AMO - ADVANCED MODELING AND OPTIMIZATION, VOLUME 15, NUMBER 2, 2013

A TWO-WAREHOUSE EOQ MODEL FOR DETERIORATING ITEMS AND STOCK DEPENDENT DEMAND UNDER CONDITIONALLY PERMISSIBLE DELAY IN PAYMENT IN IMPRECISE ENVIRONMENT

D.K.JANA[1], K.MAITY[2] AND T.K.ROY[3]

[1]Department of Applied Sciences, Haldia Institute of technology, Haldia ,Purba Medinipur-721657,INDIA, E-mail: dipakjana@gmail.com
[2]Department of Mathematics, Mugberia Gangadhar Mahavidyalaya, Purba Medinipur-721 425, West Bengal, India, E-mail: kalipada_maity@yahoo.co.in
[3]Department of Mathematics, Bengal Engineering and Science University, Shibpur,Howrah – 711103, West Bengal, INDIA. E-mail: t_k_roy@yahoo.co.in

Abstract: In this paper, a two-warehouse inventory model for deteriorating items is considered with stock demand under conditionally permissible delay in payment in imprecise environment.The capacity of any warehouse is limited, it has to rent warehouse (RW) for storing the excess units over the fixed capacity W of the own warehouse (OW) in practice. The RW is assumed to offer better preserving facilities than the OW resulting in a lower rate of deterioration and is assumed to charge higher holding cost than the OW. The purpose of this study is to find the optimal replenishment policies for minimizing the total relevant inventory costs. Here, the purchasing cost, holding cost for RW and OW and earning interest are taken fuzzy in nature. Useful Algorithms to characterize the optimal solutions have been derived. Furthermore,Graphical representation and numerical examples are provided to illustrate the proposed model.

Keywords: Inventory, Two-warehouse, Deterioration, Permissible delay in payment, Stock dependent demand, Credibility.

† AMO - Advanced Modeling and Optimization. ISSN: 1841-4311.

1. Introduction

In the past few decades, inventory problems for deteriorating items have been widely studied [1-3]. In general, deterioration is defined as the damage, spoilage, dryness, vaporization, etc., that results in decrease of usefulness of the original one. In 2006, the literature [18] defined a new phenomenon as non-instantaneous deteriorating and considered the problem of determining the optimal replenishment policy for such items with stock-dependent demand. Generally, when suppliers provide price discounts for bulk purchases or the products are seasonal, the retailers may purchase more goods than can be stored in his own warehouses (OW). Therefore, a rented warehouse (RW) is used to store the excess units over the fixed capacity W of the own warehouse. Usually, the rented warehouse is to charge higher unit holding cost than the own warehouse, but to offer a better preserving facility resulting in a lower rate of deterioration for the goods than the own warehouse. To reduce the inventory costs, it will be economical to consume the goods of RW at the earliest. Consequently, the firm stores goods in OW before RW, but clears the stocks in RW before OW. Recently, the literature [3] proposed a two-warehouse inventory model for deteriorating items under permissible delay in payments, but they assume that the deteriorating rate of two warehouses are the same. And Rong et al. [16] developed an optimization inventory policy for a deteriorating item with imprecise lead time, partially/fully backlogged shortages and price dependent demand under two-warehouse system.

Many research paper have been published in deteriorating inventory control systems. As examples, Dey, Mondal and Maiti [3] considered a finite time horizon inventory problem for a deteriorating item having two separate warehouses with interval- valued lead-time under inflation and time value of money. Musa and Sani [25] and Meher, Panda and Sahu [24] have developed deteriorating inventory models under permissible delay in payments.

In the recent competitive market, the inventory/stock is decoratively exhibited and colourably displayed through electronic media to attack the customers and thus to push the sale. For this reason, Wu, Ouyang and Yang[18], Zhou and Yang [21] and others have developed an inventory models with stock-dependent demand.

In the traditional economic order quantity (EOQ) model, it often assumed that the retailer must pay off as soon as the items are received. In fact, the supplier offers the retailer a delay period, known as trade credit period, in paying for purchasing cost, which is a very common business practice. Suppliers often offer trade credit as a marketing strategy to increase sales and reduce on hand stock level. Once a trade credit has been offered, the amount of period that the retailers capital tied up in stock is reduced, and that leads to a reduction in the retailers holding cost of finance. In addition, during trade credit period, the retailer can accumulate revenues by selling items and by earning interests. As a matter of fact, retailers, especially small businesses which tend to have a limited number of financing opportunities, rely on trade credit as a source of short-term funds. In this research field, Goyal [6] was the first to establish an EOQ model with a constant demand rate under the condition of permissible delay in payments. Khanra, Ghosh and Chaudhuri [22] have developed an EOQ model for a deteriorating item with time dependent quadratic demand under permissible delay in payment. Also, Maihami and Abadi [23] have established joint control of inventory and its pricing for non-instantaneously deteriorating items under permissible delay in payments and partial backlogging.

Many research paper have been published for two warehouse problems in imprecise environment. Maity [26], Jana et al. [27] have considered possibility and necessity representations of fuzzy inequality and its application to two warehouse production-inventory problem. Also Maiti [14] have been developed a fuzzy inventory model with two warehouses under possibility measure on fuzzy goal. None consider the earning interest rate in fuzzy in nature. In this paper this idea has been established as earned interest rate depends on different trade credit period in two warehouse model under permissible delay in payments.

The major assumptions used in the previous papers are summarized in the following Table 1. It is clear from the Table 1 that no one consider the two-warehouse problem under credit period policy with the assumption of $\alpha > \beta > 0$. Furthermore, it is inappropriate to assume that the RW has lower unit holding cost than the OW. In this study, a two-warehouse inventory model for deteriorating items is developed in which demand rate is stock dependent and delay in payment is permitted. In addition, it was assumed that the RW has higher unit holding cost than the OW. The purpose of this study is to make the model more relevant and applicable in practice. The proposed mathematical model comprises some previous models such as in Goyal [6] as special cases.
Table 1 represents the summary of related literature for two-warehouse inventory models.

Table 1 represents the summary of related literature for two-warehouse inventory models.

Author(s) and year	two warehouse deteriorating rate	Demand rate	Delay in payment	Imprecise environment
Sarma (1987)	$\alpha > \beta$	Constant	No	No
Benkherouf (1997)	$\alpha > \beta$	Time-dependent	No	No
Yang (2004)	$\alpha < \beta$	Constant	No	No
Zhou et al. (2005)	$\alpha = \beta = 0$	Stock-dependent	No	No
Yang (2006)	$\alpha < \beta$	Constant	No	No
Lee (2006)	$\alpha > \beta$	Constant	No	No
Huang (2006)	$\alpha = \beta = 0$	Constant	No	No
Hsieh et al. (2007)	$\alpha < \beta$	Constant	No	No
Chung (2007)	$\alpha = \beta \neq 0$	Constant	Yes	No
Rong et al.(2008)	$\alpha < \beta$	Price-dependent	No	Yes
Dey(2008)	$\alpha > \beta$	Time-dependent	No	Yes
Maity (2011)	$\alpha = \beta = 0$	Stock-dependent	No	Yes
Liang (2011)	$\alpha > \beta$	Constant	Yes	No
Present paper	$\alpha > \beta$	Stock-dependent	Yes	Yes

For first time, a two-warehouse inventory model for deteriorating items with stock dependent demand is developed under conditionally permissible delay in payment in imprecise environment.

2. Preliminaries

Fuzzy Set: A fuzzy set is a class of objects in which there is no sharp boundary between those objects that belong to the class and those that do not. Let X be a collection of objects and x be an element of X, then a fuzzy set $\widetilde{A}$ in X is a set of ordered pairs $\widetilde{A} = \{(x, \mu_{\widetilde{A}}(x))/x \in X\}$
where $\mu_{\widetilde{A}}(x)$ is called the membership function or grade of membership of x in $\tilde{A}$ which maps X to the membership space M which is considered as the closed interval [0,1].

Fuzzy Number: A fuzzy number $\widetilde{A}$ is a convex normalized fuzzy set on real line $\Re$ such that
(i) it exists exactly one $x_0 \in \Re$ with $\mu_{\widetilde{A}}(x_0) = 1$ (x_0 is called the mean value of $\widetilde{M}$),
(ii) $\mu_{\widetilde{A}}(x)$ is piecewise continuous.
In particular if $\widetilde{A} = (a_1, a_2, a_3)$ be a Triangular Fuzzy Number (TFN) then $\mu_{\widetilde{A}}(x)$ is

defined as follows

$$\mu_{\tilde{A}}(x) = \begin{cases} \frac{x-a_1}{a_2-a_1} & \text{for } a_1 \leq x < a_2 \\ \frac{a_3-x}{a_3-a_2} & \text{for } a_2 < x \leq a_3 \\ 0 & \text{otherwise} \end{cases}$$

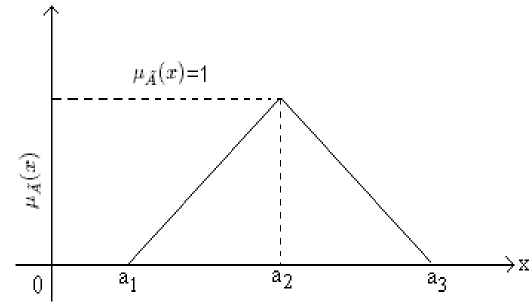

FIGURE 1. Triangular Fuzzy Number(TFN)

where a_1, a_2 and a_3 are real numbers.

2.1. **Possibility, necessity and credibility constraints.** Let $\tilde{a}$ and $\tilde{b}$ be two fuzzy numbers with membership functions $\mu_{\tilde{a}}(x)$ and $\mu_{\tilde{b}}(x)$ respectively and $\Re$ is the set of real numbers. Then according to Liu and Iwamura [9] and others

$$\begin{aligned} Pos(\tilde{a} * \tilde{b}) &= sup\{min(\mu_{\tilde{a}}(x), \mu_{\tilde{b}}(y)), x, y \in \Re, x * y\} \\ Nes(\tilde{a} * \tilde{b}) &= inf\{max(1 - \mu_{\tilde{a}}(x), \mu_{\tilde{b}}(y)), x, y \in \Re, x * y\} \end{aligned}$$

where the abbreviation *pos* represents possibility, Nes represents necessity and $*$ is any of the relations $>, <, =, \leq, \geq$.

2.2. **The relationships for possibility, necessity and credibility constraints.** The relationships between possibility and necessity measures satisfy the following condition (Dubois and Prade [4]) :

$$Nes(\tilde{a} * \tilde{b}) = 1 - Pos\overline{(\tilde{a} * \tilde{b})}$$

where the abbreviation *nes* represents necessity.
If $\tilde{a}, \tilde{b} \in \Re$ and $\tilde{c} = f(\tilde{a}, \tilde{b})$ where $f : \Re \times \Re \rightarrow \Re$ be a binary operation then according to well known Fuzzy Extension Principle, membership function $\mu_{\tilde{c}}$ of $\tilde{c}$ is defined as

$$\mu_{\tilde{c}}(z) = sup\{min(\mu_{\tilde{a}}(x), \mu_{\tilde{b}}(y)), x, y \in \Re \text{ and } z = f(x, y), \forall z \in \Re\} \tag{1}$$

Based on possibility and necessity measure, the third set function Cr , called credibility measure, analyzed by Liu and Liu [13] is as follows:

If we denote the support of $\tilde{a}$ by $R = \{r \in \Re | \mu_{\tilde{a}}(\gamma) > 0\}$, the credibility measure is given by

$$Cr(A) = \frac{1}{2}\Big[Pos(A) + Nes(A)\Big] \tag{2}$$

for any $A \in 2^R$, where 2^R is the power set of R and Cr satisfies the following two conditions:
i) $Cr(\phi) = 0$ and $Cr(R)$ =1;
ii)$A \subset B$ implies $Cr(A) \leq Cr(B)$ for any $A, B \in 2^R$
and thus, Cr is also a fuzzy measure defined on $(R, 2^R)$. Besides, Cr is self dual, i.e., $Cr(A) = 1 - Cr(A^C)$ for any $A \in 2^R$.
In this paper, based on the credibility measure (4) the following form is defined as

$$Cr(A) \ = \ \Big[\rho Pos(A) + (1-\rho)Nes(A)\Big], \tag{3}$$

for any $A \in 2^R$ and $0 < \rho < 1$, where ρ is the degree of pessimism . It also satisfies the above conditions.
For the triangular fuzzy number $\tilde{a} \ = \ (\,a_1, a_2, a_3\,))$ and the crisp number r, $Pos(\tilde{a} \geq r)$ and $Nes(\tilde{a} \geq r)$ are given by

$$Pos(\tilde{a} \geq r) = \begin{cases} 1 & \text{if} \quad r \leq a_2 \\ \dfrac{a_3 - r}{a_3 - a_2} & \text{if} \quad a_2 \leq r \leq a_3 \\ 0 & \text{if} \quad r \geq a_3 \end{cases} \tag{4}$$

$$Nes(\tilde{a} \geq r) = \begin{cases} 1 & \text{if} \quad r \leq a_1 \\ \dfrac{a_2 - r}{a_2 - a_1} & \text{if} \quad a_1 \leq r \leq a_2 \\ 0 & \text{if} \quad r \geq a_2 \end{cases} \tag{5}$$

The credibility measure for TFN can be define as

$$Cr(\tilde{a} \geq r) = \begin{cases} 1 & \text{if} \quad r \leq a_1 \\ \dfrac{a_2 - \rho a_1}{a_2 - a_1} - \dfrac{(1-\rho)r}{a_2 - a_1} & \text{if} \quad a_1 \leq r \leq a_2 \\ \dfrac{\rho(a_3 - r)}{a_3 - a_2} & \text{if} \quad a_2 \leq r \leq a_3 \\ 0 & \text{if} \quad r \geq a_3 \end{cases} \tag{6}$$

$$Cr(\tilde{a} \le r) = \begin{cases} 0 & \text{if} \quad r \le a_1 \\ \dfrac{\rho(r-a_1)}{a_2-a_1} & \text{if} \quad a_1 \le r \le a_2 \\ \dfrac{\rho a_3 - a_2}{a_3-a_2} - \dfrac{(1-\rho)r}{a_3-a_2} & \text{if} \quad a_2 \le r \le a_3 \\ 1 & \text{if} \quad r \ge a_3 \end{cases} \tag{7}$$

Definition: Let X be a normalized fuzzy variable. Then expected value of the fuzzy variable X is defined by

$$E[X] = \int_0^{\infty} Cr(X \ge r)dr - \int_{-\infty}^{0} Cr(X \le r)dr \tag{8}$$

When the right hand side of (8) is of form $\infty - \infty$,the expected value is not defined. Also, the expected value operation has been proved to be linear for bounded fuzzy variables, i.e., for any two bounded fuzzy variables X and Y, we have $E[\,aX+bY\,] = aE[X]+bE[Y]$ for any real numbers a and b.

Lemma-1:
Let $\tilde{a} = (\,a_1, a_2, a_3)\,)$ be a triangular fuzzy number and r is a crisp number. The expected value of $\tilde{a}$ is

$$E[\tilde{a}] = \frac{1}{2}\Big[(1-\rho)a_1 + a_2 + \rho a_3\Big], \quad 0 < \rho < 1. \tag{9}$$

Proof. From (8), by using (6) and (7), we have

$$\begin{aligned} E[\tilde{a}] &= \int_0^{\infty} Cr(\tilde{a} \ge r)dr - \int_{-\infty}^{0} Cr(\tilde{a} \le r)dr \\ &= \int_0^{a_1} Cr(\tilde{a} \ge r)dr + \int_{a_1}^{a_2} Cr(\tilde{a} \ge r)dr + \int_{a_2}^{a_3} Cr(\tilde{a} \ge r)dr \\ &= \int_0^{a_1} dr + \int_{a_1}^{a_2} \Big[\frac{a_2-\rho a_1}{a_2-a_1} - \frac{(1-\rho)r}{a_2-a_1}\Big]dr + \int_{a_2}^{a_3}\Big[\frac{\rho(a_3-r)}{a_3-a_2}\Big]dr \\ &= \frac{1}{2}\Big[(1-\rho)a_1 + a_2 + \rho a_3\Big].\text{(proved the Lemma-1).} \end{aligned}$$

3. Notation and Assumption:

The model under consideration is developed with the following assumptions and notations:

3.1. **Notation:**

A = ordering cost one order
$\widetilde{c}$ = unit purchasing price which is a triangular fuzzy number,
p = unit selling price per unit,
$\widetilde{h}_o$ = unit holding cost per year(excluding interest charges) for items in OW which is a triangular fuzzy number,
$\widetilde{h}_r$ = unit holding cost per year(excluding interest charges) for items in RW which is a triangular fuzzy number,
M = is the retailer's trade credit period offered by supplier in years, which is as the fraction of the year.
$\widetilde{I}_e$ = denotes earning interest per \$ per year,which is a triangular fuzzy number,
I_p = interest charges per \$ in stocks per year by the supplier.
t_1 = the time that inventory level reduce to W, T is the length of replenishment cycle.
$I_r(t)$ = the inventory level at time $[0, t_1]$ in RW,
$I_o(t)$ = the level of inventory at time $[0, T]$ in OW.
TC_i = the total relevant costs and revenue which consists of Ordering cost,Stock holding cost, Deteriorating cost, Interest payable opportunity cost, Opportunity interest earned (revenue), $(i = 1, 2, 3)$.

3.2. **Assumption:** The following assumptions and notations are used throughout this paper to develop the proposed model.
(1) Replenishment rate is infinite and the lead time is zero.
(2) no shortage is allowed.
(3) The inventory system consider a single item and the demand rate $D(.) = D+\theta I(t)$ is varying, $D \geq 0, 0 \leq \theta \leq 1$.
(4) The OW has limited capacity of W units and the RW has unlimited capacity. For economic reasons, the items of RW are consumed first and next the items of OW.
(5) The items deteriorate at a fixed rate α in OW and at β in RW, for the rented warehouse offers better facility, so $\alpha > \beta$.
(6) To guarantee the optimal solution exists, we assume that the maximum deteriorating quantity for items in OW, $\alpha W$is less than the demand rate D, that is, $\alpha W < D(.)$.
(7) The retailer can accumulate revenue by earning interest since the customer pays for the amount of purchasing cost to the retailer until the end of the trade credit

period offered by the supplier.

4. Mathematical Model formulation

4.1. **Proposed inventory model in fuzzy environment.** The inventory system goes as follows: at time t = 0, a lot size of certain units enter the system,W units are kept in OW and the rest is stored in RW. The items of OW are consumed only after consuming the goods kept in RW. In the interval $[0, t_1]$, the inventory in RW gradually decreases due to demand and deterioration and it vanishes at $t = t_1$. In OW, however, the inventory W decreases during $[0, t_1]$ due to deterioration only, but during $[t_1, T]$, the inventory is depleted due to both demand and deterioration. By the time to T, both warehouses are empty. Fig.-2 is depicted the behavior of inventory system. At time $t \in [0, t_1]$, the inventory level in RW and OW is given by the following differential equation:

$$\frac{dI_r(t)}{dt} = -D(.) - \beta I_r(t),\ 0 \leq t \leq t_1 \tag{10}$$

with the boundary condition $I_r(t_1) = 0$ and

$$\frac{dI_o(t)}{dt} = -\alpha I_o(t),\ 0 \leq t \leq t_1 \tag{11}$$

with the initial condition $Io(0) = W$,While during the interval $[t_1, T]$, the inventory level in OW, $I_0(t)$ is governed by the following differential equation:

$$\frac{dI_0(t)}{dt} = -D(.) - \alpha I_0(t),\ t_1 \leq t \leq T \tag{12}$$

with the boundary condition $I_0(T) = 0$. The solutions from Eq. (10) to Equation (12) are:

$$\begin{aligned} I_r(t) &= \frac{D}{\theta+\beta}[e^{(\theta+\beta)(t_1-t)} - 1],\ 0 \leq t \leq t_1 \\ I_0(t) &= We^{-\alpha t},\ 0 \leq t \leq t_1 \\ I_0(t) &= \frac{D}{\theta+\alpha}[e^{(\theta+\alpha)(T-t)} - 1],\ t_1 \leq t \leq T \end{aligned}$$

Considering the continuity of $I_o(t)$ at time $t = t_1$, i.e.

$$I_o(t_1) = We^{-\alpha t_w} = \frac{D}{\theta+\alpha}\left[e^{(\theta+\alpha)(T-t_1)} - 1\right] \tag{13}$$

which implies that

$$T = t_1 + \frac{1}{\theta+\alpha} ln\left[1 + \left(\frac{\theta+\alpha}{D}\right) W e^{-\alpha t_1}\right] \tag{14}$$

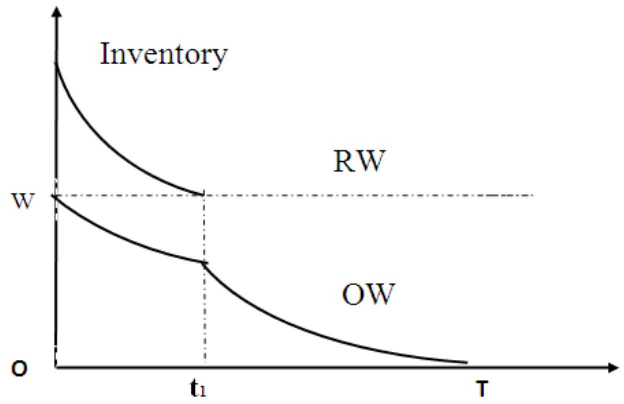

FIGURE 2. Graphical Representation of the Two-Warehouse Inventory System

Based on the assumptions and description of the model, the total annual revelent costs, TC, include the following elements:

(1) The ordering cost =A.

(2) Total stock holding cost:The cumulative inventories in RW during $[0, t_1]$ and OW during $[0, T]$ are:

$$\begin{aligned} \widetilde{h}_r \int_0^{t_1} I_r(t)dt &= \widetilde{h}_r \int_0^{t_1} \frac{D}{\theta+\beta}\left[e^{(\theta+\beta)t_1} - (\theta-\beta)t_1 - 1\right]dt \\ &= \frac{D\widetilde{h}_r}{(\theta+\beta)^2}\left[e^{(\theta+\beta)t_1} - (\theta-\beta)t_1 - 1\right] \end{aligned} \tag{15}$$

and

$$\widetilde{h}_o \int_0^{T} I_o(t)dt = \widetilde{h}_o\left[\int_0^{t_1} W e^{-\alpha t}dt + \int_{t_1}^{T} \frac{D}{\theta+\alpha}\left(e^{(\theta+\alpha)(T-t)} - 1\right)dt\right]$$

$$= \widetilde{h}_o\left[\frac{W}{\alpha}(1 - e^{-\alpha t_1}) + \frac{D}{(\theta+\alpha)^2}\left(e^{(\theta+\alpha)(T-t_1)} - (\theta+\alpha)(T-t_1) - 1\right)\right] \tag{16}$$

(3) Total deteriorating cost: The amounts of deteriorated items in both RW and OW during $[0, T]$ is

$$\widetilde{c}\Big[\beta\int_0^{t_1} I_r(t)dt+\alpha\int_0^{T} I_0(t)dt\Big] = \widetilde{c}\Big[\frac{D\beta}{(\theta+\beta)^2}\Big(e^{(\theta+\beta)t_1}-(\theta+\beta)t_1-1\Big)+W(1-e^{-\alpha t_1}) + \frac{D\alpha}{(\theta+\alpha)^2}\Big(e^{(\theta+\alpha)(T-t_1)}-(\theta+\alpha)(T-t_1)-1\Big)\Big] \quad (17)$$

(4) The interest payable opportunity cost.
There are three cases depicted as Fig. 3.

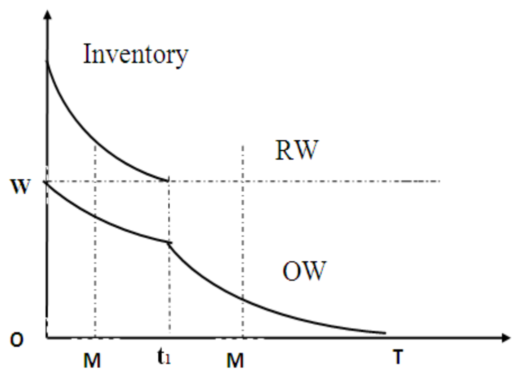

FIGURE 3. Graphical Representation of Different Trade Credit Period M.

Case 1: $M \le t_1 < T$ In this case, the annual interest payable is

$$\begin{aligned}&\widetilde{c}I_p\Big[\int_M^{t_1} I_r(t)dt+\int_M^{t_1} I_o(t)dt+\int_{t_1}^{T} I_o(t)dt\Big]\\ &=\widetilde{c}I_p\Big[\frac{D}{(\theta+\beta)^2}\Big(e^{(\theta+\beta)(t_1-M)}-(\theta+\beta)(t_1-M)-1\Big)\\ &+\frac{W}{\alpha}(e^{-\alpha M}-e^{-\alpha t_1})+\frac{D}{(\theta+\alpha)^2}\Big(e^{(\theta+\alpha)(T-t_1)}-(\theta+\alpha)(T-t_1)-1\Big)\Big] \quad (18)\end{aligned}$$

Case 2: $t_1 < M \le T$.
In this case, the total interest payable is

$$\widetilde{c}I_p\int_M^{T} I_o(t)dt=\frac{D\widetilde{c}I_p}{(\theta+\alpha)^2}\Big[e^{(\theta+\alpha)(T-M)}-(\theta+\alpha)(T-M)-1\Big] \quad (19)$$

Case 3: $M > T$
In this case, no interest charges are paid for the items.

(5) The opportunity interest earned. There are two subcases as follows:
Subcase 1: $0 < M \leq t_1$
In this case, the annual interest earned is

$$p\widetilde{I}_e \int_0^M (M-t)D(.)(t)dt = p\widetilde{I}_e\Big[\int_0^M (M-t)(D+\theta W e^{-\alpha t})dt\Big]$$
$$= p\widetilde{I}_e\Big[DM^2 + \frac{\theta W}{\alpha^2}(1+M\alpha - e^{-\alpha M})\Big] \tag{20}$$

Subcase 2: $t_1 < M \leq T$.
In this case, the annual interest earned is

$$p\widetilde{I}_e \int_0^M (M-t)D(.)(t)dt$$
$$= p\widetilde{I}_e\Big[\int_0^{t_1} (M-t)(D+\theta W e^{-\alpha t})dt + \int_{t_1}^{M} (M-t)\Big(D + \frac{D}{\theta+\alpha}[e^{(\theta+\alpha)(T-t)} - 1]\Big)dt\Big]$$
$$= p\widetilde{I}_e\Big[(M-t_1)(Dt_1 - \frac{\theta W}{\alpha}e^{-\alpha t_1}) + (\frac{1}{2}Dt_1 - \frac{\theta W}{\alpha^2}e^{-\alpha t_1})$$
$$-\frac{\theta W}{\alpha^2}(1-M\alpha) + \frac{\theta D}{\theta+\alpha}\Big(\frac{1}{2}(M^2 - t_1^2)$$
$$-\frac{e^{(\theta+\alpha)(T-M)} - e^{(\theta+\alpha)(T-t_1)}}{(\theta+\alpha)^2}\Big) - D(M-t_1)\Big\{t_1 - \frac{\theta}{\theta+\alpha}\Big(t_1 - \frac{e^{(\theta+\alpha)(T-t_1)}}{(\theta+\alpha)}\Big)\Big\}\Big] \tag{21}$$

Case 3: $M > T$
In this case, the annual interest earned is

$$p\widetilde{I}_e \int_0^T (M-t)D(.)(t)dt$$
$$= p\widetilde{I}_e\Big[\int_0^{t_1} (M-t)(D+\theta W e^{-\alpha t})dt + \int_{t_1}^{T} (M-t)\Big(D + \frac{D}{\theta+\alpha}[e^{(\theta+\alpha)(M-t)} - 1]\Big)dt\Big]$$
$$= p\widetilde{I}_e\Big[(M-t_1)(Dt_1 - \frac{\theta W}{\alpha}e^{-\alpha t_1}) + (\frac{1}{2}Dt_1^2 - \frac{\theta W}{\alpha^2}e^{-\alpha t_1}) - \frac{\theta W}{\alpha^2}(1-M\alpha)$$
$$+ \quad (M-T)\Big\{DT - \frac{\theta D}{\theta+\alpha}\Big(T + \frac{1}{\theta+\alpha})\Big\} - (M-t_1)\Big\{Dt_1 - \frac{\theta D}{\theta+\alpha}\Big(t_1 + \frac{e^{(\theta+\alpha)(T-t_1)}}{\theta+\alpha}\Big)\Big\}$$
$$+ \quad \Big\{\frac{D}{2}(T^2 - t_1^2) + \frac{\theta D}{\theta+\alpha}\Big(\frac{1 - e^{(\theta+\alpha)(T-t_1)}}{(\theta+\alpha)^2} - \frac{1}{2}(T^2 - t_1^2)\Big)\Big\}\Big] \tag{22}$$

Therefore, the annual total relevant costs for the retailer can be expressed as:
$TC(t_1, T)$= ordering cost + stock holding cost in RW + stock holding cost in OW + deteriorating cost + interest payable opportunity cost − opportunity interest earned. That is

$$TC = \begin{cases} TC_1, & \text{if } M \le t_1 < T; \\ TC_2, & \text{if } t_1 < M \le T; \\ TC_3, & \text{if } M > T. \end{cases} \tag{23}$$

Where

$$\begin{aligned} TC_1 = \frac{1}{T}\Bigg[&A + \frac{D}{(\theta+\beta)^2}\Bigg\{(\widetilde{h}_r + \widetilde{c}\beta)(e^{(\theta+\beta)t_1} - (\theta+\beta)t_1 - 1) + \widetilde{c}I_p(e^{(\theta+\beta)(T-M)} - (\theta-\beta) \\ &\times(t_1 - M) - 1)\Bigg\} + \frac{W}{\alpha}\Bigg\{(\widetilde{h}_o + \widetilde{c}\alpha)(1 - e^{-\alpha t_1}) + \widetilde{c}I_p(e^{-\alpha M} - e^{-\alpha t_1})\Bigg\} + \frac{D}{(\theta+\beta)^2}(\widetilde{h}_o + \widetilde{c}\alpha + \widetilde{c}I_p) \\ &\times(e^{(\theta+\beta)(T-t_1)} - (\theta+\beta)(T-t_1) - 1) - p\widetilde{I}_e\Bigg(DM^2 + \frac{\theta W}{\alpha^2}(1 + M\alpha - e^{-\alpha M})\Bigg)\Bigg], \end{aligned} \tag{24}$$

$$\begin{aligned} TC_2 = \frac{1}{T}\Bigg[&A + \frac{D}{(\theta+\beta)^2}\Bigg\{(\widetilde{h}_r + \widetilde{c}\beta)(e^{(\theta+\beta)t_1} - (\theta+\beta)t_1 - 1)\Bigg\} + \frac{W}{\alpha}\Bigg\{(\widetilde{h}_o + \widetilde{c}\alpha)(1 - e^{-\alpha t_1})\Bigg\} \\ &+ \frac{D}{(\theta+\beta)^2}(\widetilde{h}_o + \widetilde{c}\alpha + \widetilde{c}I_p)(e^{(\theta+\beta)(T-t_1)} - (\theta+\beta)(T-t_1) - 1) + \frac{D\widetilde{c}I_p}{(\theta+\alpha)^2}\Bigg\{e^{(\theta+\alpha)(T-M)} - 1 \\ &-(\theta+\alpha)(T-M)\Bigg\} - p\widetilde{I}_e\Bigg\{(M - t_1)(Dt_1 - \frac{\theta W}{\alpha}e^{-\alpha t_1}) + (\frac{1}{2}Dt_1 - \frac{\theta W}{\alpha^2}e^{-\alpha t_1}) \\ &-\frac{\theta W}{\alpha^2}(1 - M\alpha) + \frac{D}{2}(M^2 - t_1^2) + \frac{\theta D}{\theta+\alpha}\Bigg(\frac{1}{2}(M^2 - t_1^2) - \frac{e^{(\theta+\alpha)(T-M)} - e^{(\theta+\alpha)(T-t_1)}}{(\theta+\alpha)^2}\Bigg) \\ &-(M - t_1)\Bigg\{Dt_1 - \frac{\theta D}{\theta+\alpha}\Bigg(t_1 + \frac{e^{(\theta+\alpha)(T-t_1)}}{(\theta+\alpha)^2}\Bigg)\Bigg\}\Bigg\}\Bigg] \end{aligned} \tag{25}$$

and

$$TC_3 = \frac{1}{T}\Bigg[A + \frac{D}{(\theta+\beta)^2}\Big\{(\widetilde{h}_r + \widetilde{c}\beta)(e^{(\theta+\beta)t_1} - (\theta+\beta)t_1 - 1)\Big\} + \frac{W}{\alpha}\Big\{(\widetilde{h}_o + \widetilde{c}\alpha)(1 - e^{-\alpha t_1})\Big\}$$
$$+\frac{D}{(\theta+\alpha)^2}(\widetilde{h}_o + \widetilde{c}\alpha)(e^{(\theta+\alpha)(T-t_1)} - (\theta+\alpha)(T-t_1) - 1) - p\widetilde{I}_e\Bigg[(M-t_1)(Dt_1 - \frac{\theta W}{\alpha}e^{-\alpha t_1})$$
$$+(\frac{1}{2}Dt_1 - \frac{\theta W}{\alpha^2}e^{-\alpha t_1}) - \frac{\theta W}{\alpha^2}(1 - M\alpha) + (M-T)\Big\{DT - \frac{\theta D}{\theta+\alpha}\Big(T + \frac{1}{\theta+\alpha})\Big\}$$
$$-(M-t_1)\Big\{Dt_1 - \frac{\theta D}{\theta+\alpha}\Big(t_1 + \frac{e^{(\theta+\alpha)(T-t_1)}}{\theta+\alpha}\Big)\Big\}$$
$$+\Big\{\frac{D}{2}(T^2 - t_1^2) + \frac{\theta D}{\theta+\alpha}\Big(\frac{1 - e^{(\theta+\alpha)(T-t_1)}}{(\theta+\alpha)^2} - \frac{1}{2}(T^2 - t_1^2)\Big)\Big\}\Bigg]\Bigg]. \quad (26)$$

4.2. **Proposed inventory model in crisp environment.** For the triangular fuzzy numbers $\widetilde{c} = (c_1, c_2, c_3)$,$\widetilde{h}_r = (h_{r1}, h_{r2}, h_{r3})$, $\widetilde{h}_o = (h_{o1}, h_{o2}, h_{o3})$ and $\widetilde{I}_e = (I_{e1}, I_{e2}, I_{e3})$using lemma-1, we have the equivalent expected values as follows
$E[\tilde{c}] = \frac{1}{2}\Bigg[(1-\rho)c_1 + c_2 + \rho c_3\Bigg]$,$E[\tilde{h_r}] = \frac{1}{2}\Bigg[(1-\rho)h_{r1} + h_{r2} + \rho h_{r3}\Bigg]$, $E[\tilde{h_o}] = \frac{1}{2}\Bigg[(1-\rho)h_{o1} + h_{o2} + \rho h_{o3}\Bigg]$,$E[\tilde{I_e}] = \frac{1}{2}\Bigg[(1-\rho)I_{e1} + I_{e2} + \rho I_{e3}\Bigg]$ where $0 < \rho < 1$.
The expected total cost function is given by

$$ETC = \begin{cases} ETC_1, & \text{if } M \leq t_1 < T; \\ ETC_2, & \text{if } t_1 < M \leq T; \\ ETC_3, & \text{if } M > T. \end{cases} \quad (27)$$

Where

$$ETC_1 = \frac{1}{T}\Bigg[A + \frac{D}{(\theta+\beta)^2}\Big\{\Big(E[\widetilde{h}_r] + E[\widetilde{c}]\beta\Big)\Big(e^{(\theta+\beta)(T-t_1)} - \Big(\theta+\beta)t_1 - 1\Big)$$
$$+E[\widetilde{c}]I_p\Big(e^{(\theta+\beta)(T-M)} - (\theta-\beta)(t_1 - M) - 1\Big)\Big\} + \frac{W}{\alpha}\Big\{\Big(E[\widetilde{h}_0]$$
$$+E[\widetilde{c}]\alpha)(1 - e^{-\alpha t_1}\Big) + E[\widetilde{c}]I_p(e^{-\alpha M} - e^{-\alpha t_1})\Big\} +$$
$$\frac{D}{(\theta+\beta)^2}\Big(E[\widetilde{h}_o] + E[\widetilde{c}]\alpha + E[\widetilde{c}]I_p\Big)\Big(e^{(\theta+\beta)(T-t_1)} - (\theta+\beta)(T-t_1) - 1\Big)$$
$$-pE[\widetilde{I}_e]\Big[DM^2 + \frac{\theta W}{\alpha^2}(1 + M\alpha - e^{-\alpha M})\Big], \quad (28)$$

$$ETC_2 = \frac{1}{T}\Bigg[A + \frac{D}{(\theta+\beta)^2}\Big\{(E[\widetilde{h}_r] + E[\widetilde{c}]\beta)(e^{(\theta+\beta)t_1} - (\theta+\beta)t_1 - 1)\Big\} + \frac{W}{\alpha}\Big\{(E[\widetilde{h}_o]$$
$$+E[\widetilde{c}]\alpha)(1 - e^{-\alpha t_1})\Big\} + \frac{D}{(\theta+\beta)^2}(E[\widetilde{h}_o] + E[\widetilde{c}]\alpha + E[\widetilde{c}]I_p)(e^{(\theta+\beta)(T-t_1)} - (\theta+\beta)(T-t_1) - 1)$$
$$+\frac{DE[\widetilde{c}]I_p}{(\theta+\alpha)^2}\Big\{e^{(\theta+\alpha)(T-M)} - 1 - (\theta+\alpha)(T-M)\Big\} - E[p]\widetilde{I}_e\Big\{(M-t_1)(Dt_1 - \frac{\theta W}{\alpha}e^{-\alpha t_1})$$
$$+(\frac{1}{2}Dt_1 - \frac{\theta W}{\alpha^2}e^{-\alpha t_1}) - \frac{\theta W}{\alpha^2}(1 - M\alpha) + \frac{D}{2}(M^2 - t_1^2) + \frac{\theta D}{\theta+\alpha}\left(\frac{1}{2}(M^2 - t_1^2) - \frac{e^{(\theta+\alpha)(T-M)}}{(\theta+\alpha)^2}\right)$$
$$-(M-t_1)\Big\{Dt_1 - \frac{\theta D}{\theta+\alpha}\left(t_1 + \frac{e^{(\theta+\alpha)(T-t_1)}}{(\theta+\alpha)}\right)\Big\}\Big\}\Bigg] \tag{29}$$

and

$$ETC_3 = \frac{1}{T}\Bigg[A + \frac{D}{(\theta+\beta)^2}\Big\{(E[\widetilde{h}_r] + E[\widetilde{c}]\beta)(e^{(\theta+\beta)t_1} - (\theta+\beta)t_1 - 1)\Big\} + \frac{W}{\alpha}\Big\{(E[\widetilde{h}_o]$$
$$+E[\widetilde{c}]\alpha)(1 - e^{-\alpha t_1})\Big\} + \frac{D}{(\theta+\alpha)^2}\Big\{(E[\widetilde{h}_o] + E[\widetilde{c}]\alpha)(e^{(\theta+\alpha)(T-t_1)} - (\theta+\alpha)(T-t_1) - 1\Big\}$$
$$-pE[\widetilde{I}_e]\Bigg[(M-t_1)(Dt_1 - \frac{\theta W}{\alpha}e^{-\alpha t_1}) + (\frac{1}{2}Dt_1 - \frac{\theta W}{\alpha^2}e^{-\alpha t_1}) - \frac{\theta W}{\alpha^2}(1 - M\alpha) +$$
$$(M-T)\Big\{DT - \frac{\theta D}{\theta+\alpha}\left(T + \frac{1}{\theta+\alpha}\right)\Big\} - (M-t_1)\Big\{Dt_1 - \frac{\theta D}{\theta+\alpha}\left(t_1 + \frac{e^{(\theta+\alpha)(T-t_1)}}{\theta+\alpha}\right)\Big\}$$
$$+\Big\{\frac{D}{2}(T^2 - t_1^2) + \frac{\theta D}{\theta+\alpha}\left(\frac{1 - e^{(\theta+\alpha)(T-t_1)}}{(\theta+\alpha)^2} - \frac{1}{2}(T^2 - t_1^2)\right)\Big\}\Bigg]\Bigg] \tag{30}$$

In the next, our object is to determine the optimal values of t_1 and T such that $ETC_i(t_1, T), i = 1, 2, 3$ is minimum.

5. Analysis and optimization

(a) The necessary conditions for ETC_1 to be minimized are:

$$\frac{\partial ETC_1}{\partial t_1} = \frac{1}{T}\Bigg[\frac{D}{(\theta+\beta)}\Big\{(E[\widetilde{h}_r] + \beta E[\widetilde{c}])(e^{(\theta+\beta)t_1} - 1) + E[\widetilde{c}]I_p(e^{(\theta+\beta)(t_1-M)} - 1)\Big\} + W\Big\{(E[\widetilde{h}_o]$$
$$+E[\widetilde{c}]\alpha + E[\widetilde{c}]I_p)e^{-\alpha t_1}\Big\} + \frac{D}{(\theta+\alpha)}(E[\widetilde{h}_o] + E[\widetilde{c}]\alpha + E[\widetilde{c}]I_p)(e^{(\theta+\alpha)(T-t_1)} - 1)\Bigg] = 0 \tag{31}$$

$$\frac{\partial ETC_1}{\partial T} = \frac{1}{T}\Bigg[\frac{D}{(\theta+\alpha)}(E[\widetilde{h}_o] + E[\widetilde{c}]\alpha + I_pE[\widetilde{c}])(e^{(\theta+\alpha)(T-t_1)} - 1) - ETC_1\Bigg] = 0 \tag{32}$$

To get minimum solution of the Expect total cost function ETC_1 must satisfies the following conditions as:

$$\frac{\partial^2 ETC_1}{\partial t_1^2} = \frac{1}{T}\Bigg[D\Bigg\{(E[\widetilde{h}_r] + \beta E[\widetilde{c}])e^{(\theta+\beta)t_1} + E[\widetilde{c}]I_p e^{(\theta+\beta)(t_1-M)}\Bigg\} - \alpha W\Bigg\{E[\widetilde{h}_o]$$

$$+ \quad E[\widetilde{c}]\alpha + E[\widetilde{c}]I_p\Bigg\}e^{-\alpha t_1} + D(E[\widetilde{h}_o] + E[\widetilde{c}]\alpha + E[\widetilde{c}]I_p)e^{(\theta+\alpha)(T-t_1)}\Bigg]\Bigg|_{(t_1^{1*},T^{1*})} > 0 \quad (33)$$

$$\frac{\partial^2 ETC_1}{\partial t_1 \partial T}\Bigg|_{(t_1^{1*},T^{1*})} = -\frac{1}{T}\Bigg[D(E[\widetilde{h}_o] + E[\widetilde{c}]\alpha + E[\widetilde{c}]I_p)e^{(\theta+\alpha)(T-t_1)}\Bigg]\Bigg|_{(t_1^{1*},T^{1*})} \quad (34)$$

$$\frac{\partial^2 ETC_1}{\partial T^2} = \frac{1}{T}\Bigg[D(E[\widetilde{h}_o] + E[\widetilde{c}]\alpha + I_p E[\widetilde{c}])e^{(\theta+\alpha)(T-t_1)}\Bigg]\Bigg|_{(t_1^{1*},T^{1*})} > 0 \quad (35)$$

It was noted from Eqs. (33)-(35) that

$$H_1(t_1^{1*}, T^{1*}) = \left\{\frac{\partial^2 ETC_1}{\partial t_1^2}.\frac{\partial^2 ETC_1}{\partial T^2} - \left(\frac{\partial^2 ETC_1}{\partial t_1 \partial T}\right)^2\right\}\Bigg|_{(t_1^{1*},T^{1*})} > 0 \quad (36)$$

which implies that the matrix $H_1(t_1^{1*}, T^{1*})$is positive definite and (t_1^{1*}, T^{1*}) is the minimum solution of ETC_1^*

(b) The necessary conditions for ETC_2 to be minimized are:

$$\frac{\partial ETC_2}{\partial t_1} = \frac{1}{T}\Bigg[\frac{D}{(\theta+\beta)}\Bigg\{(E[\widetilde{h}_r] + \beta E[\widetilde{c}])(e^{(\theta+\beta)t_1} - 1) + E[\widetilde{c}]I_p(e^{(\theta+\beta)(t_1-M)} - 1)\Bigg\} +$$
$$W\Bigg\{(E[\widetilde{h}_o] + E[\widetilde{c}]\alpha + E[\widetilde{c}]I_p)e^{-\alpha t_1}\Bigg\} + \frac{D}{(\theta+\alpha)}(E[\widetilde{h}_o] + E[\widetilde{c}]\alpha + E[\widetilde{c}]I_p)(e^{(\theta+\alpha)(T-t_1)} - 1)\Bigg]$$
$$-pE[\widetilde{I}_e]\Bigg\{(M-t_1)(D + \theta W e^{-\alpha t_1}) + (M-t_1)\Bigg\{D - \frac{\theta D}{\theta+\alpha}\left(1 - e^{(\theta+\alpha)(T-t_1)}\right)\Bigg\} + \theta W e^{-\alpha t_1}$$
$$+\frac{1}{2}(Dt_1 + \frac{D\theta}{\alpha}e^{-\alpha t_1}) + \frac{\theta D}{\theta+\alpha}\left(1 - e^{(\theta+\alpha)(T-t_1)}\right)\Bigg] = 0 \quad (37)$$

and

$$\frac{\partial ETC_2}{\partial T} = \frac{1}{T}\Bigg[\frac{D}{(\theta+\alpha)}(E[\widetilde{h}_o] + E[\widetilde{c}]\alpha + I_pE[\widetilde{c}])(e^{(\theta+\alpha)(T-t_1)} - 1)$$
$$-pE[\widetilde{I}_e]\Bigg\{(M-T)(1-\frac{\theta}{\theta+\alpha})$$
$$-\Bigg(T - \frac{\theta}{\theta+\alpha}(T - \frac{1}{\theta+\alpha})\Bigg) + T + \frac{\theta D}{\theta+\alpha}\Bigg(e^{(\theta+\alpha)(T-t_1)} - T\Bigg)\Bigg] = 0 \qquad (38)$$

To get minimum solution of the Expect total cost function ETC_2 must satisfies the following conditions as:

$$\frac{\partial^2 ETC_2}{\partial t_1^2} = \frac{1}{T}\Bigg[D\Bigg\{(E[\widetilde{h}_r] + \beta E[\widetilde{c}])e^{(\theta+\beta)t_1} + E[\widetilde{c}]I_pe^{(\theta+\beta)(t_1-M)}\Bigg\} - \alpha W\Bigg\{E[\widetilde{h}_o] + E[\widetilde{c}]\alpha$$
$$+ \quad E[\widetilde{c}]I_p\Bigg\}e^{-\alpha t_1} + D(E[\widetilde{h}_o] + E[\widetilde{c}]\alpha + E[\widetilde{c}]I_p)e^{(\theta+\alpha)(T-t_1)} - pE[\widetilde{I}_e]\Bigg\{\frac{\theta W}{\theta+\alpha}(1+e^{(\theta+\alpha)(T-t_1})$$
$$+ \quad \frac{1}{2}(D - \theta We^{\theta t_1}) - \Bigg\{D\theta(M-t_1) + \theta\alpha + D + W\theta\Bigg\}e^{-\alpha t_1}$$
$$+\frac{\theta D}{\theta+\alpha}(1+e^{(\theta+\alpha)(T-t_1})\Bigg\}\Bigg|_{(t_1^{2*},T^{2*})} > 0 \qquad (39)$$

$$\frac{\partial^2 ETC_2}{\partial t_1 \partial T}\Bigg|_{(t_1^{2*},T^{2*})} = -\frac{1}{T}\Bigg[D(E[\widetilde{h}_o] + E[\widetilde{c}]\alpha + E[\widetilde{c}]I_p)e^{(\theta+\alpha)(T-t_1)}$$
$$-pE[\widetilde{I}_e]\Bigg\{D\theta - M + t_1\Bigg\}e^{(\theta+\alpha)(T-t_1})\Bigg]\Bigg|_{(t_1^{2*},T^{2*})} \qquad (40)$$

$$\frac{\partial^2 ETC_2}{\partial T^2} = -\frac{1}{T}\frac{\partial ETC_2}{\partial T} - \frac{1}{T}\Bigg[D\Bigg((E[\widetilde{h}_o] + E[\widetilde{c}]\alpha + I_pE[\widetilde{c}]\Bigg)e^{(\theta+\alpha)(T-t_1)}$$
$$-pE[\widetilde{I}_e]\Bigg\{1 + D\theta(e^{(\theta+\alpha)(T-t_1)} + 1\Bigg\}\Bigg] > 0 \qquad (41)$$

It was noted from Eqs. (39)-(41) that

$$H_2(t_1^{2*}, T^{2*}) = \Bigg\{\frac{\partial^2 ETC_2}{\partial t_1^2}.\frac{\partial^2 ETC_2}{\partial T^2} - \Bigg(\frac{\partial^2 ETC_2}{\partial t_1 \partial T}\Bigg)^2\Bigg\}\Bigg|_{(t_1^{2*},T^{2*})} > 0 \qquad (42)$$

which implies that the matrix $H_2(t_1^{2*}, T^{2*})$is positive definite and (t_1^{2*}, T^{2*}) is the optimal solution of ETC_2^*

(c) The necessary conditions for ETC_3 to be minimized are:

$$\begin{aligned}
\frac{\partial ETC_3}{\partial t_1} &= \frac{1}{T}\Bigg[\frac{D}{(\theta+\beta)}\Big\{(E[\widetilde{h}_r]+\beta E[\widetilde{c}])(e^{(\theta+\beta)t_1}-1)+E[\widetilde{c}]I_p(e^{(\theta+\beta)(t_1-M)}-1)\Big\}\\
+\quad & W\Big\{(E[\widetilde{h}_o]+E[\widetilde{c}]\alpha+E[\widetilde{c}]I_p)e^{-\alpha t_1}\Big\}+\frac{D}{(\theta+\alpha)}(E[\widetilde{h}_o]+E[\widetilde{c}]\alpha+E[\widetilde{c}]I_p)(e^{(\theta+\alpha)(T-t_1)}-1)\\
-\quad & pE[\widetilde{I}_e]\Big\{(M-t_1)(D+W\theta e^{(-\alpha t_1)}-D(M-t_1)\Big(D-\frac{\theta}{\alpha+\theta}(1-\frac{e^{(-\alpha t_1)(T-t_1)}}{\theta+\alpha})\Big)-t_1\\
& +\frac{\theta}{\alpha+\theta}\Big(1-e^{(-\alpha t_1)(T-t_1)}\Big)\Big\}\Bigg]=0 \qquad (43)
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial ETC_3}{\partial T} &= \frac{1}{T}\Bigg[\frac{D}{(\theta+\alpha)}(E[\widetilde{h}_o]+E[\widetilde{c}]\alpha+I_pE[\widetilde{c}])(e^{(\theta+\alpha)(T-t_1)}-1)\\
& -pE[\widetilde{I}_e]\Big\{D\theta(M-t_1)e^{(\alpha+\theta)(T-t_1)}+\frac{\theta}{\alpha+\theta}\Big(T+M+\frac{e^{(\theta+\alpha t_1)(T-t_1)}}{\alpha+\theta}\\
& -\frac{\theta}{\alpha+\theta}(M-T+\theta)e^{(\theta+\alpha t_1)(T-t_1)}\Big)\Big\}-ETC_3\Bigg]=0 \qquad (44)
\end{aligned}$$

To get minimum solution of the Expect total cost function ETC_3 must satisfies the following conditions as:

$$\begin{aligned}
\frac{\partial^2 ETC_3}{\partial t_1^2} &= \frac{1}{T}\Bigg[D\Big\{(E[\widetilde{h}_r]+\beta E[\widetilde{c}])e^{(\theta+\beta)t_1}+E[\widetilde{c}]I_pe^{(\theta+\beta)(t_1-M)}\Big\}-\alpha W\Big\{E[\widetilde{h}_o]\\
& +E[\widetilde{c}]\alpha+E[\widetilde{c}]I_p\Big\}e^{-\alpha t_1}+D\Big(E[\widetilde{h}_o]+E[\widetilde{c}]\alpha+E[\widetilde{c}]I_p\Big)e^{(\theta+\alpha)(T-t_1)}\\
& -pE[\widetilde{I}_e]\Big\{D\theta(M-t_1+\theta)e^{(\theta+\alpha)(T-t_1)}-(M-t_1+W\theta)e^{-\alpha t_1}\Big\}\Bigg]\Bigg|_{(t_1^{3*},T^{3*})}>0 \qquad (45)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 ETC_3}{\partial T^2} &= \frac{1}{T}\Bigg[pE[\widetilde{I}_e]\Big\{D\theta(M-t_1)e^{(\alpha+\theta)(T-t_1)}+\frac{\theta}{\alpha+\theta}\Big(1+e^{(\theta+\alpha t_1)(T-t_1)}\Big)\\
& -D\Big(E[\widetilde{h}_o]+E[\widetilde{c}]\alpha+I_pE[\widetilde{c}]\Big)e^{(\theta+\alpha)(T-t_1)}\Big\}\Bigg]>0 \qquad (46)
\end{aligned}$$

$$\left.\frac{\partial^2 ETC_3}{\partial t_1 \partial T}\right|_{(t_1^{1*},T^{1*})} = -\frac{1}{T}\Bigg[D(E[\widetilde{h}_o] + E[\widetilde{c}]\alpha + E[\widetilde{c}]I_p)e^{(\theta+\alpha)(T-t_1)} \\ -pE[\widetilde{I}_e]\Big\{D(M - t_1 + 1)e^{(\theta+\alpha)(T-t_1)}\Big\}\Bigg]\Bigg|_{(t_1^{3*},T^{3*})} \quad (47)$$

It was noted from Eqs. (42)-(45) that

$$H_3(t_1^{3*}, T^{3*}) = \left.\left\{\frac{\partial^2 ETC_3}{\partial t_1^2}.\frac{\partial^2 ETC_3}{\partial T^2} - \left(\frac{\partial^2 ETC_3}{\partial t_1 \partial T}\right)^2\right\}\right|_{(t_1^{3*},T^{3*})} > 0 \quad (48)$$

which implies that the matrix $H_1(t_1^{3*}, T^{3*})$is positive definite and (t_1^{3*}, T^{3*}) is the optimal solution of ETC_1^*

5.1. **Algorithm.** Step 0: Input the parameters.
Step 1: Solving Eqs. (31) and (32) by MATHEMATICA, getting the optimal solution t_1^{1*} and T^{1*} ,If $M \le t_1^{1*} < T^{1*}$and satisfies the eq. (38)-(40),let $t_1^* = t_1^{1*}$,$T^* = T^{1*}$,then minimum $ETC^* = ETC_1(t_1^{1*}, T^{1*})$; Otherwise, go to Step 2.
Step 2: Solving Eqs. (37) and (38) by MATHEMATICA, getting the optimal solution t_1^{2*} and T^{2*} ,If $t_1^{1*} < M \le T^{2*}$ and satisfies the eq. (37)-(40), let $t_1^* = t_1^{1*}$,$T^* = T^{2*}$,then minimum $ETC^* = ETC_2(t_1^{2*}, T^{2*})$; Otherwise, go to Step 3.
Step 3: Solving Eqs. (43) and (45) by MATHEMATICA, getting the optimal solution t_1^{3*} and T^{3*} ,If $t_1^{3*} < M \le T^{3*}$ and satisfies the eq. (45)-(48), let $t_1^* = t_1^{3*}$,$T^* = T^{3*}$,then minimum $ETC^* = ETC_3(t_1^{3*}, T^{3*})$; Otherwise, go to Step 4.
Step 4: If all steps are satisfied,then find
$(t_1^*, T^*) = \arg\, min\{ETC_1(t_1^{1*}, T^{1*}),\ ETC_2(t_1^{2*}, T^{2*}), ETC_3(t_1^{3*}, T^{3*})\}$, then the Minimum solutions are t_1^*, T^* and ETC^*;Otherwise, go to Step 0.

6. Numerical Illustration

Example: Given an inventory system with the following parameters: A = \$ 1500/order, D_1 =2000 units/year, $\theta = 0.25$ $\widetilde{h}_o = \$(0.5, 1, 1.5)$ $/unit/year, \widetilde{h}_r$ = \$ $(2, 3, 4)$/unit/year, $\widetilde{c}$ = \$ $(8, 10, 12)$/unit,p = \$15 /unit/year, I_p =0.15/\$/year, I_e= (0.02,0.03,0.04)\$/year, M = 3/12 = 0.25 year,W= 100 units,$\alpha = 0.1$, $\beta = 0.06$.
According to the Algorithm 5.1 in Section 5, it can be found the optimal solutions t_1^* = 0.1503 year; T^*= 0.7727 year and ETC^* = 2894.4\$. Thus, it is clear that the inventory in RM vanish at 0.1503 year and then after 0.6224 year own warehouses(OW) is also empty and minimal total costs 2894.4\$.

7. Conclusion

In this paper,for first time, a two-warehouse inventory model for deteriorating items with stock dependent demand is developed under conditionally permissible delay in payment in imprecise environment. Shortages are not permitted in this inventory system. It was assumed that the rented warehouse charges higher unit holding cost than the own warehouse, but to offer a better preserving facility resulting in a lower rate of deterioration for the goods than the own warehouse. In order to reduce the inventory costs, it will be economical to consume the goods of RW at the earliest. Consequently, the firm should store goods in OW before RW, but clear the stocks in RW before OW. Our aim is to find the optimal replenishment policies for minimizing the total revelent inventory costs. Some useful theorems to characterize the optimal solutions have been obtained. Numerical examples are also provided to illustrate the proposed model. Moreover, sensitivity analysis of the optimal solutions with respect to major parameters are carried out. The proposed model can be extended in several ways. First, we may extend the model to incorporate some more realistic features, such as quantity discount, the inventory holding cost and others are also fluctuating with time. Second, we could generalize the model under two-level credit period strategy in imprecise environment.

References

[1] L. Benkherouf, A deterministic order level inventory model for deteriorating items with two storage facilities, International Journal of Production Economics, 48 (1997) 167-175.

[2] K.J. Chung, T.S. Huang, The optimal retailers ordering policies for deteriorating items with limited storage capacity under trade credit financing, International Journal of Production Economics,106 (2007)127-145.

[3] J.K. Dey, S.K.Mondal, M. Maiti, Two storage inventory problem with dynamic demand and interval valued lead-time over finite time horizon under inflation and time-value of money, European Journal of Operational Research, 185 (2008) 170-194.

[4] D. Dubois, H. Prade, Ranking Fuzzy numbers in the setting of Possibility Theory, Information Sciences, 30(1983) 183-224.

[5] S.K. Goyal, Economic order quantity under conditions of permissible delay in payments, Journal of the Operational research Society, 36 (1985) 35-38.

[6] S.K. Goyal, B.C. Giri, Recent trends in modeling of deteriorating inventory, European Journal of Operational Research, 134 (2001) 1-16.

[7] T.P. Hsieh, C.Y. Dye, L.Y. Ouyang, Determining optimal lot size for a two-warehouse system with deterioration and shortages using net present value, European Journal of Operational Research, 191 (2008)182-192.

[8] Y.F. Huang, An inventory model under two levels of trade credit and limited storage space derived without derivatives, Applied Mathematical Modelling, 30 (2006) 418-436.

[9] B. Liu, K.B. Iwamura, A note on chance constrained programming with fuzzy coefficients, Fuzzy Sets and Systems, 100(1998) 229-233.

[10] C.C. Lee, S.L. Hsu, A two-warehouse production model for deteriorating inventory items with time-dependent demands, European Journal of Operational Research, 194 (2009) 700-710.

[11] C.C.Lee, Two-warehouse inventory model with deterioration under FIFO dispatching policy, European Journal of Operational Research, 174 (2006) 861-873.

[12] Y. Liang ,F. Zhou, A two-warehouse inventory model for deteriorating items under conditionally permissible delay in payment, Applied mathematicl Modelling,35 (2011) 2221-2231.

[13] B. Liu, Y.K. Liu, Expected value of Fuzzy variable and Fuzzy expected value Models, IEEE Transactions of Fuzzy Systems, 10(4), (2002), 445-450.

[14] M.K. Maiti, Fuzzy inventory model with two warehouses under possibility measure on fuzzy goal, European Journal of Operational Research, 188 (2008) 746-774.

[15] L.Y. Ouyang, K.S. Wu, C.T. Yang, A study on an inventory model for non-instantaneous deteriorating items with permissible delay in payments, Computers and Industrial Engineering, 51 (2006) 637-651.

[16] M. Rong, N.K. Mahapatra, M. Maiti, A two warehouse inventory model for a deteriorating item with partially/fully backlogged shortage and fuzzy lead time, European Journal of Operational Research, 189 (2008) 59-75.

[17] K.V.S. Sarma, A deterministic order level inventory model for deteriorating items with two storage facilities, European Journal of Operational Research ,29 (1987) 70-73.

[18] K.S. Wu, L.Y. Ouyang, C.T. Yang, An optimal replenishment policy for non-instantaneous deteriorating items with stock dependent demand and partial backlogging, International Journal of Production Economics 101 (2006) 369384.

[19] H.L. Yang, Two-warehouse inventory models for deteriorating items with shortages under inflation, European Journal of Operational Research ,157 (2004) 344-356.

[20] H.L. Yang, Two-warehouse partial backlogging inventory models for deteriorating items under inflation, International Journal of Production Economics, 103 (2006) 362-370.

[21] Y.W. Zhou, S.L. Yang, A two-warehouse inventory model for items with stock-level-dependent demand rate, International Journal of Production Economics ,95 (2005) 215-228.

[22] S. Khanra, S.K. Ghosh, K.S. Chaudhuri, An EOQ model for a deteriorating item with time dependent quadratic demand under permissible delay in payment, Applied Mathematics and Computation, 218 (2011) 1-9.

[23] R. Maihami, I.N.K. Abadi,. Joint control of inventory and its pricing for non-instantaneously deteriorating items under permissible delay in payments and partial backlogging Mathematical and Computer Modelling 55 (2012) 1722-1733.

[24] M.K. Meher, G.C. Panda, S.K Sahu,., An inventory model with weibull deterioration rate under the delay in payment in demand decling market, Applied Mathematical Sciences , 6 (2012) 1121-1133.

[25] A.Musa, B. Sani, Inventory ordering policies of delayed deteriorating items under permissible delay in payments. . International Journal of Production Economics, 136 (2012) 75-83.

[26] K. Maity, Possibility and necessity representations of fuzzy inequality and its application to two warehouse production-inventory problem , Applied Mathematical Modelling, 35 (3) (2011) 1252-1263.

[27] D.K. Jana, K. Maity, B. Das, and TK Roy, A fuzzy simulation via contractive mapping genetic algorithm approach to an imprecise production inventory model under volume flexibility, Journal of Simulation (2012), 1-11