Motivated by the definition of $P_1$-matrix ([4]), another generalization of a $P$-matrix for square singular matrices called $P_D$-matrix is proposed first. Then the uniqueness of solution of Linear Complementarity Problems for square singular matrices is proved using $P_D$-matrices. Finally some results which are true for $P$-matrices are extended to $P_D$-matrices.

**Keywords:** Drazin inverse, Group inverse, index-proper splittings, $P$-matrix, interval hull of matrices.

**AMS Subject Classification:** 15A09
1. Introduction

A matrix $A = (a_{ij}) \in \mathbb{R}^{n \times n}$ is called a $P$-matrix if every principal minor of $A$ is positive. This notion was first introduced by Fiedler and Ptak [5]. It plays important roles in studying solution properties of equations and complementarity problems, and convergence/complexity analysis of methods for solving these problems. There are numerous ways to describe a $P$-matrix, we consider only three of them for our work. The three equivalent definitions for a real square matrix $A$ are as follows:

(i) All the principal minors of $A$ are positive.

(ii) Every real eigenvalue of each principal submatrix of $A$ is positive.

(iii) The matrix $A$ does not reverse the sign of any vector; i.e., if $x \neq 0$ and $y = Ax$, then for some subscript $i$, $x_i y_i > 0$.

The equivalence of (i) and (iii) was established by Fiedler and Ptak [5]. P-matrices also arise quite frequently in systems theory. These include hermitian positive definite matrices, $M$-matrices, totally positive matrices and real diagonally dominant matrices with positive diagonal entries.

Now coming to the name “linear complementarity problem” which stems from the linearity of the mapping $W(z) = q + Az$, where $A \in \mathbb{R}^{n \times n}$ and the complementarity of real $n$-vectors $w$ and $z$. For a given $q \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n \times n}$, the linear complementarity problem (LCP) is that of finding (or concluding there is no) $z \in \mathbb{R}^n$ such that

$$w = q + Az \geq 0,$$

$$z \geq 0,$$

$$z^Tw = 0.$$

We denote the above problem by the symbol $LCP(q, A)$. A vector $z \in \mathbb{R}^n$ satisfying the above three conditions is called a solution of $LCP(q, A)$ and the set of all solutions is denoted by $SOL(q; A)$. The solution set is defined by $S(A) = \{q : SOL(q, A) \neq \emptyset\}$.

For more details on linear complementarity problems, we refer the reader to the book by Cottle, Pang and Stone [4].
Let us discuss some basic relations between the complementarity cone and $P$-matrices. The class of complementarity cone corresponding to a real square matrix $A$ is denoted by $A$, the pair of column vectors $(I_j, -A_j)$ is known as the $j$th complementary pair of vectors in $1 \leq j \leq n$. The convex cone generated by any complementary set of column vectors is known as a complementarity cone. A theorem proved by Samelson, Thrall and Wesler [13] says that the set of complementarity cone partitions $\mathbb{R}^n$ if and only if $A$ is a $P$-matrix. Later this characterization of $P$-matrices by Samelson, Thrall and Wesler [13] was improved by Murty [10]. He proved that a real $n \times n$ matrix $A$ is a $P$-matrix if and only if the $LCP(q, A)$ has a unique solution for every $q$ belonging to $\{I_1, \cdots I_n, -I_1, \cdots -I_n, A_1, \cdots A_n, e\}$, and further extended by Tamir [14]. Tamir [14] stated a $n \times n$ matrix $A$ is a $P$-matrix if and only if the $LCP(q, A)$ has a unique solution for every $q$ belonging to $\{I_1, \cdots I_n, A_1, \cdots A_n, e\}$, where $e = (1, 1, \cdots, 1)^T$ is the vector of ones of order $n$.

Very recently, Kannan and Sivakumar [9] generalized the notion of $P$-matrix for singular cases and call it as $P_1$-matrix. The definition of this is presented next.

**Definition 1.1.** *(Definition 1.1 [9]) A square matrix $A$ is said to be a $P_1$-matrix if for each non zero $x \in \mathbb{R}(A^T)$, there is an $i \in \{1, 2, \cdots, n\}$ such that $x_i(Ax_i) > 0$."

Let us introduce some more definitions and notations which are going to be used to prove our main results. Let the diagonal matrix whose entries are $t_1, t_2, \cdots t_n$ is denoted by $diag(t_1, t_2, \cdots t_n)$. Let $F$ denote the matrix whose entries are all one, and let $o$ denote the Hadamard (entry wise) product of matrices. For any $A, B \in \mathbb{R}^{n \times n}$, we define the following sets:

- $h(A, B) = \{C : C = tA + (1 - t)B, t \in [0, 1]\}$
- $r(A, B) = \{C : C = TA + (I - T)B, T = diag(t_1, t_2, \cdots t_n), t_i \in [0, 1], 1 \leq i \leq n\}$
- $c(A, B) = \{C : C = AT + B(I - T), T = diag(t_1, t_2, \cdots t_n), t_i \in [0, 1], 1 \leq i \leq n\}$
\[ i(A, B) = \{ C \in \mathbb{R}^{n \times n} : a_{ij} \leq c_{ij} \leq b_{ij} \} \]

From all the above definitions above, now it is clear that \( h(A, B) \subseteq r(A, B) \subseteq i(A, B) \) and \( h(A, B) \subseteq c(A, B) \subseteq i(A, B) \).

The interval hull \( i(A, B) \) is said to be \textit{index-range kernel regular} if \( R(A^k) = R(B^k) \) and \( N(A^k) = N(B^k) \). Let us define the set

\[ K = \{ C \in i(A, B) : R(A^k) = R(U^k) \text{ and } N(A^k) = N(U^k) \} \]

for an index-range kernel interval hull \( i(A, B) \). When \( A \) and \( B \) are invertible, then \( K \) contains only invertible matrices. Motivated by the results of Johnson and Tsatsomeros \[7\] for \( P \)-matrices, and further generalization by Kannan and Sivakumar \[9\] for \( P^\dagger \)-matrices, we establish analogous results by introducing another generalization of the notion of \( P \)-matrix.

The motto of this article is to first propose an extension of \( P \)-matrix and to study some of its properties. The organization of this paper is as follows. After a brief review of some basic definitions and notations in Section 2, we include the definition of a \( P_D \)-matrix and few of its properties in Section 3. We first state linear complementarity problems (LCP), secondly a relation between \( P_D \)-matrices and LCP, and then some results. Section 4 contains the characterization of the inclusion \( r(A, B) \subseteq K \) with \( P_D \)-matrices, and an analogous result for \( c(A, B) \subseteq K \) in Section 4. Finally, the case \( i(A, B) = K \) and the generalization of Theorem 3.8, \[7\] are discussed. Section 5 presents the inclusion
\(h(A, B) \subseteq K\) and its link with certain constrained eigenvalue condition. Then a new result for the nonnegativity of Drazin inverse of an interval is proved in Section 6.

2. Preliminaries

Let \(\mathbb{R}^n\) denote the \(n\) dimensional real Euclidean space and \(\mathbb{R}^n_+\) denote the nonnegative orthant in \(\mathbb{R}^n\). For a real \(m \times n\) matrix \(A\), i.e., \(A \in \mathbb{R}^{m \times n}\), the matrix \(G\) satisfying the four equations known as Penrose equations: \(AGA = A\), \(GAG = G\), \((AG)^T = AG\) and \((GA)^T = GA\) is called the Moore-Penrose inverse of \(A\) (\(B^T\) denotes the transpose of \(B\)). It always exists and unique, and is denoted by \(A^\dagger\). A \(A \in \mathbb{R}^{m \times n}\) is said to be semimonotone if \(A^\dagger \geq O\) (here the comparison is entry wise and \(O\) is the null matrix of respective order). For a real \(n \times n\) matrix \(A\). The index of a real square matrix \(A\) is the least nonnegative integer \(k\) such that \(\text{rank}(A^{k+1}) = \text{rank}(A^k)\). It is denoted by \(\text{ind}(A)\). Then \(\text{ind}(A) = k\) if and only if \(\text{rank}(A^k) \bigoplus \text{N}(A^k) = \mathbb{R}^n\). For \(A \in \mathbb{R}^{n \times n}\) the matrix \(G\) satisfying the three equations : \(A^kGA = A^k\), \(GAG = G\), \(AG = GA\) known as Drazin inverse of \(A\), where \(k\) is the index of \(A\). It always exists and unique, and is denoted by \(A^D\). When \(k = 1\), then the Drazin inverse is known as group inverse and is denoted as \(A^\#\). The spectral radius of \(A \in \mathbb{R}^{n \times n}\), denoted by \(\rho(A)\) is defined by \(\rho(A) = \max_{1 \leq i \leq n} |\lambda_i|\), where \(\lambda_1, \lambda_2, \cdots, \lambda_n\) are the eigenvalues of \(A\).

The next theorem is a part of Perron–Frobenius theorem.
Theorem 2.1. (Theorem 2.20, [15]) Let \( A \geq O \). Then \( A \) has a nonnegative real eigenvalue equal to its spectral radius.

Another result which relates spectral radius of two nonnegative matrices is given below.

Theorem 2.2. (Theorem 2.21, [15]) Let \( A \geq B \geq O \). Then \( \rho(A) \geq \rho(B) \).

The theory of splitting plays a major role in finding solution of system of linear equations. Many authors have proposed several splittings. Chen-Chen [3] proposed the following splitting.

Definition 2.3. A splitting \( A = U - V \) of \( A \in \mathbb{R}^{n \times n} \) is called an index-proper splitting (3) if \( R(A^k) = R(U^k) \) and \( N(A^k) = N(U^k) \), \( k = \text{ind}(A) \).

3. \( P_D \)-matrices

We begin this section with another generalization of a singular \( P \)-matrix which we call as a \( P_D \)-matrix, and the definition is presented below.

Definition 3.1. A square matrix \( A \) is said to be a \( P_D \)-matrix if for each non zero \( x \in \mathbb{R}(A^k) \), \( k = \text{ind}(A) \) there is an \( i \in \{1, 2, \ldots, n\} \) such that \( x_i(Ax_i) > 0 \).

In other words, for any \( x \in \mathbb{R}(A^k) \) the inequality \( x_i(Ax_i) \leq 0 \) for \( i \in \{1, 2, \ldots, n\} \) imply \( x = 0 \). Trivially, every \( P \)-matrix is a \( P_D \)-matrix.

Example 3.2. Let \( A = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \). Then \( \text{ind}(A) = 2 \). Also \( R(A^2) = \text{span of} \)

\[
\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} : \alpha \in \mathbb{R} \right\} .
\]

Taking \( x = (1, -1, 0)^T \) and calculating \( x_i(Ax)_i \), we get \( x_i(Ax)_i > 0 \). So \( A \) is a \( P_D \)-matrix.

\( P_D \)-matrix reduces to \( P_{\#} \)-matrix when \( k = 1 \), and the definition is as follows.

Definition 3.3. (Definition 5.1, [9])

A square matrix \( A \) is said to be a \( P_{\#} \)-matrix if for each non zero \( x \in R(A) \) there is an \( i \in \{1, 2, \ldots, n\} \) such that \( x_i(Ax_i) > 0 \).
In other words, for any \( x \in R(A) \) the inequality \( x_i(Ax_i) \leq 0 \) for \( i \in \{1, 2, \cdots, n\} \) imply \( x = 0 \).

**Example 3.4.** Let \( A = \begin{pmatrix} 3 & 1 & 0 \\ 3 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix} \). Here \( \text{ind}(A) = 1 \) and \( R(A) = \text{span} \) of \( \begin{Bmatrix} \begin{pmatrix} 3 \\ 3 \\ 0 \end{pmatrix} , \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} \end{Bmatrix} \). Then \( x_i(Ax_i) \leq 0 \) for any \( x \in R(A) \). Hence \( A \) is a \( P_\# \)-matrix.

We discuss below some useful properties of \( P_D \)-matrices. The first one is the Drazin inverse analogue of Theorem 2.3, [9].

**Theorem 3.5.** \( A \) is a \( P_D \)-matrix if and only if \( A^D \) is a \( P_D \)-matrix.

**Proof.** Suppose that \( A \) is a \( P_D \)-matrix. So for each \( 0 \neq y \in R((A^D)^k) = R(A^k) \), there is an \( i \in \{1, 2, \cdots, n\} \) such that \( y_i(Ay)_i > 0 \). Let \( y \in R((A^D)^k) = R(A^k) \) then \( y = AA^Dx \), for some \( x \in \mathbb{R}^n \), so \( y_i(A^Dy)_i = (AA^Dx)_i(A^DA^Dx)_i = (A^Dx)_i(A^DA^Dx)_i = (A^Dx)_i(AAx)_i = (A^Dx)_i(AAx)_i = u_i(Au)_i > 0 \). Set \( u = A^Dx \in R(A^D) = R(A^k) \). So \( A^D \) is a \( P_D \)-matrix.

Conversely: Let \( A^D \) be a \( P_D \)-matrix. In order to show that \( A \) is a \( P_D \)-matrix, we have to prove that \( y_i(Ay)_i > 0 \) for \( y \in R(A^k) \), \( i \in \{1, 2, \cdots, n\} \). Since \( A^D \) is a \( P_D \)-matrix, \( x_i(A^Dx)_i > 0 \), \( 0 \neq x \in R((A^D)^k) = R(A^k) \). Therefore \( y_i(Ay)_i = (A^Dx)_i(A^DA^Dx)_i = (A^DAA^Dx)_i(A^DA^Dx)_i = (A^Dx)_i(AAx)_i = (A^Dx)_i(AAx)_i = u_i(Au)_i > 0 \), where \( u = AA^Dx \). Again \( A^D \) is a \( P_D \)-matrix, \( y_i(Ay)_i > 0 \), \( y \in R(A^k) \). Hence \( A \) is a \( P_D \)-matrix. \( \square \)

When \( \text{ind}(A) = 1 \), we then have Theorem 5.1, [9] as a corollary. However, we give a different proof for the existence of the group inverse.

**Corollary 3.6.** \( A \) is a \( P_\# \)-matrix if and only if \( A^\# \) is a \( P_\# \)-matrix.

**Proof.** The proof is same as the proof for \( P_D \)-matrices for \( k = 1 \), but here only to show \( A^\# \) exists. For this, suppose that \( A \) is a \( P_\# \)-matrix. Let \( x \in R(A) \). Then, \( x_i(Ax)_i = 0 \) for each \( i \in \{1, 2, \cdots, n\} \), so \( R(A) = 0 \Rightarrow r(A) = 0 \). Again \( R(A) = 0 \Rightarrow R(A^2) = 0 \). So \( r(A^2) = 0 \). Hence \( \text{ind}(A) = 1 \). Therefore \( A^\# \) exists. \( \square \)
Next theorem says that a $P_D$-matrix has a nonnegative eigenvalue decomposition under a given condition.

**Theorem 3.7.** Let $A$ be a $P_D$-matrix. Suppose $Ax = \lambda x$, $0 \neq x \in R(A^k)$ and $\lambda \in \mathbb{R}$. Then $\lambda > 0$.

**Proof.** Assume that $Ax = \lambda x$, $0 \neq x \in R(A^k)$ and $\lambda \in \mathbb{R}$ and $A$ is a $P_D$-matrix. Then $\lambda x_i^2 = \lambda x_i x_i = x_i(Ax)_i > 0$, for some $i \in \{1, 2, \cdots n\}$. Hence $\lambda > 0$. \hfill \Box

The above theorem admits the following corollary.

**Corollary 3.8.** Let $A$ be a $P_\#$-matrix. Suppose $Ax = \lambda x$, $0 \neq x \in R(A)$ and $\lambda \in \mathbb{R}$. Then $\lambda > 0$.

A characterization of a $P_D$-matrix is presented next.

**Theorem 3.9.** Let $A \in \mathbb{R}^{n \times n}$. Then $A$ be a $P_D$-matrix if and only if for each $x \in R(A^k)$ there is a positive diagonal matrix $D_x \in \mathbb{R}^{n \times n}$ such that $x^T(D_x Ax) > 0$.

**Proof.** Necessity: Let $A$ be a $P_D$-matrix. So for each $0 \neq x \in R(A^k)$, there is an $i_0 \in \{1, 2, \cdots n\}$ such that $x_{i_0}(Ax)_{i_0} > 0$. Then there exists $\epsilon > 0$ such that $x_{i_0}(Ax)_{i_0} + \epsilon \sum_{j=1,j \neq i_0}^n x_j(Ax)_j > 0$. Let $D_x = diag(d_1, d_2, \cdots d_n)$ with $d_{i_0} = 1$ and $d_j = \epsilon$ for all $j \neq i_0$. Hence $x^T(D_x Ax) > 0$.

Sufficiency: Suppose for each $x \in R(A^k)$ there is a positive diagonal matrix $D_x \in \mathbb{R}^{n \times n}$ such that $x^T(D_x Ax) > 0$. So, $D_x Ax = (d_1 \sum_{j=1}^n a_{ij} x_j, \cdots d_n \sum_{j=1}^n a_{nj} x_j)^T$. Since $x^T(D_x Ax) > 0$ and $d_i > 0$, $x_i(Ax)_i > 0$ for each $i$. Hence $A$ is a $P_D$-matrix. \hfill \Box

For $\text{ind}(A) = 1$, the above theorem yields a characterization of a $P_\#$-matrix. With this, we proceed to present the definition of a sign-change matrix.

**Definition 3.10.** A diagonal matrix $S$ is called a sign-change matrix if the diagonals of $S$ are 1 or $-1$.

A relationship between a $P$-matrix and a block $P_D$-matrix is shown next.
Theorem 3.11. Let \( A = \begin{pmatrix} L & O \\ O & O \end{pmatrix} \in \mathbb{R}^{m \times m} \) be a partition matrix such that \( L \in \mathbb{R}^{n \times n} \) with \( m \geq n \).

(a) If \( L \) is a \( P \)-matrix, then \( A \) is a \( P_D \)-matrix.

(b) \( A \) is a \( P_D \)-matrix and \( L \) is invertible, then \( L \) is a \( P \)-matrix. In this case, \( A^T \) and \( SAS \) also are \( P_D \)-matrices, where \( S \) is a sign-change matrix.

Proof. (a): Let \( 0 \neq x = (x_1, x_2, \cdots, x_m)^T \in R(A^k) \). Define \( u = (x_1, x_2, \cdots, x_m)^T \). Then \( u \in R(A^k) \), \( L \) is a \( P \)-matrix. Hence there exists at least one \( i \in \{1, 2, \cdots, n\} \) such that \( u_i(Lu)_i > 0 \). \( x_i(Ax)_i = u_i(Lu)_i \) for each \( 1 \leq i \leq n \), then it follows that \( A \) is a \( P_D \)-matrix.

(b) Let \( 0 \neq x = (x_1, x_2, \cdots, x_n)^T \). Define \( v = (x_1, x_2, \cdots, x_n, 0, 0, \cdots, 0)^T \in R(A^k) \). Hence there exists at least one \( i \in \{1, 2, \cdots, n\} \) such that \( v_i(Av)_i > 0 \), since for \( n+1 \leq i \leq m \), \( x_i = 0 \). As \( v_i(Av)_i = x_i(Lx)_i \) for each \( 1 \leq i \leq n \), it then follows that \( L \) is a \( P \)-matrix. Also \( L^T \) and \( SLS \) are \( P \)-matrices. \( \square \)

Then the above theorem produces the following corollary.

Corollary 3.12. Let \( A = \begin{pmatrix} L & O \\ O & O \end{pmatrix} \in \mathbb{R}^{m \times m} \) be a partition matrix such that \( L \in \mathbb{R}^{n \times n} \) with \( m \geq n \).

(a) If \( L \) is a \( P \)-matrix, then \( A \) is a \( P_\# \)-matrix.

(b) \( A \) is a \( P_\# \)-matrix and \( L \) is invertible, then \( L \) is a \( P \)-matrix. In this case, \( A^T \) and \( SAS \) also are \( P_\# \)-matrices, where \( S \) is a sign-change matrix.

Let us recall the definition of a \( Z \)-matrix and an \( M \)-matrix. A square matrix whose off-diagonal elements are non-positive is called a \( Z \)-matrix. It follows that a \( Z \)-matrix \( A \) can be written as \( A = sI - B \), where \( B \geq 0 \), \( s \geq \rho(B) \). A \( Z \)-matrix \( A \) is called a \( M \)-matrix if \( s \geq \rho(B) \). A \( Z \)-matrix \( A \) is called a nonsingular \( M \)-matrix if \( A \) is monotone. It is well known that if \( A \) is a \( Z \)-matrix then \( A \) is a \( P \)-matrix if and only if \( A \) is an invertible \( M \)-matrix. The matrix in Example 3.2 is a \( Z \)-matrix and is also a \( P_D \)-matrix.
However, it is not always true that $A^D \geq 0$. In order to study this property we have to apply the well-known result stated in [6] and is recalled below.

**Theorem 3.13.** (Theorem 3.9, [6])

Let $A$ be a Z-matrix having all principal minors are nonnegative. Then $A^\dagger \geq 0$ if and only if there exists a permutation matrix $S$ such that $SAS^T = \begin{pmatrix} L & O \\ O & O \end{pmatrix}$ where $L$ is an invertible M-matrix.

In Theorem 3.13, if $L$ is an invertible M-matrix, then $L$ is an $P$-matrix. Next theorem says about a relation between a Z-matrix and a $PD$-matrix.

**Theorem 3.14.** Let $A \in \mathbb{R}^{n \times n}$ be a Z-matrix having all principal minors are nonnegative and $A^D \geq 0$. Then there exists a permutation matrix $S$ such that $SAS^T$ is a $PD$-matrix.

**Proof.** Let $B = SAS^T = \begin{pmatrix} L & O \\ O & O \end{pmatrix}$ where $L \in \mathbb{R}^{n \times n}$ is an invertible M-matrix, i.e., $P$-matrix. We will show that $B$ is a $PD$-matrix. Let $0 \neq x = (x_1, x_2, \cdots x_m)^T \in R(B^k)$. Taking $v = (v_1, v_2, \cdots v_n)^T$. Since every $P$-matrix is a $PD$-matrix. So $v \in R(L^k)$. Hence there exists at least one $i \in \{1, 2, \cdots n\}$ such that $v_i(Lv)_i > 0$ and $x_i(Bx)_i = v_i(Lv)_i$ for each $1 \leq i \leq n$ which follows that $B$ is a $PD$-matrix. \qed

The corollary of the above theorem comes when we take $x \in R(A)$.

**Corollary 3.15.** Let $A \in \mathbb{R}^{n \times n}$ be a Z-matrix having all principal minors are nonnegative, $A^#$ exists and $A^# \geq 0$. Then there exists a permutation matrix $S$ such that $SAS^T$ is a $P^#$-matrix.

Next theorem relates Drazin monotonicity and $PD$-matrices.

**Theorem 3.16.** Let $A \in \mathbb{R}^{n \times n}$ be any matrix. Then the following statements are equivalent.

(i) $A$ is a $PD$-matrix.

(ii) $A^D$ is a $PD$-matrix and $A^D \geq 0$. 
Proof. (i) $\Rightarrow$ (ii): Suppose that $A$ is a $P_D$-matrix. Then by Theorem 3.5 $A^D$ is a $P_D$-matrix. Next, to show $A^D \geq 0$. Let $u \in R(A^k)$, $u > 0$ and $y = A^D u$. Then $y \in R(A^k)$ and $Ay = AA^D u = u > 0$. Hence $Ay > 0$. Since $A$ is a $P_D$-matrix, so for $0 \neq y \in R(A^k)$ there is an $i \in \{1, 2, \cdots n\}$ such that $y_i(Ay)_i > 0$. So $y_i > 0$ where $y_i$ is the $ith$ component of $y$, as $Ay > 0$. Therefore $A^D \geq 0$.

(ii) $\Rightarrow$ (i): The proof is same as in Theorem 3.5. \hfill \Box

We then have the following corollary for $P_\#$-matrices.

Corollary 3.17. Let $A \in \mathbb{R}^{n \times n}$ be any matrix. Then the following statements are equivalent.

(i) $A$ is a $P_\#$-matrix.

(ii) $A^\#$ is a $P_\#$-matrix, $A^\#$ exists and $A^\# \geq 0$.

4. A Connection with Linear Complementarity Problems

It is well-known that a $P$-matrix $A$ is characterized by the condition that the standard linear complementarity problem $LCP(q, A)$ has a unique solution for all $q \in \mathbb{R}^n$ in [4]. A relation between them is shown next.

Theorem 4.1. $LCP(q, A)$ has unique solution for each $q \in \mathbb{R}^n$ if and only if $A$ is a $P$-matrix.

Motivated by the work of Kannan and Sivakumar [9], we are now going to prove the existence of solution of LCP with the help of $P_D$-matrices.

Theorem 4.2. $LCP(q, A)$ has unique solution for each $q \in \mathbb{R}^n$ if and only if $A$ is a $P_D$-matrix.

Proof. The proof of this theorem is similar to the proof for $P$-matrix (see page: 274-275, [2]). \hfill \Box

The following result is well-known in the theory of linear complementarity problems.
Theorem 4.3. (Theorem 3.4.4, [4])

Let $A \in \mathbb{R}^{n \times n}$. Then the following statements are equivalent.

(a) For all $q \in S(A)$, if $x^1, x^2 \in \text{SOL}(q, A)$, then $Ax^1 = Ax^2$.

(b) Every vector whose sign is reversed by $A$ must belong to $N(A^k)$, i.e., if $x_i(Ax)_i \leq 0$ for all $i$, then $x \in N(A)$.

Using this result we show a sufficient condition for a matrix to be a $P_D$-matrix.

Theorem 4.4. Let $A \in \mathbb{R}^{n \times n}$. Suppose that for all $q \in S(A)$, and for every $x^1, x^2 \in \text{SOL}(q, A)$, it follows that $Ax^1 = Ax^2$. Then $A$ is a $P_D$-matrix.

Proof. Let $x \in R(A^k)$ be such that $x_i(Ax)_i \leq 0$ for all $i$. Then by the condition (b) of the Theorem 4.3, it follows that $x \in N(A^k)$. Hence $x = 0$. So $A$ is a $P_D$-matrix. □

We conclude this section with the remark that all the theorems mentioned in this section are also true for $P_\#$-matrices.

5. Characterization of $P_D$-matrices with $i(A, B)$

In this section, first we discuss the inclusion $r(A, B) \subseteq K$ with $P_D$-matrices. We also state a result of Johnson and Tsatsomero, [7] for $P$-matrices, and then extend it for index-range symmetric matrices.

Theorem 5.1. (Theorem 3.3, [7])

Let $A, B \in \mathbb{R}^{n \times n}$ be such that $A$ and $B$ are invertible. Then $r(A, B) \subseteq K$ if and only if $BA^{-1}$ is $P$-matrix.

The extension of the above result is proposed next.

Theorem 5.2. Let $A, B \in \mathbb{R}^{n \times n}$ be such that $R(A^k) = R(B^k)$ and $N(A^k) = N(B^k)$ and $A, B$ are commutative. Then $r(A, B) \subseteq K$ if and only if $BA^D(AB^D)$ is a $P_D$-matrix.

Proof. Necessity: Let $r(A, B) \subseteq K$ and suppose $BA^D$ is not a $P_D$-matrix. Then, there exists $0 \neq x \in R((BA^D)^k) = R(B^k(AB^D)^k) \subseteq R(B^k) = R(A^k)$ such that $x_i(BA^Dx)_i \leq 0$
for all $i$. For $1 \leq i \leq n$, consider the function $f_i : [0, 1] \rightarrow \mathbb{R}$ defined by $f_i(t) = tx_i + (1 - t)(BA^D)x$. Then by intermediate value theorem, there exists $t_i \in [0, 1]$ such that $tx_i + (1 - t)(BA^D)x = 0$. Let $L = diag(t_1, t_2, \ldots, t_n)$. Then, $Lx + (I - L)(BA^D)x = 0$. Since $x \in R(A^k)$, then $x = AA^Dx$ for some $x \in \mathbb{R}^n$. Hence, $0 = Lx + (I - L)(BA^D)x = LAA^Dx + (I - L)(BA^D)AA^Dx = (LAA^D + (I - L)(BA^D)AA^D)x = (LAA^D + (I - L)BA^D)x = (LA + (I - L)B)A^Dx$. This implies $A^Dx = 0$ implies $x \in N(A^D) = N(A^k)$ (as $LA + (I - L)B \in r(A, B)$), a contradiction. So $BA^D$ is a $P_D$-matrix.

Sufficiency: Let $t_i \in [0, 1]$, $i = \{1, 2, \ldots, n\}$, $L = diag(t_1, t_2, \ldots, t_n)$ and $(LA + (I - L)B)x = 0$ for some $x \in R(A^k)$. Since $x \in R(A^D) = R(A^k)$, we have $x = A^Dy$ for some $y \in R(A^k)$. Therefore $(LA + (I - L)B)x = 0$ implies $(LA + (I - L)B)A^Dy = 0$ which again yields $LAA^Dy + (I - L)BA^Dy = 0$. If $y \in R(A^k)$, then $LAA^Dy + (I - L)BA^Dy = Ly + (I - L)BA^Dy = 0$. Also, $(BA^D)^D(BA^D)y = y$, since $R((BA^D)^k) \subseteq R(A^k)$. Thus, $y \in R((BA^D)^k)$. The fact $L \geq 0$ and $(I - L) \geq 0$, so $y_i$ and $(BA^D)y_i$ are opposite in signs for each $i$, i.e., $y_i(BA^D)y_i \leq 0$. So $BA^D$ is not a $P_D$-matrix, a contradiction. Hence $r(A, B) \subseteq K$. □

**Corollary 5.3.** Let $A, B \in \mathbb{R}^{n \times n}$ be such that $R(A) = R(B)$ and $N(A) = N(B)$. Then $r(A, B) \subseteq K$ if and only if $A^\#(B^\#)$ exists and $BA^\#(AB^\#)$ is a $P_\#$-matrix.

**Theorem 5.4.** Let $A, B \in \mathbb{R}^{n \times n}$ be such that $R(A^k) = R(B^k)$ and $N(A^k) = N(B^k)$. Then $c(A, B) \subseteq K$ if and only if $B^D A(A^D B)$ is a $P_D$-matrix.

The proof is similar to proof of Theorem 5.2 and this theorem carries a corollary, given next.

**Corollary 5.5.** Let $A, B \in \mathbb{R}^{n \times n}$ be such that $R(A) = R(B)$ and $N(A) = N(B)$. Then $c(A, B) \subseteq K$ if and only if $A^\#(B^\#)$ exists and $B^\# A(A^\# B)$ is a $P_\#$-matrix.

Combining Theorem 5.2 and 5.4, we have the following theorem.
Corollary 5.6. Let $A, B \in \mathbb{R}^{n \times n}$ be such that $R(A^k) = R(B^k)$ and $N(A^k) = N(B^k)$. Then $r(A, B) \subseteq K$ and $c(A, B) \subseteq K$ if and only if $BA^D$, $AB^D$, $A^DB$ and $B^DA$ are $P_D$-matrices.

Now, we produce the following result which is proved by the authors Rohn [12], and Johnson and Tsatsomeros [7] for the matrices whose interval hull contains no singular matrices.

Theorem 5.7. Let $A, B \in \mathbb{R}^{n \times n}$ such that each matrix in $i(A, B)$ is invertible. Then $BA^{-1}$, $A^{-1}B$, $B^{-1}A$ and $AB^{-1}$ are $P$-matrices.

Now, we present the generalization of above theorem to singular case.

Theorem 5.8. Let $A, B \in \mathbb{R}^{n \times n}$ be such that $R(A^k) = R(B^k)$ and $N(A^k) = N(B^k)$. Further, let $i(A, B) = K$. Then $BA^D$, $AB^D$, $A^DB$ and $B^DA$ are $P_D$-matrices.

Proof. Suppose $BA^D$ is not a $P_D$-matrix. Then, there exists $0 \neq x \in R((BA^D)^k)$ such that $x_i(BA^Dx)_i \leq 0$ for all $i$. Let $C_i$ denotes the $i$th row of $C \in \mathbb{R}^{n \times n}$ defined by $C_i = B_i + t_i(A_i - B_i)$, where $A_i$, $B_i$ are the $i$th rows of $A$, $B$ respectively. Let $t_i = 1$ if $x_i = 0$ and if $x_i \neq 0$, then $t_i$ be an arbitrary root of the continuous function $\phi(t) = x_i(B + t(A - B))_iA^Dx$ in $[0, 1]$; such a root exists, since $\phi(0) = x_i(BA^Dx)_i \leq 0$ and $\phi(1) = x_i(AA^Dx)_i = x_i^2 \geq 0$. So $C_i$ is a convex combination of $A_i$ and $B_i$ for each $i = \{1, 2, \cdots n\}$, hence $C \in i(A, B)$. Now, we will show $C \in K$. Let $A^Dx \in N(C) \subseteq N(C^k)$. If $x_i = 0$, then $(CA^Dx)_i = C_iA^Dx = A_i(A^Dx) = (AA^Dx)_i = x_i = 0$, and if $x_i \neq 0$, then $(CA^Dx)_i = (C_iA^Dx) = \frac{\phi(t_i)}{x_i} = 0$. Hence, $A^Dx \in N(C) \subseteq N(C^k)$. If $A^Dx \in N(A) \subseteq N(A^k)$, then $x_i = 0$, a contradiction. So, $N(C^k) \neq N(A^k)$, again a contradiction. Hence $BA^D$ is a $P_D$-matrix.

Corollary 5.9. Let $A$, $B \in \mathbb{R}^{n \times n}$ be such that $R(A) = R(B)$ and $N(A) = N(B)$. Further, let $i(A, B) = K$. Then $BA^\#, AB^\#, A^\#B$ and $B^\#A$ are $P_D$-matrices provided $A^\#$ and $B^\#$ exists.
6. A Characterization for the Inclusion \( h(A, B) \subseteq K \)

In this section, we present a theorem which creates a relation between the inclusion \( h(A, B) \subseteq K \) and a constrained eigenvalue condition of the matrix \( A^D B \). Then the result obtained by taking \( J = [A, B] \) and \( h(A, B) \subseteq K \) is discussed. Finally, a new result is proved which is based on the inclusion \( h(A, B) \subseteq K \), and nonnegativity of Drazin inverse of certain element in \( J = [A, B] \).

**Theorem 6.1.** Let \( A, B \in \mathbb{R}^{n \times n} \) be such that \( R(A^k) = R(B^k) \) and \( N(A^k) = N(B^k) \). Then the following conditions are equivalent:

(a) \( h(A, B) \subseteq K \).

(b) \( A^D B x = \lambda x, \ 0 \neq \lambda \in \mathbb{R} \). Then \( \lambda > 0 \).

**Proof.** (a) \( \Rightarrow \) (b): Suppose that (a) holds. Assume that \( A^D B x = \lambda x \) holds for some \( \lambda < 0 \). Then \( B x = P_{R(A^k),N(A^k)} B x = A A^D B x = \lambda A x \). If \( B x = 0 \), then \( x = 0 \). So \( B x \neq 0 \) and \( A x \neq 0 \). Set \( u = \frac{-\lambda}{1-\lambda} \) and \( C^k = (u A + (1 - u) B) \in h(A, B) \). Then, \( u \in (0, 1) \) and \( A^D C^k x = A^D (u A + (1 - u) B) x = \frac{-1}{1-\lambda} (-\lambda I + A^D B) x = 0 \). So \( C^k x \in N(A^D) = N(A^k) \) and \( C^k x \in R(A^k) \). Therefore \( C^k x = 0 \). Thus \( N(C^k) \not\subseteq N(A^k) \) and then \( h(A, B) \not\subseteq K \), a contradiction. So \( \lambda > 0 \).

(b) \( \Rightarrow \) (a): Suppose that (b) holds and assume that \( h(A, B) \not\subseteq K \). Then, \( (u A + (1 - u) B) \not\in h(A, B) \) for some \( u \in (0, 1) \). As \( N(A^k) \subseteq N(u A + (1 - u) B) \). Suppose that \( N(A^k) \neq N(u A + (1 - u) B) \). Then, \( (u A + (1 - u) B) x = 0 \) for some \( x \not\in N(A^k) \). Let \( x = x^1 + x^2 \), where \( x^1 \in N(A^k) \) and \( 0 \neq x^2 \in R(A^k) \). Then, \( (u A + (1 - u) B) x^2 = 0 \). Pre-multiplying by \( A^D \), we get \( (u A^D A + (1 - u) A^D B) x^2 = 0 \). By setting \( \lambda = \frac{-u}{1-u} < 0 \), it follows that \( A^D B x^2 = \lambda x^2 \), a contradiction. \( \square \)

For \( k = 1 \), the property index range-symmetric reduces to range-symmetric and we have the following corollary.

**Corollary 6.2.** Let \( A, B \in \mathbb{R}^{n \times n} \) be such that \( R(A) = R(B) \) and \( N(A) = N(B) \). Then the following conditions are equivalent:
(a) \( h(A, B) \subseteq K \).

(b) \( A^\# B x = \lambda x, \ 0 \neq \lambda \in \mathbb{R} \). Then \( \lambda > 0 \).

An square interval matrix is defined as the set of matrices of the form \( J = [A, B] = \{ C : A \leq C \leq B \} \) for \( A \in \mathbb{R}^{n \times n}, \ B \in \mathbb{R}^{n \times n} \) and \( A \leq B \). We shall often use the center matrix \( J_C = \frac{1}{2}(B + A) \) and the radius matrix \( \Delta = \frac{1}{2}(B - A) \). Thus, \( A = J_C - \Delta, \ B = J_C + \Delta \) and \( \Delta \geq 0 \) which yields \( J = [J_C - \Delta, J_C + \Delta] \).

An interval matrix \( J = [A, B] \) where \( A, B \in \mathbb{R}^{n \times n} \) is called index range-kernel regular if for all \( C \in J \), \( R(C^k) = R(A^k) \) and \( N(C^k) = N(A^k) \). When \( k = 1 \), it coincides with range-kernel regular, i.e., \( R(C) = R(A) \) and \( N(C) = N(A) \).

**Theorem 6.3.** Let \( J = [A, B] \). If \( h(A, B) \subseteq K \), then \( \rho(J_c^D \Delta) < 1 \).

**Proof.** Since \( J_c \in K \), then it follows that \( R(A^k) = R(J_c^k) \) and \( N(A^k) = N(J_c^k) \). Suppose on contrary, \( \beta = \rho(J_c^D \Delta) \geq 1 \). Then, there exists \( 0 \neq x \in \mathbb{R}^n \) such that \( J_c^D \Delta = \beta x \). Then, \( x \in R(J_c^D) = R(J_c^k) = R(A^k) \). Also, \( P_{R(J_c^k)} N(J_c^k)(J_c - A)x = J_c J_c^D \Delta x = \beta J_c x \) so that \( (J_c - A - \beta J_c)x = 0 \). Dividing it by \( \frac{1}{\beta} \) and taking \( \eta = 1 + \frac{1}{\beta} \geq 0 \), then we have \( (\eta J_c + (1 - \eta)A)x = 0 \). Let \( P = (\eta J_c + (1 - \eta)A) \). Then, \( P \in K \) and \( x \in N(P^k) = N(A^k) \). As \( x \in R(A^k) \), thus \( x = 0 \), a contradiction. \( \square \)

**Corollary 6.4.** Let \( J = [A, B] \). If \( h(A, B) \subseteq K \), then \( \rho(J_c^\# \Delta) < 1 \).

We next present an analogous result to Theorem 3.5, [8] for square singular matrices using the Drazin inverse.

**Theorem 6.5.** Let \( J \) be index range-kernel regular. Then the following are equivalent.

(i) \( C^D \geq 0 \) whenever \( C \in K \),

(ii) \( B^D \geq 0 \) and \( A^D \geq 0 \),

(iii) \( B^D \geq 0 \) and \( \rho(B^D(B - A)) < 1 \).

**Proof.** (i) \( \Rightarrow \) (ii) Follows from definition of \( J \).

(ii) \( \Rightarrow \) (iii) \( A = B - (B - A) \) is an index-proper splitting of \( A \). Then \( B^D(B - A) \geq 0 \). So by Theorem 2.1, \( \rho(B^D(B - A)) < 1 \).
(iii) $\Rightarrow$ (i) Let $C = B(I - B^D(B - C))$. Now to show $(I - B^D(B - C))$ is invertible. Let $(I - B^D(B - C))x = 0$ then, $x = B^D(B - C)x \in R(B^D) = R(B^k)$. So $x = BB^Dx$ and hence $x = BB^Dx - B^D C x = x - B^D C x$. Therefore $B^D C x = 0$. Thus, $Cx \in N(B^D) = N(C^D)$ it implies that $x = CC^D x = 0$ implies $x = 0$. Hence $(I - B^D(B - C))$ is invertible. As $C = B(I - B^D(B - C))$. Next to show, $C^D = (I - B^D(B - C))^{-1}B^D$.

For this, let $X = B$, $Y = (I - B^D(B - C))$. Then, $(XY)^D = Y^{-1}X^D$ if and only if $YY^kX^k = X^D X Y Y^k X^k$. We have $R(Y Y^k X^k) = R(Y (X Y)^k) = R((I - B^D(B - C))C^k) = R(C^k - B^D(B - C))C^k \subseteq R(C^k) = R(B^k) \subseteq R(B) = R(X)$ (since $P_{L,M}A = A$ if and only if $R(A) \subseteq L$). Therefore $C^D = (I - B^D(B - C))^{-1}B^D = \sum_{k=0}^{\infty} (B^D(B - C))^k B^D \geq 0 \quad \square$

Using the above result, the next result follows.

**Theorem 6.6.** Let $J = [A, B]$ be index range-kernel regular. Then the following are equivalent.

(a) $B^D \geq 0$ and $A^D \geq 0$.
(b) $h(A, B) \subseteq K$ and $C^D \geq 0$ for all $C \in h(A, B)$.

**Proof.** (a) $\Rightarrow$ (b): Let $C = \lambda A + (1 - \lambda)B$ for some $\lambda \in [0, 1]$. Then, $N(A^k) \subseteq N(C^k)$ and $R(A^k) \subseteq R(C^k)$. Also, we have $0 \leq B^D(B - C) \leq B^D(B - A)$ and hence $\rho(B^D(B - A)) < 1$. Thus, $\rho(B^D(B - C)) < 1$ and $(I - B^D(B - C))$ is invertible. Now, $(I - B^D(B - C)) = I - P_{\rho(B^D(B - C))} + B^D C = -P_{\rho(B^D(B - C))} + B^D C = E$. Then, $BE = BP_{\rho(B^D(B - C))} + BB^D C = BB^D C = C$. So, $B = CE^{-1}$ and hence $R(B^k) = R(C^k)$. By rank-nullity dimension theorem, it follows that $N(A^k) = N(C^k)$. So, $C \in K$. Then, by Theorem 6.3, we have $C^D \geq 0$.

(a) $\Rightarrow$ (b): Since $h(A, B) \subseteq K$ and $C^D \geq 0$ for all $C \in h(A, B)$, then $C = \lambda A + (1 - \lambda)B$ for some $\lambda \in [0, 1]$. Again as $C^D \geq 0$ for all $C \in h(A, B)$, it is obviously true that $B^D \geq 0$ and $A^D \geq 0$. \quad \square

**Corollary 6.7.** Let $J = [A, B]$ be range-kernel regular. Then the following are equivalent.

(a) $B^\# \geq 0$ and $A^\# \geq 0$.
(b) $h(A, B) \subseteq K$ and $C^\# \geq 0$ for all $C \in h(A, B)$. 
References


