes the proof.

Theorem 2.3. The proposed entropy $E(A,\lambda)$ is a increasing function of $\lambda(\lambda\in[0,+\infty))$.

Proof. Obviously.

3 Illustrative examples

In this section, we consider the following examples to illustrate the applications of the proposed measures of IvIFSs to pattern recognition problems. For two given IvIFSs

$$A=\{\langle x,[\mu_A^-(x),\mu_A^+(x)],[v_A^-(x),v_A^+(x)]\rangle | x\in X\},$$

$$B=\{\langle x,[\mu_B^-(x),\mu_B^+(x)],[v_B^-(x),v_B^+(x)]\rangle | x\in X\},$$

based on the entropy (6) and the transformation method in [16], we can get the following similarity measures:

$$S(A,B)=\frac{1}{n}\sum_{i=1}^{n}\cos\frac{AB1+AB2}{4+\lambda(2-AB1-AB2)}\pi, \tag{14}$$

where

$$AB1=|\mu_A^- - \mu_B^-|\vee|v_A^- - v_B^-|, AB2=|\mu_A^+ - \mu_B^+|\vee|v_A^+ - v_B^+|.$$

Example Let us consider the following pattern recognition problem as discussed in [15]. Assume $R_i(i=1,2,\ldots,5)$ are given five known patterns, which correspond to five decision alternatives $d_i(i=1,2,\ldots,5)$, respectively. The patterns are denoted by the following IvIFSs in $X=\{x_1,x_2,x_3,x_4\}$.

$R_1=\{\langle x_1,[0.4,0.5],[0.3,0.4]\rangle,\langle x_2,[0.4,0.6],[0.2,0.4]\rangle,\langle x_3,[0.3,0.4],[0.4,0.5]\rangle,\langle x_4,[0.5,0.6],[0.1,0.3]\rangle,$

$R_2 = \{\langle x_1, [0.5, 0.6], [0.2, 0.3]\rangle, \langle x_2, [0.6, 0.7], [0.2, 0.3]\rangle, \langle x_3, [0.5, 0.6], [0.3, 0.4]\rangle, \langle x_4, [0.4, 0.7], [0.1, 0.2]\rangle,$

$R_3 = \{\langle x_1, [0.3, 0.5], [0.3, 0.4]\rangle, \langle x_2, [0.1, 0.3], [0.5, 0.6]\rangle, \langle x_3, [0.2, 0.5], [0.4, 0.5]\rangle, \langle x_4, [0.2, 0.3], [0.4, 0.6]\rangle,$

$R_4 = \{\langle x_1, [0.2, 0.5], [0.3, 0.4]\rangle, \langle x_2, [0.4, 0.7], [0.1, 0.2]\rangle, \langle x_3, [0.4, 0.5], [0.3, 0.5]\rangle, \langle x_4, [0.5, 0.8], [0.1, 0.2]\rangle,$

$R_5 = \{\langle x_1, [0.3, 0.4], [0.1, 0.3]\rangle, \langle x_2, [0.7, 0.8], [0.1, 0.2]\rangle, \langle x_3, [0.5, 0.6], [0.2, 0.4]\rangle, \langle x_4, [0.6, 0.7], [0.1, 0.2]\rangle.$

Given an unknown sample

$$R = \{\langle x_1, [0.5, 0.6], [0.1, 0.3]\rangle, \langle x_2, [0.7, 0.8], [0.1, 0.2]\rangle, \langle x_3, [0.5, 0.6], [0.2, 0.4]\rangle, \langle x_4, [0.6, 0.8], [0.1, 0.2]\rangle.$$

Our purpose is to distinguish which class the unknown pattern R belongs to.

Calculate the similarity degree $S(R_i, R)(i = 1, 2, \ldots, 5)$ between R_i and R by (14), we have

$$S(R_1, R) = 0.9862, S(R_2, R) = 0.9976, S(R_3, R) = 0.9254, S(R_4, R) = 0.9908, S(R_5, R) = 0.9985.$$

Then select the largest one, that is the similarity degree $S(R_5, R) = 0.9985$ between R_5 and R. Hence R belongs to pattern R_5, which is the same as that obtained in [15].

4 Conclusion

Many information measures have been developed, however, there is scope that better measures can be developed, which will find applications in variety of fields. In this paper, we propose a set of entropy measures on IvIFSs based on the entropy measure defined in [21]. We also demonstrate the efficiency of proposed entropy and similarity measures through pattern recognition.

In future research, we shall continue working in the extension and application of the developed measures to other domains. As pointed in Remark 2.1, the different entropies may affect some specific application problems. Thus, how to determine an adaptive value of λ deserves further studying.

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