Feasible short-step interior point algorithm for linear complementarity problem based on kernel function

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Abstract. In this paper we deal with the study of the polynomial complexity analysis and numerical implementation for a short-step interior point algorithm for monotone linear complementarity problems (\textit{LCP}) based on kernel function. The analysis is based on a new class of search directions. We establish the global convergence of the algorithm. Furthermore, it is shown that the algorithm has $O(n^{2.5}L)$, iteration complexity. For its numerical tests some strategies are used and indicate that the algorithm is efficient.

Keyword(s). Quadratic programming, Convex nonlinear programming, Interior point methods

AMS subject classification. 90C30, 90C51

1. Introduction

Let us consider the linear complementarity problem (\textit{LCP}): find vectors $x$ and $y$ in real space $\mathbb{R}^n$ that satisfy the following conditions:

$$x \geq 0, \ y = Mx + q \geq 0 \text{ and } x^ty = 0,$$

where $q$ is a given vector in $\mathbb{R}^n$ and $M$ is a given $n \times n$ real matrix. \textit{LCP} have important applications in mathematical programming and various areas of engineering. Interior-point methods (\textit{IPMs}) for solving Linear Optimization (\textit{LO}) problems were initiated by Karmarkar \cite{2}. They not only have polynomial complexity, but are also highly efficient in practice. Feasible \textit{IPMs} start with a strictly feasible interior-point and maintain feasibility during the solution process. Feasible \textit{IPMs} require that the starting points satisfy exactly the equality constraints and are strictly positive, i.e., they lie in the interior of a region defined by constraints. Extending methods for \textit{LO} to \textit{LCP} has been successful in many cases. See, e.g.,\cite{3, 4}. Recently, Peng et al. \cite{5, 6} designed...
primal–dual feasible IPMs by using self-regular functions for LO and also extended the approach to LCP. In this paper we deal with the complexity analysis and the numerical implementation of a short-step interior point algorithm. This algorithm is based on the strategy of the central path and on a method for finding a new search directions, where we show that this short-step algorithm deserves the best current polynomial complexity namely $O(n^{2.5}L)$. The paper is organized as follows. In the next section, the statement of the problem is presented, we deal with the weighted vector introduced to ensure that the initial point $(x^0, y^0, \mu^0)$ verified $(x^0, y^0, \mu^0) = 0$, (proximity measure define bellow). In Section 3, we deal with the new search directions and the description of the algorithm. In Section 4, we state its polynomial complexity. Section 5 contains the numerical experiments. In Section 6, a conclusion and remarks are given.

We use the classical notation. In particular, $\mathbb{R}^n$ denotes the $n$-dimensional Euclidean space. Given $u, v \in \mathbb{R}^n$, $u^t v = \sum_{i=1}^n u_i v_i$ is their inner product, and $\|u\| = \sqrt{u^t u}$ is the Euclidean norm. Given a vector $d \in \mathbb{R}^n$, $D = \text{diag}(d)$ is the $n \times n$ diagonal matrix. $I$ is the identity matrix and $e$ is the identity vector.

2. Statement of the problem

The feasible set, the strictly feasible set and the solution set of (1.1) are denoted, respectively by

$$F = \{(x, y) \in \mathbb{R}^{2n} : y = Mx + q, x \geq 0, y \geq 0\},$$

$$F_{\text{int}} = \{(x, y) \in F : x > 0, y > 0\},$$

and

$$\Omega = \{(x, y) \in F : x \geq 0, y \geq 0, x^t y = 0\}.$$

In this paper, we assume that the following assumptions hold.

**Assumption 1.** $F_{\text{int}} \neq \emptyset$.

**Assumption 2.** $M$ is a positive semidefinite matrix.

In addition (1.1), is equivalent to the following convex quadratic problem, see, e.g.,[7].

$$\min \left\{ x^t y : x \geq 0, y \geq 0, y = Mx + q \right\}. \tag{2.1}$$

Hence, finding the solution of (1.1) is equivalent to find the minimizer of (2.1) with its objective value is zero.

In order to introduce an interior point method to solve (2.1), we associate with it the following barrier minimization problem

$$\min \left\{ f_{\mu r}(x, y) : y = Mx + q, x > 0, y > 0 \right\}, \tag{2.2}$$
where \( f_r(x, y) = x^t y - \mu \sum_{i=1}^{n} r_i \log(x_i y_i) \), \( \mu > 0 \) be the barrier parameter and \( r = (r_1, \ldots, r_n) \in \mathbb{R}_+^n \) is a weighted vector introduced to ensure that the initial point \((x^0, y^0)\) verified \( \delta(x^0, y^0, \mu^0) = 0 \) (proximity measure define bellow), if \( r_i = 1, i = 1, \ldots, n \), then the weighted central path coincides with the classical one. Hence, this approach can be seen as a generalization of central path methods.

The problem (2.2) is a convex optimization problem and then its first order optimality conditions are:

\[
\begin{aligned}
Mx + q &= y, \\
x > 0, y > 0.
\end{aligned}
\]  

(2.3)

If the Assumptions 1 and 2 hold then for a fixed \( \mu > 0 \), the problem (2.2) and the system (2.3) have a unique solution [7] denoted as \((x(\mu), y(\mu))\), with \( x(\mu) > 0 \) and \( y(\mu) > 0 \). We call \((x(\mu), y(\mu))\), with \( \mu > 0 \), the \( \mu \)-centers of (2.3). The set of the \( \mu \)-centers defines the so-called the central path of (1.1).

In the next section, we introduce a method for tracing the central path based a new class of search directions.

### 3. A new search directions

Now, the basic idea behind this approach is to replace the non linear equation:

\[
\frac{xy}{\mu r} = e,
\]

in (2.3) by an equivalent equation

\[
\psi\left(\frac{xy}{\mu r}\right) = \psi(e),
\]

where \( \psi \), is a real valued function on \([0, \infty)\) and differentiable on \([0, \infty)\) such that \( \psi(t) \) and \( \psi'(t) > 0 \), for all \( t > 0 \). Then the system (2.3) can be written as the following equivalent form:

\[
\begin{aligned}
Mx + q &= y, \ x > 0, y > 0 \\
\psi\left(\frac{xy}{\mu r}\right) &= \psi(e).
\end{aligned}
\]  

(3.1)

Suppose that we have \((x, y) \in F_{int}\). Applying Newton’s method for the system (2.3), we obtain a new class of search directions:

\[
\begin{aligned}
M \Delta x &= \Delta y, \\
\frac{\partial}{\partial \mu} \psi(e) \Delta x + \frac{\partial}{\partial \mu} \psi\left(\frac{xy}{\mu r}\right) \Delta y &= \psi(e) - \psi\left(\frac{xy}{\mu r}\right).
\end{aligned}
\]  

(3.2)

Now, the following notations are useful for studying the complexity of the proposed algorithm.

The vectors
\[ v = \sqrt{\frac{xy}{\mu r}}, \quad d = \sqrt{xy^{-1}}, \]

these notations lead to

\[ \frac{d^{-1}x}{\sqrt{\mu r}} = \frac{dy}{\sqrt{\mu r}} = v. \]

Denote by

\[ d_x = \frac{d^{-1}\triangle x}{\sqrt{\mu r}}, \quad d_y = \frac{d\triangle y}{\sqrt{\mu r}}, \quad (3.3) \]

and hence, we have

\[ \mu rv(d_x + d_y) = y\triangle x + x\triangle y, \quad (3.4) \]

and

\[ d_x d_y = \frac{\triangle x\triangle y}{\mu r} \quad (3.5) \]

So using (3.3) and (3.4), the system (3.2) becomes

\[ \begin{cases} M d_x = d_y, \\ d_x + d_y = pv, \end{cases} \]

where \( M = MDM \) with \( D = \text{diag}(d) \)

and

\[ pv = \frac{\psi(e) - \psi(v^2)}{v\psi'(v^2)}. \]

We shall consider the following function:

\[ \psi(t) = \frac{1}{2}(t^2 - 1), \quad \text{with } \psi'(t) = t \quad \text{for all } t > 0. \]

Hence, the Newton directions in (3.2) is

\[ \begin{cases} M \triangle x = \triangle y, \\ d_x + d_y = \frac{1}{2}(v^{-3} - v), \end{cases} \quad (3.6) \]

with

\[ pv = \frac{1}{2}(v^{-3} - v), \]

and we define for all vector \( v \) the following proximity measure by
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\[
\delta(xy, \mu) = \frac{\|p_v\|_2}{2}, \\
= \|v^{-3} - v\|_2, \\
= \left\| \left( \frac{xy}{\mu^r} \right)^{-3} - \sqrt{\frac{xy}{\mu^r}} \right\|_2.
\]

Now, the generic short-step primal-dual algorithm to solve \( LCP \) has the following form

3.1. Algorithm

Begin algorithm

Input:
an accuracy parameter \( \varepsilon > 0 \),
an update parameter \( 0 < \theta < 1 \) (default \( \theta = \frac{1}{2\sqrt{n}} \)),
a strictly feasible point \((x^0, y^0)\) and \( \mu^0 = (\frac{x^0}{Ry^0})^T R y^0 \).

\( \sigma = \frac{\|X^0 Y^0 e\|}{\sqrt{n}}, r = \frac{X^0 Y^0 e}{\sigma}.R = diag(r_i) \)

\( k = 0 \)

While \((n \mu^k) > \varepsilon\) do

1°) Compute \((\triangle x, \triangle y)\),

2°) Update \((x^{k+1}, y^{k+1}) = (x^k, y^k) + (\triangle x, \triangle y)\)

3°) Set \( \mu^{k+1} = (1-\theta)\mu^k = (1-\theta)\frac{x^k R y^k}{n} \) and \( k = k + 1 \).

End While.

End algorithm.

4. Complexity analysis

Let

\( p_v = d_x + d_y, q_v = d_x - d_y, \)

and, we have

\( d_x = \frac{1}{2}(p_v + q_v), d_y = \frac{1}{2}(p_v - q_v), \)

hence,
\[ d_x d_y = \frac{1}{4}(p_v^2 - q_v^2) \text{ and } \|q_v\| \leq \|p_v\|. \]

This last result follows directly from the equality

\[ \|p_v\|^2 = \|q_v\|^2 + 4d_x d_y, \]

since,

\[ d_x d_y = d_x \overline{M} d_x \geq 0, \text{ because } \overline{M} \text{ is positive semidefinite.} \]

We have

\[ \delta(v, \mu) \geq \|q_v\|. \]

In the following lemma, we state a condition which ensures the feasibility of the full Newton step.

Let

\[ x^+ = x + \triangle x, \]
\[ y^+ = y + \triangle y, \]

be the new iterate after a full Newton step.

**Lemma 1.** Let \((x, y)\) is a strictly feasible iteration. If \(e + d_x d_y > 0\) then

\[ (x^+, y^+) = (x + \triangle x, y + \triangle y) \]

**Proof.** Let \(0 < \alpha \leq 1\) is step lengt. We define:

\[ x(\alpha) = x + \alpha \triangle x, \quad y(\alpha) = y + \alpha \triangle y, \]

we have

\[ x(\alpha)y(\alpha) = (x + \alpha \triangle x)(y + \alpha \triangle y) = xy + \alpha(x \triangle y + s \triangle x) + \alpha^2 \triangle x \triangle y = xy + \alpha(\mu r - xy) + \alpha^2 \triangle x \triangle y. \]

We assume that \(e + d_x d_y > 0\), we deduce \(\mu r + \triangle x \triangle y > 0\), which is equivalent to \(\triangle x \triangle y > -\mu r\), by substitution we obtain

\[ x(\alpha)s(\alpha) > xs + \alpha(\mu r - xs) - \alpha^2 \mu r = (1 - \alpha)xy + (\alpha - \alpha^2)\mu r = (1 - \alpha)xy + \alpha(1 - \alpha)\mu r. \]
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Since

\[ xy > 0 \text{ and } \mu r > 0, \]  

it follows that \( x(\alpha)y(\alpha) > 0 \) for all \( \alpha \in [0, 1] \).

Now for convenience, we may write

\[ (v^+)^2 = \frac{x^+y^+}{\mu r} = e + d_x d_y. \]

**Lemma 2.** If \( \delta(xy, \mu) < 1 \), then \( x^+ > 0 \) and \( y^+ > 0 \).

**Proof.** In the Lemma 1, we have \((x^+, y^+)\) are strictly feasible if \((e + d_x d_y) > 0\). So \((e + d_x d_y) > 0\) holds if \((1 + (d_x d_y))_i > 0\) for all \(i \in \mathbb{R}^n\).

We have

\[(1 + (d_x d_y))_i \geq (1 - |(d_x d_y)|_i), \text{ for all } i \in \mathbb{R}^n\]

\[ \geq (1 - \delta^2). \]

Thus \((e + d_x d_y) > 0\) if \(\delta(xy, \mu) < 1\).

In the next lemma we proved the local quadratic convergence for our algorithm

**Lemma 3.** Let \( \delta = \delta(xy, \mu) < 1 \) then

\[ \delta(x^+ y^+, \mu^+) \leq \frac{\delta^2}{\sqrt{2(1 - \delta^2)}}. \]

**Proof.** Letting \(\alpha = 1\), we have

\[ 4\delta^2_+ = \left\| (v^+)^{-1} - v^+ \right\|^2 \]

\[ = \left\| (v^+)^{-1}(e - (v^+)^2) \right\|^2, \]

where \((v^+)^2 = (e - d_x d_y)\) and \((v^+)^{-1} = \frac{1}{\sqrt{(e + d_x d_y)}}\), then it follows that

\[ 4\delta^2_+ = \left\| \frac{d_x d_s}{\sqrt{(e + d_x d_y)}} \right\|^2 \]

\[ = \left\| \frac{d_x d_s}{\sqrt{(e + d_x d_y)}} \right\|^2 \]

\[ \leq \frac{\|d_x d_y\|^2}{(1 - \|d_x d_y\|_\infty)}. \]
We deduce that

$$4\delta_+^2 \leq \frac{2\delta_+^4}{(1 - \delta_+^2)}.$$ 

This proves the lemma. \(\Box\)

**Lemma 4.** Let \(\delta(xy, \mu) < \frac{1}{\sqrt{2}}\) and \(\mu^+ = (1 - \theta)\mu,\ 0 < \theta < 1\). Then

$$\delta^2(x^+y^+, \mu^+) \leq (1 - \theta)\delta_+^2 + \frac{\theta^2(n + 1)}{4(1 - \theta)} + \frac{\theta}{2}.$$ 

Furthermore, if \(\delta \leq \frac{1}{\sqrt{2}},\ \theta = \frac{1}{2\sqrt{n}}\) and \(n \geq 2\), then we have \(\delta(x^+y^+, \mu^+) \leq \frac{1}{\sqrt{2}}\).

**Proof.** Let \(v^+ = \sqrt{\frac{x^+y^+}{\mu r}}\) and \(\mu^+ = (1 - \theta)\mu\), then

$$4\delta^2(x^+y^+, \mu^+) = \left\| \sqrt{\frac{\mu^+ r}{x^+y^+}} - \sqrt{\frac{x^+y^+}{\mu^+ r}} \right\|_2^2$$

$$= \left\| \sqrt{1 - \theta} (v^+)^{-1} - \frac{1}{\sqrt{1 - \theta}} v^+ \right\|_2^2$$

$$= \left\| \sqrt{1 - \theta} ((v^+)^{-1} - v^+) - \frac{\theta}{\sqrt{1 - \theta}} v^+ \right\|_2^2$$

$$= (1 - \theta) \left\| ((v^+)^{-1} - v^+) \right\|_2^2 + \frac{\theta^2}{1 - \theta} \left\| v^+ \right\|_2^2 - 2\theta \left((v^+)^{-1} - v^+\right)^t v^+$$

$$= (1 - \theta) \left\| ((v^+)^{-1} - v^+) \right\|_2^2 + \frac{\theta^2}{1 - \theta} \left\| v^+ \right\|_2^2 - 2\theta (v^+)^{-t} v^+ + v^t v^+$$

$$= 4(1 - \theta)\delta_+^2 + \frac{\theta^2}{1 - \theta} \left\| v^+ \right\|_2^2 - 2\theta n + 2\theta \left\| v^+ \right\|_2^2$$

since, \((v^+)^{-t} v^+ = n\) and \((v^+)^t v^+ = \left\| v^+ \right\|_2^2\), and we have \(\delta_+^2 \leq \frac{\delta^4}{2(1 - \delta_+^2)}\). Then

$$4\delta^2(x^+y^+, \mu^+) \leq 4(1 - \theta) \frac{\delta^4}{2(1 - \delta_+^2)} + \frac{\theta^2}{1 - \theta} \left\| v^+ \right\|_2^2 - 2\theta n + 2\theta \left\| v^+ \right\|_2^2,$$

since, \(x^+s^+ = \mu r(e + dx dy)\),

and if \(\delta < \frac{1}{\sqrt{2}}\), it follows

$$\left\| v^+ \right\|_2^2 = \frac{x^+y^+}{\mu r} = e + dx dy \leq 1 + n.$$
Consequently,
\[ 4\delta^2(x^+y^+, \mu^+) \leq 4(1-\theta)\frac{\delta^4}{2(1-\delta^2_+)} + \frac{\theta^2}{1-\theta} \|v^+\|^2 - 2\theta n + 2\theta(n+1), \]

and
\[ \delta^2(x^+y^+, \mu^+) \leq (1-\theta)\delta^4_+ + \frac{\theta^2(n+1)}{4(1-\theta)} + \frac{\theta}{2}. \]

The last statement the proof goes as follows. If \( \delta < \frac{1}{\sqrt{2}} \), then \( \delta^2_+ = \frac{1}{4} \) and this yields the following upper bound for \( \delta(x^+y^+, \mu^+) \) as:
\[ \delta^2(x^+y^+, \mu^+) \leq \frac{(1-\theta)}{4} + \frac{\theta^2(n+1)}{4(1-\theta)} + \frac{\theta}{2}. \]

Now, taking \( \theta = \frac{1}{2\sqrt{n}} \) then \( \theta^2 = \frac{1}{4n} \) if follows that
\[ \delta^2(x^+y^+, \mu^+) \leq \frac{(n+1)}{4n} + \frac{(1-\theta)}{4} + \frac{\theta}{2}, \]

since \( \frac{(n+1)}{4n} \leq \frac{3}{8} \) for all \( n \geq 2 \) then we have
\[ \delta^2(x^+y^+, \mu^+) \leq \frac{3}{32(1-\theta)} + \frac{\theta+1}{4}. \]

Now for \( n \geq 2 \), we have \( 0 < \theta \leq \frac{1}{2\sqrt{2}} \) and since the function \( f(\theta) = \frac{3}{32(1-\theta)} + \frac{\theta+1}{4} \) is continuous and monotonic increasing on \( 0 < \theta \leq \frac{1}{2\sqrt{2}} \), consequently,
\[ f(\theta) \leq f\left(\frac{1}{2\sqrt{2}}\right) < \frac{1}{2}, \text{ for all } 0 < \theta \leq \frac{1}{2\sqrt{2}}. \]

Hence,
\[ \delta(x^+y^+, \mu^+) < \frac{1}{2}. \]

The following theorem gives an upper bound for the total number of iteration for our algorithm.

**Theorem 5.** Let \( \varepsilon > 0 \) be an accuracy parameter. The algorithm has a complexity bound of \( O(\sqrt{nL}) \) iterations and the total complexity bound of the algorithm is:
\( O(n^{2.5}L) \), where \( L = \log \frac{1}{\varepsilon} \).

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Proof. We have

\[ \mu^k = (1 - \theta)^k \mu^0, \]

thus,

\[ (1 - \theta)^k \mu^0 \leq \varepsilon. \]

Now taking logarithms of \((1 - \theta)^k \mu^0 \leq \varepsilon\), we may write

\[ \log((1 - \theta)^k \mu^0) \leq \log \varepsilon \]

equivalent

\[ k \log(1 - \theta) \leq \log \varepsilon - \log \mu^0, \]

using the fact that \(\log(1 - \theta) \leq \theta\), for \(0 \leq \theta \leq 1\), then the above inequality holds if

\[ k \geq \frac{1}{\theta} \left[ \log \frac{\mu^0}{\varepsilon} \right]. \]

Let \(L = \log \frac{\mu^0}{\varepsilon}\), then at most \(k = 2\sqrt{n} \log \frac{\mu^0}{\varepsilon} = O(\sqrt{n} \log \frac{\mu^0}{\varepsilon}) = O(\sqrt{n}L)\) iterations in the algorithm, we can obtain \(\varepsilon\)-solution of (2.3). However, in every step, the complexity bound of computing the linear system is \(O(n^2)\). Therefore, the total complexity bound of the algorithm is \(O(n^{2.5}L)\).

5. Numerical implementation

In this section, we deal with the numerical implementation of this algorithm applied to some problems of monotone \(LCPs\). Here we used \((x^0, y^0)\) to denote the feasible starting point of the algorithm, \(\delta(x^0, y^0, \mu^0) < \frac{1}{2}\), the proximity condition, \((x^*, y^*)\) the optimal solution of \(LCP\) and \(\text{Iter}\) means the iterations number produced by the algorithm. \(z^*\) denotes the value of the objective function at \((x^*, y^*)\) and \(\mu^*\) denotes the value when the algorithm terminates. The implementation is manipulated in \texttt{DEV C++}. Our tolerance is \(\varepsilon = 10^{-6}\). For the update parameter we have vary \(0 < \theta < 1\). Finally we note that the linear system of Newton in (3.2) is solved thanks to the Gauss elimination procedure.

**Problem 1.**

\[
M = \begin{pmatrix}
0 & 0 & 2 & 1 & 0 \\
0 & 0 & 1 & 2 & 1 \\
-2 & -1 & 0 & 0 & 0 \\
-1 & -2 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0
\end{pmatrix},
q = (-4 & -5 & 8 & 7 & 3)\]

The feasible starting point is

\[
x^0 = (2 & 2 & 2 & 2), \quad y^0 = (-4 & -5 & 8 & 7 & 3)
\]
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\[ \delta(x^0y^0, \mu^0) = 2.517539 > \frac{1}{2}, \]
then the classical method is diverge.

The numerical results with this problem are summarized in the table below:

<table>
<thead>
<tr>
<th>( \theta = 0.15 )</th>
<th>( \theta = 0.20 )</th>
<th>( \theta = 0.60 )</th>
<th>( \theta = 0.95 )</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>iter</strong> 87</td>
<td><strong>iter</strong> 65</td>
<td><strong>iter</strong> 24</td>
<td><strong>iter</strong> 19</td>
</tr>
<tr>
<td>( x^* ) (2.999995, 2.000002, 1.000001, 2.000001, 0.000005)</td>
<td>( x^* ) (2.999994, 2.000002, 1.000001, 2.000000, 0.000005)</td>
<td>( x^* ) (2.999994, 2.000002, 1.000001, 2.000001, 0.000004)</td>
<td>( x^* ) (2.999993, 2.000003, 1.000002, 2.000000, 0.000004)</td>
</tr>
<tr>
<td>( y^* ) (0.000003, 0.000007, 0.000009, 0.000002, 0.999998)</td>
<td>( y^* ) (0.000003, 0.000008, 0.000010, 0.000002, 0.999998)</td>
<td>( y^* ) (0.000003, 0.000006, 0.000008, 0.000002, 0.999998)</td>
<td>( y^* ) (0.000002, 0.000006, 0.000012, 0.000002, 0.999997)</td>
</tr>
<tr>
<td>( z^* ) 0.000041</td>
<td>( z^* ) 0.000045</td>
<td>( z^* ) 0.000037</td>
<td>( z^* ) 0.000036</td>
</tr>
<tr>
<td>( \mu^* ) 0.000009</td>
<td>( \mu^* ) 0.000010</td>
<td>( \mu^* ) 0.000008</td>
<td>( \mu^* ) 0.000008</td>
</tr>
</tbody>
</table>

**Problem 2.**

\[
M = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 3 & 0.8 & 0.32 & 1.128 & 0.0512 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0.8 & 0.32 & 0.128 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0.8 & 0.32 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0.8 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-0.8 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-0.32 & -0.8 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1.128 & -0.32 & -0.8 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-0.0512 & -1.128 & -0.32 & -0.8 & -1 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix},
\]

\[
q = \begin{pmatrix}
-0.0256 & -0.064 & -0.16 & 5.59 & -1 & 1 & 1 & 1 & 1 & 1 \\
\end{pmatrix},
\]

The feasible starting point is

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\[ x^0 = (0.18 \ 0.18 \ 0.18 \ 0.18 \ 0.25 \ 3 \ 4 \ 5 \ 6 \ 9), \]
\[ y^0 = (21.0032 \ 11.008 \ 12.52 \ 12.8 \ 8 \ 0.46 \ 0.676 \ 0.6184 \ 0.41536 \ 0.336144). \]
\[ \delta(x^0 y^0, \mu^0) = 2.278802 > \frac{1}{2}, \]
then the classical method is diverge.

The numerical results with this problem are summarized in the table below:

<table>
<thead>
<tr>
<th>( \theta = 0.15 )</th>
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<tr>
<td>( x^* )</td>
<td>( x^* )</td>
<td>( x^* )</td>
<td>( x^* )</td>
</tr>
<tr>
<td>( \begin{pmatrix} 0.000036 &amp; 0.886520 &amp; 0.000003 &amp; 0.000001 &amp; 0.000001 \ 0.000005 &amp; 0.000088 &amp; 0.000039 &amp; 0.000013 &amp; 8.035358 \end{pmatrix} )</td>
<td>( \begin{pmatrix} 0.385925 &amp; 0.000008 &amp; 2.411364 &amp; 12.028297 &amp; 7.035360 \end{pmatrix} )</td>
<td>( \begin{pmatrix} 0.000005 &amp; 0.000008 &amp; 2.411364 &amp; 12.028297 &amp; 7.035360 \end{pmatrix} )</td>
<td>( \begin{pmatrix} 0.000004 &amp; 0.000008 &amp; 2.411364 &amp; 12.028297 &amp; 7.035360 \end{pmatrix} )</td>
</tr>
<tr>
<td>( y^* )</td>
<td>( y^* )</td>
<td>( y^* )</td>
<td>( y^* )</td>
</tr>
<tr>
<td>( \begin{pmatrix} 0.999892 &amp; 0.113452 &amp; 0.290770 &amp; 0.716269 &amp; 0.000001 \end{pmatrix} )</td>
<td>( \begin{pmatrix} 0.999892 &amp; 0.113452 &amp; 0.290770 &amp; 0.716269 &amp; 0.000001 \end{pmatrix} )</td>
<td>( \begin{pmatrix} 0.999892 &amp; 0.113452 &amp; 0.290770 &amp; 0.716269 &amp; 0.000001 \end{pmatrix} )</td>
<td>( \begin{pmatrix} 0.999892 &amp; 0.113452 &amp; 0.290770 &amp; 0.716269 &amp; 0.000001 \end{pmatrix} )</td>
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<tr>
<td>( z^* )</td>
<td>( z^* )</td>
<td>( z^* )</td>
<td>( z^* )</td>
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<td>0.000085</td>
<td>0.000085</td>
<td>0.000085</td>
</tr>
<tr>
<td>( \mu^* )</td>
<td>( \mu^* )</td>
<td>( \mu^* )</td>
<td>( \mu^* )</td>
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<tr>
<td>0.000009</td>
<td>0.000010</td>
<td>0.000009</td>
<td>0.000007</td>
</tr>
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</table>
Feasible short-step interior point algorithm for linear complementarity problem based on kernel function

**Problem 3.** Let $M \in \mathbb{R}^{n \times n}$ and $q \in \mathbb{R}^n$ are given by:

$$M = \begin{pmatrix}
1 & 2 & 2 & \ldots & 2 \\
0 & 1 & 2 & \ldots & 2 \\
0 & 0 & \ldots & \ldots & \ldots \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & \ldots & 2 \\
0 & 0 & \ldots & \ldots & 0 & 1
\end{pmatrix},
q = \begin{pmatrix}
-1 & \ldots & -1
\end{pmatrix},$$

Case 1: $n = 10$.

The feasible starting point is

$$x^0 = \begin{pmatrix}
0.0009 & 0.0009 & 0.0009 & 0.0009 & 0.0009 & 0.0009 & 0.0009 & 0.0009 & 0.0009 & 1.0009
\end{pmatrix},
\quad y^0 = \begin{pmatrix}
1.0171 & 1.0153 & 1.0135 & 1.0108 & 1.0099 & 1.0081 & 1.0063 & 1.0045 & 1.0027 & 0.0009
\end{pmatrix},
\delta(x^0, y^0, \mu^0) = 0.032154 < \frac{1}{2},$$

then the classical method is converge.

The numerical results with this problem are summarized in the table below:
### Results of the Algorithm ($r = e$)

\[ \delta(x^0, y^0, \mu^0) = 0.032154 < \frac{1}{2} \]

<table>
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<th>31</th>
</tr>
</thead>
<tbody>
<tr>
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<tr>
<td>$y^*$</td>
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<td>1.000168 1.000151 1.000133 0.999215 1.000097</td>
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<tr>
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<td>0.000089</td>
</tr>
<tr>
<td>$\mu^*$</td>
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<tr>
<td>$y^*$</td>
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<tbody>
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</tr>
<tr>
<td>$y^*$</td>
<td></td>
<td>1.000129 1.000115 1.000102 0.999188 1.000074</td>
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<tr>
<td>$z^*$</td>
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<tr>
<td>$\mu^*$</td>
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</table>

<table>
<thead>
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<th>iter</th>
<th>7</th>
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<tbody>
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<td>$x^*$</td>
<td></td>
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</tr>
<tr>
<td>$y^*$</td>
<td></td>
<td>1.000138 1.000124 1.000109 0.999194 1.000080</td>
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<td>$z^*$</td>
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<td>0.000073</td>
</tr>
<tr>
<td>$\mu^*$</td>
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<td>0.000007</td>
</tr>
</tbody>
</table>

Case 2: $n = 15$.

The feasible starting point is

\[
x^0 = \begin{pmatrix}
0.0009 & 0.0009 & 0.0009 & 0.0009 & 0.0009 & 0.0009 & 0.0009 & 0.0009 & 0.0009 & 0.0009 & 0.0009 \\
0.0009 & 0.0009 & 0.0009 & 0.0009 & 0.0009 & 0.0009 & 1.0009 & 0.0009 & 0.0009 & 0.0009 & 0.0009
\end{pmatrix},
\]

\[
y^0 = \begin{pmatrix}
1.0261 & 1.0243 & 1.0225 & 1.0198 & 1.0189 & 1.0171 & 1.0153 & 1.0135 & 1.0108 & 1.0099 \\
1.0081 & 1.0063 & 1.0045 & 1.0027 & 0.0009 &
\end{pmatrix}.
\]

\[\delta(x^0, y^0, \mu^0) = 0.059169 < \frac{1}{2}, \text{ then the classical method is converge.}\]

The numerical results with this problem are summarized in the table below:
<table>
<thead>
<tr>
<th>$\theta$</th>
<th>$\text{iter}$</th>
<th>$x^*$</th>
<th>$y^*$</th>
<th>$z^*$</th>
<th>$\mu^*$</th>
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<td>0.95</td>
<td>4</td>
<td>0.000060 0.000059 0.000059 0.000059 0.000059 0.000059 0.000059 0.000059 0.000059 0.000058 0.000058 0.000058 0.000058 0.000058 0.000058 0.000058 0.000058 0.000058 0.000058 0.000058 0.000058</td>
<td></td>
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</tr>
</tbody>
</table>

Results of the Algorithm ($r = e$)

$\delta(x^0y^0, \mu^0) = 0.032154 < \frac{1}{2}$
6. Conclusion

In this paper, we have proposed a feasible short-step interior point algorithm for solving monotone linear complementarity problem. The algorithm deserves the best well-known theoretical iteration bound $O(n^{2.5}L)$ when the starting point is $(x^0, y^0)$ is strictly feasible and verified proximity measure condition. This choice of initial point can be done by the technique of Djamel Benterki [1]. For the numerical tests we vary the parameter $\theta$, and we note that each problem addressed when the parameter $\theta$ crosses we get the good numerical behavior. Future research might extended the algorithm for other optimization problems.

Acknowledgements

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References


