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A batch arrival two phases queueing system with random relapse, admissibility restricted and a single vacation policy

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Abstract

This paper deals with a batch arrival Poisson input, two heterogeneous services with randomly relaps and the second phase of service having many options. The first phase of service is essential for all customers, but with probability γ_1 a tagged customer chose second phase, with probability η_1 relapses to tail of original queue or with probability $\zeta_1 = 1 - \gamma_1 - \eta_1$ leave the system. Also, after completion of the second phase, with probability γ_2 the customer leaves the system, or with probability$1 - \gamma_2$ relapses to tail of original queue. In addition, we assume restricted admissibility of arriving batches in which not all batches are allowed to join the system at all times. After completion of the first phase or second phase of service, the server either goes for a vacation with probability $\theta(0 \leq \theta \leq 1)$, or may continue to serve the next unit with probability $1 - \theta$, if any. Otherwise, it remains in the system until a customer arrives. In this paper we derive the steady- state equations, PGF's of the system, and measures of sysytem.

Keywords:M/G/1 Queue, Two phase of heterogeneous service, relapse, admissibility restricted, Bernoulli vacation, Mean queue size, Mean response time.

1. Introduction

Qeues with relapse and vacation and optional services have been widely used to model problems in telephones, computers and communication systems. In general, queueing theory is an important subject in computers and operations research. Buffers /queues are used to store information that can not be transmitted instantaneously. Classical queueing systems assume that customers are in continuous contact with the server, that is, they can see whether or not the server is busy and thus commence service immediately whenever the service station becomes idle. However, queueing systems with relapse differ in that customers do not know the server state and consequently must verify if the server is idle from time to time. It is natural for telephone callers to break contact when the line is occupied and re-apply for connection later as a vacation. Madan [10], Madan and Choudhury [14] studied the $M/G/1$ queueing system with two phases of heterogeneous

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services such that the first phase of service follows by the second phase of service, under a Bernoulli vacation schedule.

An $M^X/G/1$ queue system with an additional service channel were analyzed by Choudhury [8]. Furthermore, a similar work can be found in Artalejo[2] and Ke[9].

Madan and Choudhury [13] proposed an $M^X/G/1$ queueing system, assuming batch arrivals with restricted admissibility of arriving batches and Bernoulli schedule server vacation. Earlier, Madan and Abu-Dayyeh [11,12] dealt with this type of model and studied some aspects of batch arrivals Bernoulli vacation models with restricted admissibility, where all arriving batches are not allowed into the system at all time. Furtheremore in Chaudhury[5] and Artalejo[2] the $M/G/1$ queueing systems with optional service are analyzed.

Recently in Badamchizadeh and Shahkar[3], Badamchizadeh[4], Salehirad and Badamchizadeh[16] author of this paper has extended this models.

In many applications such as hospital services, production systems, bank services, computer and communication networks, there is many phases of services such that after completion of services, customers may leave the system or may immediately go for the next phase of service, or the services must be repeated.

For simplicity we assume two phases of services such that the second phase of service consisting of many optional cases, where customers may choose one of them with certain probability. However, this model are extended to k phases of services. Unlike the usual batch arrival queueing systems, the policy of restricted admissibility of batches in which not all batches are allowed to join the system at all times, has been assumed in this model. In other words, an arriving batch will be allowed to join the system during the server's non-vacation and vacation periods with constant probabilities. Also in this system for overhauling or maintenance purpooses of the system, or serving other customers, the server being fatigue or for other reasons not mentioned here, the server may go to vacation.

In this paper our aim is toanalyze a single server queue with a batch arrival Poisson input, two phases of heterogeneous service in which the second phase has optional cases, restricted admissibility of batches, randomly relapse in services, and Bernoulli vacation for server. In section 2 we deal with the mathematical model and definitions. Steady-State conditions and generating functions are discussed in section 3. Mean queue size and mean response time are computed in section 4, where in section 6 some special cases are investigated. Finaly wiyh some numerical method the validity of model has been examined.

2. **Mathematical model and definitions**

We consider a queueing system such that:

i) Customers arrive at the system one by one in a compound Poisson process with a batch of random size X and mean rate $\lambda > 0$. Size of succesive arriving batches are

X_1,X_2,..., where i.i.d random variables, distributed with probability mass function(p.m.f) $d_n = Prob[X_i = n]; n \geqslant 1$, probability genrerating function(PGF) $d(z) = E[z_1^X]$. The first and second moments $d^{(1)} = E[X]$ and $d^{(2)} = E[X^2]$; respectively, are assumed to be finite.

ii)The server provides two phases of heterogenous service in succession. The service discipline is assumed to be on the basis of first come, first serve(FCFS). The first phase of service is essential for all customers, but as soon as the essential service is completed, a tagged customer moves for second phase with probability γ_1, relapses to tail of original queue with probability η_1 or leaves the system with probability $\zeta_1 = 1 - \gamma_1 - \eta_1$.

Similarly after completion of the second phase with probability γ_2 the customer leaves the system or with probability $\zeta_2 = 1 - \gamma_2$ relapses to the tail of original queue.

The second phase may have k cases(alternatives) where the customer chooses every case with probability p_i respectively such that $\Sigma p_i = 1$.
The service times for two phases are independent random variables, denoted by B_1, B_2. Their Laplace-Stieltjes transform (LST)are $B_1^*(s), B_2^*(s)$ where we assume they have finite moments $E(B_i^l)$ for $l \geqq 1$ and i=1,2. Also for the second phase of the service the random variables S_j for $j = 1, 2, ..., k$ denotes the service time of cases, respectively. Their corresponding LST are shown as $S_j^*(s)$. Also we assume that the $E(S_j^l)$ is finite for $l \geq 1$. In other words:

$$B_2 = \textstyle\sum_{j=1}^{k} p_j S_j$$

and

$$B_2^*(s) = \sum_{j=1}^{k} p_j S_j^*(s) \tag{1}$$

iii)There is a policy restricted admissibility of batches in which not all batches are allowed to join the system at all times. Let $\alpha(0 \leqq \alpha \leqq 1)$ and $\beta(0 \leqq \beta \leqq 1)$ be the probability that an arriving batch will be allowed to join the system during the period of the server's non-vacation period(system's turn off period, setup time and service time) and vacation period respectively.

iv)As soon as the first phase of a customer is completed or the second phase is completed, the server may go for a vacation of random length V with probability $\theta(0 \leq \theta \leq 1)$ or it may continue to serve the next customer, if any, with probability $(1 - \theta)$, otherwise it remains in the system and waits for a new arrival. We denote $V(x), V^*(s)$ and $E(V^l)$ for distribution function (DF),LST and $l'th$ finite moment of V, respectively, where $l \geq 1$.

v)The random variables B_1, B_2, V and also S_j are all independent variables.

Definition 2.1. The modified service time or the time required by a customer to complete the service cycle is given by:

(2) $$B = \begin{cases} B_0 + V & \text{with probability } \theta \\ B_0 & \text{with probability } (1-\theta) \end{cases}$$

then the LST $B^*(s)$ of B is given by

(3) $$B^*(s) = \theta B_0^*(s)V^*(s) + (1-\theta)B_0^*(s)$$

Also

(4) $$E(B) = -\frac{dB^*(s)}{ds} \mid_{s=0} = E(B_0) + \theta E(V)$$

In this system, random variable B_0 with

(5) $$B_0 = \begin{cases} B_1 & \text{with probability } \gamma_1 \\ B_1 + B_2 & \text{with probability } (1-\gamma_1) \end{cases}$$

(6) $$B_0^*(s) = \gamma_1 B_1^*(s)B_2^*(s) + (1-\gamma_1)B_1^*(s)$$

(7) $$\begin{aligned} E(B_0) = -\frac{dB_0^*(s)}{ds} \mid_{s=0} &= E(B_1) + \gamma_1 E(B_2) \\ &= E(B_1) + \gamma_1 [\sum_{j=1}^{k} p_j E(S_j)] \end{aligned}$$

(8) $$E(B_0^2) = \frac{(-1)^2 d^2 B_0^*(s)}{ds^2} \mid_{s=0} = E(B_1^2) + 2\gamma_1 E(B_1)E(B_2) + \gamma_1 E(B_2^2)$$

represents the required time without relapse and random variable B_f with

(9) $$E(B_f) = \zeta_1 E(B_1) + \gamma_1\gamma_2[E(B_1) + E(B_2)]$$

represents required time with relapse.

Further,for$i = 1, 2$ we assume that; $\mathrm{B}_i(0) = 0$, $\mathrm{B}_i(\infty) = 1$ and $B_i(x)$ are continuous at $x = 0$, so that

(10) $$\mu(x)_i dx = \frac{dB_i(x)}{1 - B_i(x)}$$

is the first order differential equation(hazard rate functions) of B_i.

Also, $V(0) = 0$, $V(\infty) = 1$ and $V(x)$ is continuous at $x = 0$, so that

(11) $$\nu(x)dx = \frac{dV(x)}{1 - V(x)}$$

is hazard rate function of V.

Definition 2.2. Let $N_Q(t)$ be the queue size at time 't' and the supplementary variables are defined as :

$$B_1^0(t)[B_2^0(t)] \equiv \text{the elapsed first [second] phase of service at time 't'}$$
$$V^0(t) \equiv \text{the elapsed vacation time at time 't'}$$

Now let us introduce the following random variables :

$$(12)\quad Y(t) = \begin{cases} 0 & \text{if the server is idle at time 't'}, \\ 1[2] & \text{if the server is busy with first[second]phase of service at time 't'}, \\ 3 & \text{if the server is on vacation at time 't'}. \end{cases}$$

From this we have a bivariate Markov process $\{N_Q(t), L(t)\}$ where $L(t) = 0$ if $Y(t) = 0$; $L(t) = B_i^0(t)$ if $Y(t) = i$ for $i = 1, 2$ and $L(t) = V^0(t)$ if $Y(t) = 3$. Now for $i = 1, 2$ the following probabilities are defined as:

$Q_n(x,t) = Prob[N_Q(t) = n, L(t) = V^0(t); x < V^0(t) \leqslant x + dx] \quad x > 0, \quad n \geqslant 0$
$P_{i,n}(x,t) = Prob[N_Q(t) = n, L(t) = B_i^0(t); x < B_i^0(t) \leqslant x + dx] \quad x > 0, \quad n \geqslant 1$
and

$$R_0(t) = Prob[N_Q(t) = 0, L(t) = 0]$$

Now the analysis of the limiting behaviour of this queueing process at a random epoch can be performed with the help of Kolmogorov forward equations, provided the following limits exist and are independent of initial state :

$$R_0 = \lim_{t\to\infty} R_0(t)$$
$$P_{i,n}(x)dx = \lim_{t\to\infty} P_{i,n}(x,t)dx \quad i = 1, 2 \quad x > 0,\ n \geqslant 0$$
$$Q_n(x)dx = \lim_{t\longrightarrow\infty} Q_n(x,t)dx \quad x > 0, \quad n > 0$$

Definition 2.3. For $i = 1, 2$ the PGF of these probabilities are defined as follow:

$$(13)\qquad P_i(x,z) = \sum_{n=1}^{\infty} z^n P_{i,n}(x) \quad |z| \leqslant 1, \quad x > 0$$

$$(14)\qquad P_i(0,z) = \sum_{n=1}^{\infty} z^n P_{i,n}(0) \quad |z| \leqslant 1$$

Also

$$(15)\qquad Q(x,z) = \sum_{n=0}^{\infty} z^n Q_n(x) \quad |z| \leqslant 1, \quad x > 0$$

$$(16)\qquad Q(0,z) = \sum_{n=0}^{\infty} z^n Q_n(0)$$

3. **Steady-state probability generating function**

From Kolmogorov forward equations , for $i = 1, 2$ the steady-state conditions can be written as follow

$$\frac{d}{dx}P_{i,n}(x)+[\lambda+\mu_i(x)]P_{i,n}(x) = \lambda(1-\alpha)P_{i,n}(x)+\lambda\alpha\sum_{k=1}^{n} a_k P_{i,n-k}(x) \quad n \geq 1, \quad x > 0 \tag{17}$$

$$\frac{d}{dx}Q_n(x) + [\lambda + \nu(x)]Q_n(x) = \lambda(1-\beta)Q_n(x) + \lambda\beta\sum_{k=1}^{n} a_k Q_{n-k}(x) \quad n \geq 1, \quad x > 0 \tag{18}$$

$$\frac{d}{dx}Q_0(x) + [\lambda + \nu(x)]Q_0(x) = \lambda(1-\beta)Q_0(x) \tag{19}$$

also

$$\lambda\alpha\beta R_0 = \beta(1-\theta)(1-\gamma_1-\eta_1)\int_0^{+\infty}\mu_1(x)P_{1,1}(x)dx+\beta(1-\theta)\gamma_2\int_0^{+\infty}\mu_2(x)P_{2,1}(x)dx+\alpha\int_0^{+\infty}\nu(x)Q_0(x)dx \tag{20}$$

For $n \geq 1$ these sets of equations are to be solved under the following boundary conditions at $x = 0$

$$\begin{aligned} \beta P_{1,n}(0) &= \lambda\alpha\beta a_n R_0 + \beta(1-\theta)(1-\gamma_1-\eta_1)\int_0^{+\infty}\mu_1(x)P_{1,n+1}(x)dx + \beta\eta_1\int_0^{+\infty}\mu_1(x)P_{1,n}(x)dx \\ &+\beta\gamma_2(1-\theta)\int_0^{+\infty}\mu_2(x)P_{2,n+1}(x)dx + \beta(1-\gamma_2)\int_0^{+\infty}\mu_2(x)P_{2,n}(x)dx + \alpha\int_0^{+\infty}\nu(x)Q_n(x)dx \end{aligned} \tag{21}$$

and

$$P_{2,n}(0) = \gamma_1\int_0^{+\infty}\mu_1(x)P_{1,n}(x)dx, \quad n \geqslant 1 \tag{22}$$

also for $n \geq 0$

(23)

$$\alpha Q_n(0) = \beta(1-\gamma_1-\eta_1)\theta\int_0^{+\infty}\mu_1(x)P_{1,n+1}(x)dx + \beta\theta\gamma_2\int_0^{+\infty}\mu_2(x)P_{2,n+1}(x)dx, \quad n \geqslant 0$$

Finally the normalizing condition is

$$R_0 + \sum_{i=1}^{2}\sum_{n=1}^{\infty}\int_o^{+\infty}P_{i,n}(x)dx + \sum_{n=0}^{\infty}\int_0^{+\infty}Q_n(x)dx = 1 \tag{24}$$

For $i = 1, 2$ from (3.17) we have

$$P_i(x,z) = P_i(0,z)[1-B_i(x)]e^{-\lambda\alpha(1-d(z))x} \quad x > 0 \tag{25}$$

and from (3.18),(3.19)

$$Q(x,z) = Q(0,z)[1-V(x)]e^{-\lambda\beta(1-d(z))x} \quad x > 0 \tag{26}$$

Now for $i = 1, 2$ let

$$B_i^*(\lambda\alpha(1-d(z)) = \int_0^{+\infty}e^{-\lambda\alpha(1-d(z))x}dB_i(x) \tag{27}$$

$$V^*(\lambda\beta(1-d(z)) = \int_0^{+\infty} e^{-\lambda\beta(1-d(z))x} dV(x) \tag{28}$$

be the z-transform of B_i and V respectively, then by multiplying (3.21) in z^n and summation from $n = 1$ to $+\infty$, adding (3.20) to result and using (3.25) and (3.26) we have:

$$\begin{aligned}\beta z P_1(0,z) &= \lambda\alpha\beta R_0 z(d(z)-1) + \beta(1-\theta)(1-\gamma_1-\eta_1)P_1(0,z)B_1^*(\lambda\alpha(1-d(z)))\\ &+ z\beta\eta_1 P_1(0,z)B_1^*(\lambda\alpha(1-d(z))) + \beta\gamma_2(1-\theta)P_2(0,z)B_2^*(\lambda\alpha(1-d(z)))\\ &+ \beta(1-\gamma_2)P_2(0,z)B_2^*(\lambda\alpha(1-d(z))) + z\alpha Q(0,z)V^*(\lambda\beta(1-d(z)))\end{aligned} \tag{29}$$

Also by multiplying (3.22) in z^n and summation on $n = 1$ to $+\infty$ we have:

$$P_2(0,z) = \gamma_1 P_1(0,z)B_1^*(\lambda\alpha(1-d(z))) \tag{30}$$

similarly from (3.23)

$$z\alpha Q(0,z) = \beta(1-\gamma_1-\eta_1)\theta P_1(0,z)B_1^*(\lambda\alpha(1-d(z))) + \beta\theta\gamma_2 P_2(0,z)B_2^*(\lambda\alpha(1-d(z))) \tag{31}$$

In the rest of this section for simplifying the formulas we omit $(\lambda\alpha(1-d(z)))$ from B_i^* and

$(\lambda\beta(1-d(z)))$ from V^*.

Remark 3.1. We set $v^*(z) = [(1-\theta)+\theta V^*(\lambda\beta(1-d(z)))]$. In the systems with vacation, this function has the main role. Also if $b^*(z) = (\zeta_1 + \gamma_1\gamma_2 B_2^*)B_1^*$, then by substituting $P_2(0,z)$ and $Q(0,z)$ for (3.30), (3.31) in (3.29) we have

$$P_1(0,z) = \frac{\lambda\alpha z R_0(d(z)-1)}{z[1-(\eta_1+(1-\gamma_2)\gamma_1 B_2^*)B_1^*] - b^*(z)v^*(z)} \tag{32}$$

Since

$$P_1(z) = \int_0^{+\infty} P_1(x,z)dx$$

hence from (3.25) for $i = 1$, using (3.32) and integration by part we have

$$P_1(z) = \frac{R_0 z(1-B_1^*)}{b^*(z)v^*(z) - z[1-(\eta_1+(1-\gamma_2)\gamma_1 B_2^*)B_1^*]} \tag{33}$$

Similarly from (3.25) for $i = 2$, (3.30) and (3.32) we have

$$\begin{aligned}P_2(z) = \int_0^{+\infty} P_2(x,z)dx &= \int_0^{+\infty} P_2(0,z)[1-B_2(x)]e^{-\lambda\alpha(1-X(z))x}dx\\ &= \int_0^{+\infty} \gamma_1 P_1(0,z)B_1^*[1-B_2(x)]e^{-\lambda\alpha(1-X(z))x}dx\\ &= \frac{R_0 z\gamma_1 B_1^*(1-B_2^*)}{b^*(z)v^*(z) - z[1-(\eta_1+(1-\gamma_2)\gamma_1 B_2^*)B_1^*]}\end{aligned} \tag{34}$$

Finally from (3.26), (3.31) and with the same method we have

$$Q(z) = \int_0^{+\infty} Q(x,z)dx = \frac{R_0\theta b^*(z)[1-V^*]}{b^*(z)v^*(z) - z[1-(\eta_1+(1-\gamma_2)\gamma_1 B_2^*)B_1^*]} \tag{35}$$

REMARK 3.2. The unknown constant R_0 can be determined by using normalizing condition (3.24) which is

$$R_0 + P_1(1) + P_2(1) + Q(1) = 1 \tag{36}$$

where for $i = 1, 2$

$P_i(1)$=Prob[The server is busy with $i-th$ phase of service]

and

$Q(1)$=Prob[The server is on vacation]

from (3.33),(3.34) and (3.35) by using L'Hopital rule we have

$$P_1(1) = R_0 \frac{\frac{\lambda E(X)}{\zeta_1+\gamma_1\gamma_2}\alpha E(B_1)}{1-\frac{\lambda E(X)}{\zeta_1+\gamma_1\gamma_2}[\alpha E(B_0)+\beta\theta E(V)(\zeta_1+\gamma_1\gamma_2)]}$$
$$P_2(1) = R_0 \frac{\frac{\lambda E(X)}{\zeta_1+\gamma_1\gamma_2}\alpha\gamma_1 E(B_2)}{1-\frac{\lambda E(X)}{\zeta_1+\gamma_1\gamma_2}[\alpha E(B_0)+\beta\theta E(V)(\zeta_1+\gamma_1\gamma_2)]}$$
$$Q(1) = R_0 \frac{\lambda E(X)[\theta\beta E(V)]}{1-\frac{\lambda E(X)}{\zeta_1+\gamma_1\gamma_2}[\alpha E(B_0)+\beta\theta E(V)(\zeta_1+\gamma_1\gamma_2)]}$$

hence by substituting in (3.36) and simplifying we have $R_0 = 1-\rho$ where

$$\rho = \frac{\lambda E(X)}{\zeta_1+\gamma_1\gamma_2}[\alpha E(B_0) + \beta\theta E(V)(\zeta_1+\gamma_1\gamma_2)] \tag{37}$$

R_0 is the steady-state probability that the server is idle but available in the system, hence $\rho < 1$ can be the stability condition under which the steady state solution exists.

From (3.33),(3.34) and (3.35) the PGF of the queue size distribution at a random epoch is

$$\begin{aligned} P_Q(z) &= R_0 + P_1(z) + P_2(z) + zQ(z) \\ &= (1-\rho)\frac{(z-1)b^*(z)v^*(z)}{z[1-(\eta_1+(1-\gamma_2)\gamma_1 B_2^*)B_1^*] - b^*(z)v^*(z)} \end{aligned} \tag{38}$$

4. **Mean queue size and other measures of system**

Let L_Q be the mean number of customers in the queue (i.e mean queue size), then we have

$$L_Q = \frac{dP_Q(z)}{dz}\,|_{z=1} \tag{39}$$

The denominator and numerator of $P_Q(z)$ are zero at $z = 1$. If $f(z) = (z-1)b^*(z)v^*(z)$ and $g(z) = z[1-(\eta_1+(1-\gamma_2)\gamma_1 B_2^*)B_1^*] - v^*(z)b^*(z)$, then $\lim_{z\to 1} f(z) = \lim_{z\to 1} g(z) = 0$. Hence using L'Hopital rule we have

$$L_Q = (1-\rho)\frac{f''(1)g'(1) - f'(1)g''(1)}{2[g'(1)]^2} \tag{40}$$

By computing $f'(1)$,$f''(1)$,$g'(1)$ and $g''(1)$ we have

(41)
$$L_Q=\theta\beta\lambda E(X)E(V)+\frac{2\lambda\alpha E(X)[E(B_0)-\rho E(B_f)]+\lambda^2E(X)^2[\alpha^2E(B_0^2)+2\theta\alpha\beta E(B_f)E(V)+(\zeta_1+\gamma_1\gamma_2)\theta\beta^2E(V^2)]}{2(1-\rho)(\zeta_1+\gamma_1\gamma_2)}$$

Now for computing the mean waiting-time of a test customer in this model, by using Little's formula, this measure of system is equal

$$W_Q=\frac{L_Q}{\lambda_X} \tag{42}$$

where, following the admissibility assumption of our model, λ_X the actual arrival rate of batches is given by

$$\lambda_X=\lambda\alpha(\text{proportion of non-vacation time})+\lambda\beta(\text{proportion of vacation time})$$

But from remark 3.2 we have

$$\text{the proportion of vacation time}=Q(1)=\tfrac{\theta\lambda\beta E(X)}{(\zeta_1+\gamma_1\gamma_2)}E(V)$$

Hence the proportion of non-vacation time including the first and second service times and idle time, is equal $1-\frac{\theta\lambda\beta E(X)}{(\zeta_1+\gamma_1\gamma_2)}E(V)$. Consequently

$$\lambda_X=\lambda\alpha+\frac{1}{(\zeta_1+\gamma_1\gamma_2)}(\beta-\alpha)\theta\lambda^2\beta E(X)E(V) \tag{43}$$

4.1. **Particular case.** I) If $\eta\to 0$ then $\zeta_1=1-\gamma_1$, and also $\gamma_2\to 1$, we have the system in Badamchizadeh[3] such that $\zeta_1+\gamma_1\gamma_2=1$ and $E(B_f)=E(B_0)$, also

$\rho=\lambda E(X)[\alpha E(B_0)+\theta\beta E(V)$ and $b^*(z)=(1-\gamma_1)B_1^*+\gamma_1\gamma_2B_1^*B_2^*$ and

$$L_Q=\rho+\frac{\lambda^2E(X)^2[\alpha^2E(B_0^2)+2\alpha\beta\theta E(B_0)E(V)+\theta\beta^2E(V^2)]}{2(1-\rho)} \tag{44}$$

II)If $\theta\to 0$; i.e there is no vacation in the system, then $v^*(z)=1$ and from (3.38) we have

$$P_Q(z)=(1-\rho)\frac{(z-1)b^*(z)}{z[1-(\eta_1+(1-\gamma_2)\gamma_1B_2^*)B_1^*]-b^*(Z)} \tag{45}$$

where

$$\rho=\lambda\alpha E(X)E(B_0) \tag{46}$$

and using (4.41) we reach

$$L_Q=\rho+\frac{2\lambda\alpha E(X)[E(B_0)-\rho E(B_f)]+\lambda^2\alpha^2E(X)^2E(B_0^2)}{2(1-\rho)(\zeta_1+\gamma_1\gamma_2)} \tag{47}$$

5. Special cases and numerical results

Analyzing a queueing system via actual cases are very important and useful way to confirm the models. In this section we chose known distributions for service times and vacation time, so with this, and by some numerical approches the validity of the system are examained. Also this approch explains that our model can function reasonably well for certain practical problems.

case 1 : Let the distribution of first service time be a-Erlang as follows:

$$\mathrm{dB}_1(x) = \frac{(a\mu_1)^{\mu_1} x^{a-1} e^{-a\mu_1 x}}{(a-1)!} dx \qquad x > 0, a \geq 1$$

hence

$$B_1^*(\lambda - \lambda d(z)) = \frac{(a\mu_1)^a}{[\lambda(d(z)-1) + a\mu_1)]^a}$$

so $E(B_1) = \dfrac{1}{\mu_1}$ and $E(B_1^2) = \dfrac{a+1}{a\mu_1^2}$.

To simplify, we assume $p_1 = p_2 = .5$ and for $i = 1,2$ distribution of S_i is b_i-Erlang

$$dS_i(x) = \frac{(b_i s_i)^{s_i} x^{b_i - 1} e^{-b_i s_i x}}{(b_i - 1)!} dx \qquad x > 0, b_i \geq 1$$

hence

$$S_i^*(\lambda - \lambda d(z)) = \frac{(b_i s_i)^{b_i}}{[\lambda(d(z)-1) + b_i s_i]^{b_i}}$$

and $E(S_i) = \dfrac{1}{s_i}$ and $E(S_i^2) = \dfrac{b_i + 1}{b_i s_i}$.

Also we assume the distribution of vacation time be c-Erlang

$$dV(x) = \frac{(c\nu)^{\nu} x^{c-1} e^{-c\nu x}}{(c-1)!} dx \qquad x > 0, c \geq 1$$

hence

$$V^*(\lambda - \lambda d(z)) = \frac{(c\nu)^c}{[\lambda(d(z)-1) + c\nu]^c}$$

so $E(V) = \dfrac{1}{\nu}$ and $E(V^2) = \dfrac{c+1}{c\nu^2}$. If we chose geometric distribution for batch size, i.e $d_n = d(1-d)^{n-1}, 0 < d < 1$, then $E(X) = \dfrac{1}{d}$ and $E(X^2) = \dfrac{2-d}{d^2}$. Now for numerical result we assume the following values for parametrs such that the steady state condition($\rho < 1$) obtained

Table 1. values of parameters

γ_1	ζ_1	γ_2	ζ_2	a	μ_1	b_1	b_2	s_1	s_2	c	ν	α	β	d
.2	7.	.1	.9	2	2	1	1	2	3	1	1	.2	.5	.5

In this case using above values and (3.37), $p_1 = p_2 = .5$, if $\theta = 0$ then steady state condition is

$\rho = .32\lambda < 1$, so $\lambda < 3.1$. By using (4.41)

$$L_Q = \frac{.8\lambda + .14\lambda^2}{1 - .8\lambda}$$

The graph of model is in figure 1.

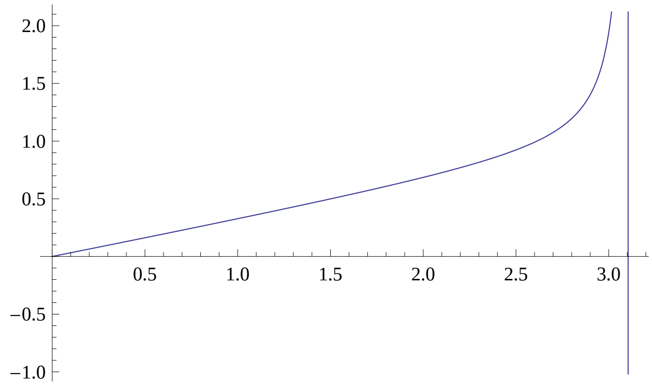

FIGURE 1. L vis-a-vis λ for $\theta = 0$

In this case if $\theta = .1$, then steady state condition is $\rho = .52\lambda < 1$, so $\lambda < 1.91$. By using (4.41)

$$L_Q = .1\lambda + \frac{.322\lambda + .1\lambda^2}{1 - .53\lambda}$$

Figure 2 shows the graph of model.

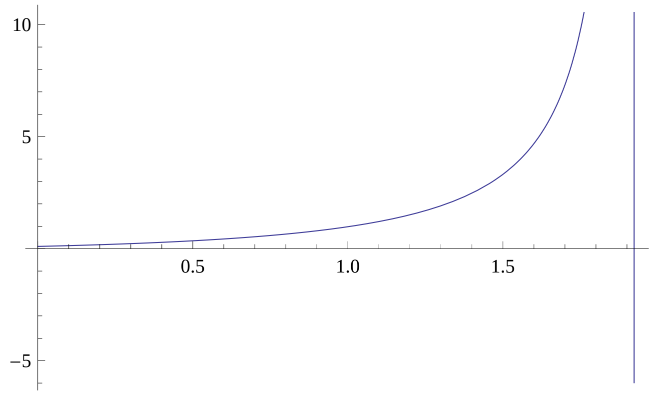

FIGURE 2. L vis-a-vis λ for $\theta = .1$

Now we analyze L with respect to θ. Using values of table 1, $p_1 = p_2 = .5$, and $\lambda = 1$ the steady-state condition is $\rho = .32 + .995\theta < 1$, hence $\theta < .68$. Also

$$L_Q = \theta + \frac{.3 + \theta}{.68 - .995\theta}$$

Figure 3 shows the graph of model. Also in table 2 some values of L against θ are computed. After $\theta = .5$ the system blows up.

TABLE 2. values of L with respect θ

θ	.1	.2	.3	.4	.5	.6	.65
L	.78	1.2	1.8	2.8	4.8	11.4	29.22

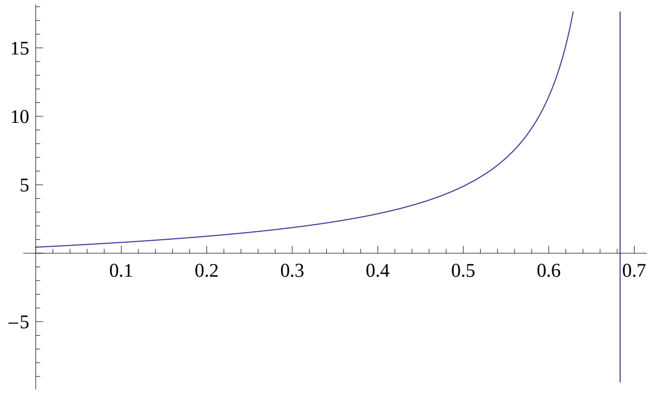

FIGURE 3. L vis-a-vis θ for $\lambda = 1$

case 2: In this case we assume the distribution of service times and vacation time are exponential as follow

$$dB_1(x) = \mu_1 e^{-\mu_1 x}dx, \quad E(B_1) = \tfrac{1}{\mu_1}\ , \quad E(B_1^2) = \tfrac{2}{\mu_1^2}$$

and for $i = 1, 2$

$$dS_i(x) = \mu_i e^{-\mu_i x}dx, \quad E(S_i) = \tfrac{1}{i}\ , \quad E(S_i^2) = \tfrac{2}{s_i^2}$$

$$dV(x) = \nu e^{-\nu x}dx, \quad E(V) = \tfrac{1}{\nu}\ , \quad E(V^2) = \tfrac{2}{\nu^2}$$

With geometric distribution for batch size according to case 1 and following values for parameters in table 3 the steady state condition is $\rho = \dfrac{.27}{\mu_1} + .023 < 1$ or $\mu_>.27$. Also

$$L_Q = \frac{\frac{1.99}{\mu_1} + \frac{.001}{\mu_1^2} + .292}{.977 - \frac{.27}{\mu_1}}$$

TABLE 3. values of parameters

γ_1	ζ_1	γ_2	ζ_2	λ	s_1	s_2	ν	α	β	d
.2	7.	.1	.9	2	2	3	1	.2	.5	.5

and the graph of model is in figure 4. According to this curve and values of table 4, L decresses with respet μ, and after $\mu = 1$ the system is stable.

TABLE 4. values of L with respect μ_1

μ_1	.3	.5	1	2	3	4
L	90	9.7	3.2	1.5	1	.8

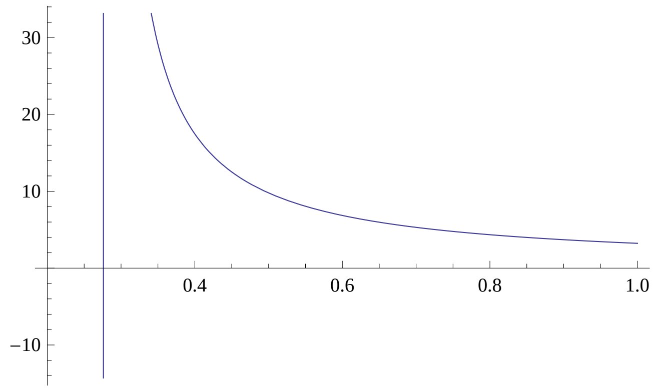

FIGURE 4. L vis-a-vis μ

Now, in this case we assume θ is unknown. With the values in table 5 the steady state condition is $\rho = .322 + \theta < 1$ or $\theta < .67$.

TABLE 5. values of parameters

γ_1	ζ_1	γ_2	ζ_2	λ	μ_1	s_1	s_2	ν	α	β	d
.2	7.	.1	.2	1	2	2	3	1	.2	.5	.5

Also

$$L = \theta + \frac{.32 + .25\theta}{.678 - \theta}$$

The graph of models is in figure 5.

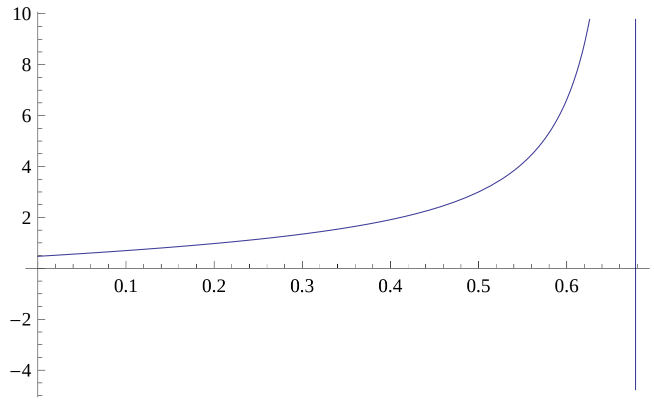

FIGURE 5. L vis-a-vis θ

Table 6 shows some values of L against θ. After $\theta = .5$, the system blows up.

TABLE 6. values of L with respect θ

θ	.1	.2	.3	.4	.5	.6	.65
L	.69	.97	1.34	1.9	3	6.6	17.8

6. Concluding Remarks

In this paper we have studied a batch arrival two phase queueing system with randomly relapse and option in services, admissibility restricted and server's vacation which generalized classical $M/G/1$ queue. An application of this model can be found in mobile network where the messages are in batch form, the service may have many phases such that services may be unaccepted and customer may repeat the services. Also, because of admissibility restriction in service or system, all batches don't enter in service. Our investigations are concerned with not only queue size distribution but also waiting time distribution. This model extends the systems for example in Artalejo[2], Badamchizadeh and Shahkar[3], Badamchizadeh[4], Choudhury[7], Madan and Choudhury[13] and Madan and Choudhury[14]. A practical generalization for this system is to consider many cases of services, optional vacation.

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