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A new approach for solving the one-phase Stefan problem with temperature-boundary specification

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Abstract

The classical one-phase one-dimensional Stefan problem is a boundary value problem involving a parabolic partial-differential equation, along with two boundary conditions on a moving boundary. In Stefan problem the moving boundary s and the distribution of temperature u in the domain are unknown and must be determined. In this paper, an approximate analytical method which is based on a scheme introduced by Cannon [1] is presented to obtain the solution of the problem. At first, by subdividing the time interval $[0, T]$ into subintervals of equal length θ, a family of approximations (s^{θ}, u^{θ}) to the solution of the problem is constructed and then by taking θ sufficiently small, these approximations are converged to the solution of the Stefan problem. The method is illustrated using the Stefan problem concerning the heat transfer in an ice-water medium. The computational results have been compared with the exact values and are found to be in good agreement with each other.

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1 Introduction

The one-phase Stefan problem is one of the simplest examples of a free boundary-value problem for the heat equation. Many problems which describe the thermal processes with phase change such as solidification of metals, freezing of water and soil, deep freezing of foodstuffs and melting of ice have been modeled by the introduction of Stefan problems. Phase change problems have still remained an active area of research. In recent decades, Stefan problems have attracted much attention of many researchers and investigators [1-7].

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Here, we consider the free boundary problem for the heat equation, which is the determination of a pair (u, s) of functions, $u = u(x,t)$ and $s = s(t)$, which satisfy

$$\begin{aligned} &(a)\ \ Lu \equiv u_{xx} - u_t = 0,\ \ 0 < x < s(t),\ \ 0 < t \le T \\ &(b)\ \ u(0,t) = g(t),\ \ 0 < t \le T, \\ &(c)\ \ u(s(t),t) = 0,\ \ 0 < t \le T, \\ &(d)\ \ u(x,0) = f(x),\ \ 0 \le x \le s(0) = b, \end{aligned} \tag{1}$$

$$\dot{s}(t) = -u_x(s(t),t),\ \ 0 < t \le T, \tag{2}$$

where T is a positive constant which can be selected arbitrarily, $b > 0$, $0 \le f(x) \le N(b-x)$, $0 \le x \le b$, N is a positive constant, and f and g are nonnegative piecewise-continuous functions.

Because of the presence of a free or moving boundary, analytical solutions of Stefan problems have remained very limited and usually not available. Goodman [8], Reynolds and Dolton [9], Gupta and Banik [10] have investigated approximate analytical methods that yield solutions of Stefan problems in simple closed forms. Slota and Zielonka [11] used the variational iteration method for an approximate solution of a one-phase Stefan problem. Jafari, Saeidy and Firozjaei [12] applied the homotopy analysis method to solve a Stefan problem. Kushwaha [13] and Rajeev [14] found an approximate analytical solution of a moving boundary problem with variable latent heat by using Adomian decomposition method and homotopy perturbation method, respectively.

In this paper, we present an approximate analytical approach which results in finding the sequence of approximations, convergent to the exact solution. This method is based on a scheme introduced by Cannon [1]. This paper is organized as follows. In Section 2, we define what is meant by a solution of (1)-(2) and state the existence and uniqueness theorem. Our purpose in Section 3 is to study the initial-boundary-value problem for the heat equation in the domain $D_T = \{(x,t)|0 < x < s(t), 0 < t \le T\}$. In Section 4, by a retardation of the argument in the free boundary equation (2), we construct a family of uniformly convergent approximations to the solution of (1)-(2). In Section 5, the obtained results of test problem solved by the proposed method is reported. Conclusions are finally made in Section 6.

2 Preliminaries

Definition 2.1 *(see [1]). The pair (u, s) is a solution of the Stefan problem (1)-(2) if*

- *s is continuously differentiable for $0 < t \le T$ and continuous for $0 \le t \le T$,*
- *u is continuous in $0 \le x \le s(t)$, $0 \le t \le T$, except at points of discontinuity of g or f, and at points of continuity has the boundary and initial values indicated in (1)-b, (1)-c and (1)-d,*

- *u is bounded in $0 \le x \le s(t)$, $0 \le t \le T$, and*
- *u and s satisfy (2).*

Definition 2.2 *(see [1]). If I is an interval, we denote by $\mathcal{C}^0(I)$, $\mathcal{C}^1(I)$, and $\mathcal{C}^\beta(I)$, $0 < \beta \le 1$, the continuous, the continuously differentiable, and the Hölder continuous (with exponent β) functions on I, respectively. For $I = (0, C]$, $C > 0$, we define $\mathcal{C}^0_{(\nu)}(I)$, $0 < \nu \le 1$, as the subspace of $\mathcal{C}(I)$ that consists of those functions ψ such that*

$$||\psi||_C^{(\nu)} = \sup_{t\in I} t^{1-\nu}|\psi(t)| < \infty.$$

Also $\mathcal{C}^0_{(\nu)}(I)$ is a Banach space under the norm $||\psi||_C^{(\nu)}$.

Theorem 2.3 *(see [1]). Under the assumptions given in introduction, there exists a unique solution (u, s) to the Stefan problem (1)-(2). Moreover, the free boundary s which is $\mathcal{C}^1$ and nondecreasing, satisfies*

$$0 \le \dot{s}(t) \le C$$

where $C = \max\{Mb^{-1}, N\}$, $M = \max(\max_{0\le t\le T} g(t), \max_{0\le x\le b} f(x))$.

3 Initial-Boundary-Value Problem for the Heat Equation in $D_T = \{(x,t)|0 < x < s(t), 0 < t \le T\}$

Theorem 3.1 *(see [1]). For the problem*

$$\begin{aligned}
&u_{xx} = u_t,\ \ 0 < x < s(t),\ \ 0 < t \le T\\
&u(x,0) = f(x),\ \ 0 < x < b,\ \ s(0) = b,\\
&u(0,t) = g(t),\ \ 0 < t \le T,\\
&u(s(t),t) = h(t),\ \ 0 < t \le T,
\end{aligned} \tag{3}$$

where $s \in \mathcal{C}^\gamma([0,T])$, $\gamma > \frac{1}{2}$, $\delta_1 = \inf_{0\le t\le T} s(t) > 0$, $g \in \mathcal{C}^0_{(\nu)}((0,T])$, $0 < \nu \le 1$, $h \in \mathcal{C}^{\gamma_1}([0,T])$, $\gamma_1 > \frac{1}{2}$, $f \in \mathcal{C}([0,b])$, $h(0) = f(b) = 0$, are known and

$$|f(\xi)| = |f(\xi) - f(b)| \le C_f|\xi - b|^\beta, \quad 0 < \beta \le 1,\ \ C_f > 0,$$

there exists a solution u that has the representation

$$u(x,t) = v(x,t) + z_\varphi(x,t) + w_\varphi(x,t;s) \tag{4}$$

where

$$\begin{aligned}
w_\varphi(x,t;s) &= \int_0^t K(x - s(\tau), t-\tau)\varphi(\tau)d\tau,\\
z_\varphi(x,t) &= -2\int_0^t \frac{\partial K}{\partial x}(x, t-\tau)\{g(\tau) - w_\varphi(0,\tau;s)\}d\tau,
\end{aligned}$$

$$v(x,t) = \int_0^b G(x,\xi,t) f(\xi) d\xi,$$

and

$$G(x,\xi,t) = K(x-\xi,t) - K(x+\xi,t),$$
$$K(x,t) = \frac{1}{\sqrt{4\pi t}} \exp(-\frac{x^2}{4t}).$$

The representation (4) for u is equivalent to the existence of a continuous $\varphi = \varphi(t)$, which satisfies the integral equation

$$h(t) = v(s(t),t) + z_\varphi(s(t),t) + w_\varphi(s(t),t;s), \tag{5}$$

which is a Volterra integral equation of the first kind for φ.

Lemma 3.2 *(see [1]). If $s \in \mathcal{C}^\gamma([0,T])$, $\frac{1}{2} < \gamma \leq 1$, $g \in \mathcal{C}^0_{(\nu)}((0,T])$, $0 < \nu \leq 1$, $h \in \mathcal{C}^{\gamma_1}([0,T])$, $\gamma_1 > \frac{1}{2}$, $f \in \mathcal{C}([0,b])$, $h(0) = f(b) = 0$,*

$$|f(\xi)| < C_f |\xi - b|^\beta, \quad 0 < \beta \leq 1, \ C_f > 0$$

, and $\varphi \in \mathcal{C}^0_{(\nu_1)}((0,T])$, $0 < \nu_1 \leq 1$, then the Abel operator

$$(AF)(t) = \frac{1}{\pi} \frac{d}{dt} \int_0^t \frac{1}{\sqrt{t-\eta}} F(\eta) d\eta$$

can be applied to the integral equation (5) and an equivalent Volterra integral equation of the second kind

$$\varphi(t) = G(t) + \int_0^t H(t,\tau) \varphi(\tau) d\tau \tag{6}$$

is obtained where

$$\begin{aligned} G(t) &= \frac{2}{\sqrt{\pi}} \int_0^t g(\tau) \{ \int_\tau^t (t-\eta)^{\frac{-3}{2}} (\eta-\tau)^{\frac{-1}{2}} \{ (t-\tau)^{\frac{1}{2}} \frac{\partial K}{\partial x}(s(t), t-\tau) \\ &\quad - (\eta-\tau)^{\frac{1}{2}} \frac{\partial K}{\partial x}(s(\eta), \eta-\tau) \} d\eta \} d\tau \\ &\quad - \frac{1}{\sqrt{\pi}} \{ \int_0^t (t-\eta)^{\frac{-3}{2}} \eta^{\frac{-1}{2}} \{ t^{\frac{1}{2}} v(s(t),t) - \eta^{\frac{1}{2}} v(s(\eta),\eta) \} d\eta \}, \end{aligned}$$

$$\begin{aligned} H(t,\tau) &= \frac{-1}{2\pi} \{ \int_\tau^t (t-\eta)^{\frac{-3}{2}} (\eta-\tau)^{\frac{-1}{2}} \{ \exp(\frac{-(s(t)-s(\tau))^2}{4(t-\tau)}) \\ &\quad - \exp(\frac{-(s(\eta)-s(\tau))^2}{4(\eta-\tau)}) \} d\eta \} \\ &\quad - \frac{1}{\sqrt{\pi}} \{ \int_\tau^t (t-\eta)^{\frac{-3}{2}} (\eta-\tau)^{\frac{-1}{2}} \{ (t-\tau)^{\frac{1}{2}} L(t,\tau) \\ &\quad - (\eta-\tau)^{\frac{1}{2}} L(\eta,\tau) \} d\eta \}, \end{aligned}$$

and

$$K(x,t) = \frac{1}{\sqrt{4\pi t}} \exp(-\frac{x^2}{4t}), \qquad \frac{\partial K}{\partial x}(x,t) = \frac{-x}{2t\sqrt{4\pi t}} \exp(-\frac{x^2}{4t}),$$
$$v(x,t) = \int_0^b \{K(x-\xi,t) - K(x+\xi,t)\} f(\xi) d\xi,$$
$$L(t,\tau) = 2\int_\tau^t \frac{\partial K}{\partial x}(s(t), t-\sigma) K(s(\sigma), \sigma-\tau) d\sigma.$$

When we consider the Banach space $\mathcal{C}^0_{(\nu_2)}((0,T])$, where $\nu_2 = \nu_2(\nu,\gamma,\gamma_1,\beta)$, we see that

$$B_\varphi(t) = G(t) + \int_0^t H(t,\tau)\varphi(\tau) d\tau \tag{7}$$

is a contraction of $\mathcal{C}^0_{(\nu_2)}((0,T])$ into itself for T sufficiently small. Hence there exists a unique solution $\varphi \in \mathcal{C}^0_{(\nu_2)}((0,T])$ of (7) for T sufficiently small and the fixed point iteration method (8) converges for any initial function $\varphi_0 \in \mathcal{C}^0_{(\nu_2)}((0,T])$.

$$\varphi_n(t) = G(t) + \int_0^t H(t,\tau)\varphi_{n-1}(\tau) d\tau, \;\; t \in (0,T], \; n \geq 1, \tag{8}$$

Theorem 3.3 *(see [1]). Under the hypothesis of Theorem 3.1, the solution u of problem (3), possesses a two-dimensionally continuous u_x at $x = s(t)$, $0 < t \leq T$.*

4 Method of Solution

We begin by first subdividing the time interval $[0,T]$ into N equal subintervals of length $\theta = \frac{T}{N}$. Next, we let $s^{n\theta}(t)$ denote the approximate value of s on each subinterval $n\theta \leq t \leq (n+1)\theta$ and define $u^{n\theta}(x,t)$ to be the unique solution of (1) in the region $0 \leq x \leq s^{n\theta}(t)$, $n\theta \leq t \leq (n+1)\theta$.
For each, $0 < \theta < b$, let

$$f^\theta(x) = \begin{cases} f(x), & 0 \leq x \leq b-\theta \\ 0, & b-\theta < x \leq b \end{cases}$$

In the first interval $0 \leq t \leq \theta$, we set $s^{0\theta}(t) \equiv b$ and solve the following problem by utilizing Section 3

$$\begin{array}{l} u^{0\theta}_{xx} = u^{0\theta}_t, \;\; 0 < x < s^{0\theta}(t), \;\; 0 < t \leq \theta \\ u^{0\theta}(x,0) = f^\theta(x), \;\; 0 < x < b, \\ u^{0\theta}(0,t) = g(t), \;\; 0 < t \leq \theta, \\ u^{0\theta}(s(t),t) = 0, \;\; 0 < t \leq \theta. \end{array}$$

From Theorem 3.3, $u^{0\theta}_x(s^{0\theta}(t),t)$ exists and is continuous for $0 \leq t \leq \theta$. On each subinterval $n\theta \leq t \leq (n+1)\theta$, $n \geq 1$, at first, we define

$$s^{n\theta}(t) \;\; = \;\; b - \sum_{i=0}^{n-2} \int_{-(n-i)\theta}^{-(n-i-1)\theta} u^{i\theta}_x(s^{i\theta}(\eta+n\theta), \eta+n\theta) d\eta$$

$$-\int_{-\theta}^{t-(n+1)\theta} u_x^{(n-1)\theta}(s^{(n-1)\theta}(\eta+n\theta),\eta+n\theta)d\eta \tag{9}$$

and solve the following problem by utilizing Section 3

$$\begin{array}{l} u_{xx}^{n\theta}=u_t^{n\theta},\ \ 0<x<s^{n\theta}(t),\ \ 0<t\leq\theta \\ u^{n\theta}(x,0)=f^{\theta}(x),\ \ 0<x<b, \\ u^{n\theta}(0,t)=g(t+n\theta),\ \ 0<t\leq\theta, \\ u^{n\theta}(s^{n\theta}(t),t)=0,\ \ 0<t\leq\theta. \end{array}$$

Then, by a translation of the solution to the interval $n\theta \leq t \leq (n+1)\theta$, the approximation $(s^{n\theta}, u^{n\theta})$ can be constructed in the region $0 \leq x \leq s^{n\theta}(t)$, $n\theta \leq t \leq (n+1)\theta$. Clearly $u_x^{n\theta}(s^{n\theta}(t), t)$ exists and is continuous for $n\theta \leq t \leq (n+1)\theta$, $n \geq 1$.
In this way, we can construct the approximations $(s^{n\theta}, u^{n\theta})$ to the solution of the Stefan problem (1)-(2) in all regions $0 \leq x \leq s^{n\theta}(t)$, $n\theta \leq t \leq (n+1)\theta$, $n \geq 1$.

Remark 4.1 *Differentiating of the left and right sides of (9) over the time interval $n\theta \leq t \leq (n+1)\theta$, $n \geq 1$, yields*

$$\dot{s}^{n\theta}(t) = -u_x^{(n-1)\theta}(s^{(n-1)\theta}(t-\theta), t-\theta),$$

which amounts to a retardation of the argument in the free boundary equation (2).

Remark 4.2 *The approximations $(s^{n\theta}, u^{n\theta})$ are uniformly convergent to the solution of the Stefan problem (1)-(2), if the length of each subinterval tends to zero. This means that, by taking θ sufficiently small, these approximations are uniformly convergent to the solution of the Stefan problem (1)-(2) (see [1]).*

5 Example

The solution procedure by the proposed method, presented in the previous section, will be illustrated with the following theoretical Example, for which the exact solution is known. For simplicity, all the integrals in Section 3 or Section 4 are calculated numerically using the trapezoidal or midpoint rules. The results are carried out using MATLAB software and depicted through figure and table.

Example 5.1 *Assume that: $b = \frac{1}{2}$, $f(x) = \exp(-x + \frac{1}{2}) - 1$, $g(t) = \exp(t + \frac{1}{2}) - 1$, $T = 0.03$. For such data, the exact solution of the Stefan problem (1)-(2) are the following functions*

$$u(x,t) = \exp(t - x + \frac{1}{2}) - 1, \quad s(t) = t + \frac{1}{2}. \tag{10}$$

The exact and approximate solutions describing the free boundary s with $N = 3$ and $n = 1$, $\varphi_0(t) = 1$ in iteration formula (8) have been shown in Figure 1. Also in Table 1, the numerical values of the temperature $u(x,t)$ at selected times and at point $x = 0.2$, are found to be in good agreement with the exact solution in (10).

Table 1: A comparison of numerical values of $u(0.2, t)$ with the exact solution at selected times

t	Approximate	Exact
0.005	0.3352	0.3566
0.01	0.3751	0.3634
0.015	0.3355	0.3703
0.02	0.3765	0.3771
0.025	0.3357	0.3840
0.03	0.3779	0.3910

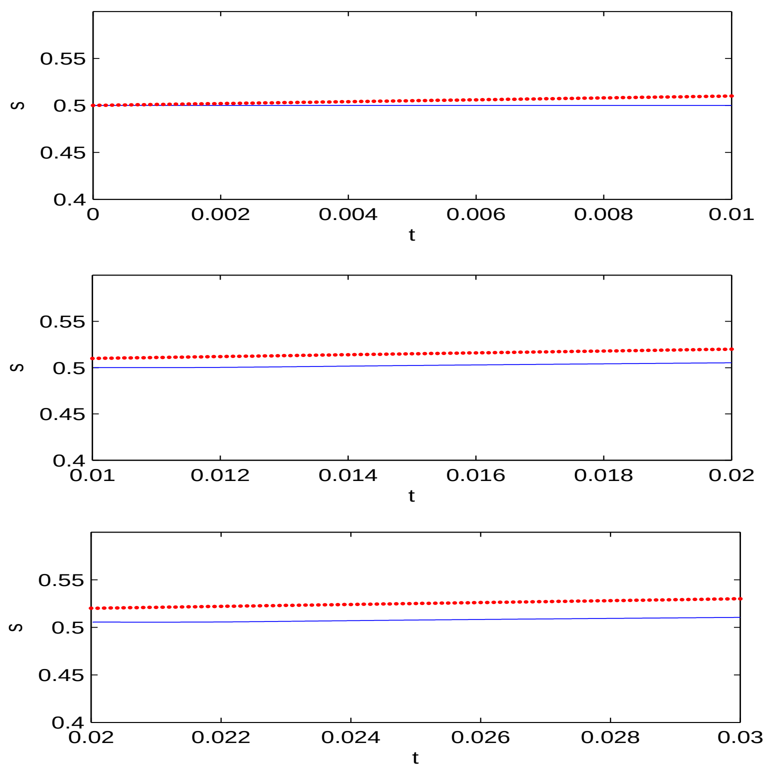

Figure 1: Comparison between the exact position of the free boundary (dots) and its approximation (solid line).

6 Conclusion

In this paper the solution of the one-phase Stefan problem was presented. This problem consists in a calculation of temperature distribution and of a function which describes the position of the moving interface. The proposed method is based on a scheme introduced by Cannon [1]. The calculations show that this method is effective for solving the problems under consideration. The advantage of the proposed method comparing it with classical methods, for example finite difference method or finite element method, consists in obtaining the interface position and temperature distribution in the form of continuous functions, instead of a discrete form. As a result of the current approach we receive the sequence of approximations, convergent to the exact solution. We may conclude that this method will be very much useful for solving moving and other many physical problems.

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