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P_D-MATRICES AND LINEAR COMPLEMENTARITY PROBLEMS

LITISMITA JENA AND SABYASACHI PANI

School of Basic Sciences,

Indian Institute of Technology Bhubaneswar,

Bhubaneswar - 751 013, India.

Email: litumath@gmail.com

Abstract. Motivated by the definition of $P_\dagger$-matrix ([9]), another generalization of a P-matrix for square singular matrices called P_D-matrix is proposed first. Then the uniqueness of solution of Linear Complementarity Problems for square singular matrices is proved using P_D-matrices. Finally some results which are true for P-matrices are extended to P_D-matrices.

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1. Introduction

A matrix $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ is called a *P-matrix* if every principal minor of A is positive. This notion was first introduced by Fiedler and Ptak [5]. It plays important roles in studying solution properties of equations and complementarity problems, and convergence/complexity analysis of methods for solving these problems. There are numerous ways to describe a P-matrix, we consider only three of them for our work. The three equivalent definitions for a real square matrix A are as follows:
(i) All the principal minors of A are positive.
(ii) Every real eigenvalue of each principal submatrix of A is positive.
(iii) The matrix A does not reverse the sign of any vector; i.e., if $x \neq 0$ and $y = Ax$, then for some subscript i, $x_i y_i > 0$.
The equivalence of (i) and (iii) was established by Fiedler and Ptak [5]. P-matrices also arise quite frequently in systems theory. These include hermitian positive definite matrices, M-matrices, totally positive matrices and real diagonally dominant matrices with positive diagonal entries.

Now coming to the name "linear complementarity problem" which stems from the linearity of the mapping $W(z) = q + Az$, where $A \in \mathbb{R}^{n\times n}$ and the complementarity of real n-vectors w and z. For a given $q \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n\times n}$, the linear complementarity problem (LCP) is that of finding (or concluding there is no) $z \in \mathbb{R}^n$ such that

$$w = q + Az \geq 0,$$

$$z \geq 0,$$

$$z^T w = 0.$$

We denote the above problem by the symbol $LCP(q, A)$. A vector $z \in \mathbb{R}^n$ satisfying the above three conditions is called a *solution* of $LCP(q, A)$ and the set of all solutions is denoted by $SOL(q; A)$. The solution set is defined by $S(A) = \{q : SOL(q, A) \neq \phi\}$. For more details on linear complementarity problems, we refer the reader to the book by Cottle, Pang and Stone [4].

Let us discuss some basic relations between the complementarity cone and P-matrices. The class of complementarity cone corresponding to a real square matrix A is denoted by $\mathcal{A}$, the pair of column vectors $(I_{.j}, -A_{.j})$ is known as the jth complementary pair of vectors in $1 \leq j \leq n$. The convex cone generated by any complementary set of column vectors is known as a complementarity cone. A theorem proved by Samelson, Thrall and Wesler [13] says that the set of complementarity cone partitions $\mathbb{R}^n$ if and only if A is a P-matrix. Later this characterization of P-matrices by Samelson, Thrall and Wesler [13] was improved by Murty [10]. He proved that a real $n \times n$ matrix A is a P-matrix if and only if the $LCP(q, A)$ has a unique solution for every q belonging to $\{I_{.1}, \cdots I_{.n}, -I_{.1}, \cdots - I_{.n}, A_{.1}, \cdots A_{.n} - A_{.1}, \cdots - A_{.n}, e$, and further extended by Tamir [14]. Tamir [14] stated a $n \times n$ matrix A is a P-matrix if and only if the $LCP(q, A)$ has a unique solution for every q belonging to $\{I_{.1}, \cdots I_{.n}, A_{.1}, \cdots A_{.n} - A_{.1}, \cdots - A_{.n}, e$, where $e = (1, 1, \cdots, 1)^T$ is the vector of ones of order n.

Very recently, Kannan and Sivakumar [9] generalized the notion of P-matrix for singular cases and call it as $P_\dagger$-matrix. The definition of this is presented next.

Definition 1.1. *(Definition 1.1* [9]*) A square matrix A is said to be a $P_\dagger$-matrix if for each non zero $x \in \mathbb{R}(A^T)$, there is an $i \in \{1, 2, \cdots, n\}$ such that $x_i(Ax_i) > 0$.*

Let us introduce some more definitions and notations which are going to be used to prove our main results. Let the diagonal matrix whose entries are $t_1, t_2, \cdots t_n$ is denoted by $diag(t_1, t_2, \cdots t_n)$. Let F denote the matrix whose entries are all one, and let o denote the Hadamard (entry wise) product of matrices. For any $A, B \in \mathbb{R}^{n\times n}$, we define the following sets:

$h(A, B) = \{C : C = tA + (1 - t)B, t \in [0, 1]\}$

$r(A, B) = \{C : \ C = TA + (I - T)B, \ T = diag(t_1, t_2, \cdots t_n), t_i \in [0, 1],$
$1 \leq i \leq n\}$

$c(A, B) = \{C : \ C = AT + B(I - T), \ T = diag(t_1, t_2, \cdots t_n), t_i \in [0, 1],$
$1 \leq i \leq n\}$

$i(A,B)=\{C: C=ToA+(F-T)oB,\ T=diag(t_1,t_2,\cdots t_n), t_i\in[0,1],$
$1\leq i\leq n\}$

$J=i(A,B)$, if $A\leq B$.

$r(A,B)$ denotes the set of matrices whose rows (columns) are independent convex combinations of the corresponding rows of A and B. While $c(A,B)$ denotes the set of matrices whose columns are independent convex combinations of the corresponding columns of A and B. The *interval hull* ($i(A,B)$) for any two matrices $A,B\in\mathbb{R}^{n\times n}$ is defined as

$$i(A,B)=\{C\in\mathbb{R}^{n\times n}: min\{a_{ij},b_{ij}\}\leq c_{ij}\leq max\{a_{ij},b_{ij}\}\}.$$

From all the above definitions above, now it is clear that $h(A,B)\subseteq r(A,B)\subseteq i(A,B)$ and $h(A,B)\subseteq c(A,B)\subseteq i(A,B)$.

The interval hull $i(A,B)$ is said to be *index-range kernel regular* if $R(A^k)=R(B^k)$ and $N(A^k)=N(B^k)$. Let us define the set

$$K=\{C\in i(A,B): R(A^k)=R(U^k) \text{ and } N(A^k)=N(U^k)\}$$

for an index-range kernel interval hull $i(A,B)$. When A and B are invertible, then K contains only invertible matrices. Motivated by the results of Johnson and Tsatsomeros [7] for P-matrices, and further generalization by Kannan and Sivakumar [9] for $P_{\dagger}$-matrices, we establish analogous results by introducing another generalization of the notion of P-matrix.

The motto of this article is to first propose an extension of P-matrix and to study some its properties. The organization of this paper is as follows. After a brief review of some basic definitions and notations in Section 2, we include the definition of a P_D-matrix and few of its properties in Section 3. We first state linear complementarity problems (LCP), secondly a relation between P_D-matrices and LCP, and then some results. Section 4 contains the characterization of the inclusion $r(A,B)\subseteq K$ with P_D-matrices, and an analogous result for $c(A,B)\subseteq K$ in Section 4. Finally, the case $i(A,B)=K$ and the generalization of Theorem 3.8, [7] are discussed. Section 5 presents the inclusion

$h(A,B) \subseteq K$ and its link with certain constrained eigenvalue condition. Then a new result for the nonnegativity of Drazin inverse of an interval is proved in Section 6.

2. Preliminaries

Let $\mathbb{R}^n$ denote the n dimensional real Euclidean space and $\mathbb{R}^n_+$ denote the nonnegative orthant in $\mathbb{R}^n$. For a real $m \times n$ matrix A, i.e., $A \in \mathbb{R}^{m\times n}$, the matrix G satisfying the four equations known as Penrose equations: $AGA = A$, $GAG = G$, $(AG)^T = AG$ and $(GA)^T = GA$ is called the *Moore-Penrose inverse* of A (B^T denotes the transpose of B). It always exists and unique, and is denoted by $A^\dagger$. $A \in \mathbb{R}^{m\times n}$ is said to be *semimonotone* if $A^\dagger \geq O$ (here the comparison is entry wise and O is the null matrix of respective order). For a real $n \times n$ matrix A. The index of a real square matrix A is the least nonnegative integer k such that $rank(A^{k+1}) = rank(A^k)$. It is denoted by ind A. Then $ind(A) = k$ if and only if $R(A^k) \bigoplus N(A^k) = \mathbb{R}^n$. For $A \in \mathbb{R}^{n\times n}$ the matrix G satisfying the three equations : $A^kGA = A^k$, $GAG = G$, $AG = GA$ known as Drazin inverse of A, where k is the index of A. It always exists and unique, and is denoted by A^D. When $k = 1$, then the Drazin inverse is known as group inverse and is denoted as $A^\#$. $A \in \mathbb{R}^{n\times n}$ is said to be *Drazin monotone* if $A^D \geq O$. When A is a square nonsingular, then $A^\dagger = A^\# = A^D = A^{-1}$, and a semimonotone (or Drazin monotone) matrix becomes a *monotone matrix* (i.e., A^{-1} exists and $A^{-1} \geq O$). (See the book by Berman and Plemmons, [2] for more details on monotone matrices and their generalizations.) For $A, B, C \in \mathbb{R}^{m\times n}$, we say A is *nonnegative* if $A \geq O$, and $B \geq C$ if $B - C \geq O$. We denote a *nonnegative vector* x as $x \geq 0$. Let L and M be complementary subspaces of $\mathbb{R}^n$. Let $P_{L,M}$ be a projector on L along M. Then $P_{L,M}A = A$ if and only if $R(A) \subseteq L$ and $AP_{L,M} = A$ if and only if $N(A) \subseteq M$, where $R(A)$ and $N(A)$ denote the range space and the null space of A. Some well-known index properties of A^D ([1]) are: $R(A^k) = R(A^D)$; $N(A^k) = N(A^D)$ and $AA^D = P_{R(A^k),N(A^k)}$. In particular, if $x \in R(A^k)$, then $x = A^DAx$. The *spectral radius* of $A \in \mathbb{R}^{n\times n}$, denoted by $\rho(A)$ is defined by $\rho(A) = \max_{1\leq i\leq n} |\lambda_i|$, where $\lambda_1, \lambda_2, \cdots, \lambda_n$ are the eigenvalues of A.

The next theorem is a part of Perron$-$Frobenius theorem.

Theorem 2.1. *(Theorem 2.20,* [15]*) Let $A \geq O$. Then A has a nonnegative real eigenvalue equal to its spectral radius.*

Another result which relates spectral radius of two nonnegative matrices is given below.

Theorem 2.2. *(Theorem 2.21,* [15]*) Let $A \geq B \geq O$. Then $\rho(A) \geq \rho(B)$.*

The theory of splitting plays a major role in finding solution of system of linear equations. Many authors have proposed several splittings. Chen-Chen [3] proposed the following splitting.

Definition 2.3. *A splitting $A = U - V$ of $A \in \mathbb{R}^{n\times n}$ is called an index-proper splitting (*[3]*) if $R(A^k) = R(U^k)$ and $N(A^k) = N(U^k)$, $k = ind(A)$.*

3. P_D-MATRICES

We begin this section with another generalization of a singular P-matrix which we call as a P_D-matrix, and the definition is presented below.

Definition 3.1. *A square matrix A is said to be a P_D-matrix if for each non zero $x \in \mathbb{R}(A^k)$, $k = ind(A)$ there is an $i \in \{1, 2, \cdots, n\}$ such that $x_i(Ax_i) > 0$.*

In other words, for any $x \in \mathbb{R}(A^k)$ the inequality $x_i(Ax_i) \leq 0$ for $i \in \{1, 2, \cdots, n\}$ imply $x = 0$. Trivially, every P-matrix is a P_D-matrix.

Example 3.2. *Let $A = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$. Then $ind(A) = 2$. Also $R(A^2) =$ span of $\left\{\alpha \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} : \alpha \in \mathbb{R}\right\}$. Taking $x = (1, -1, 0)^T$ and calculating $x_i(Ax)_i$, we get $x_i(Ax)_i > 0$. So A is a P_D-matrix.*

P_D-matrix reduces to $P_\#$-matrix when $k = 1$, and the definition is as follows.

Definition 3.3. *(Definition 5.1,* [9]*)*
A square matrix A is said to be a $P_\#$-matrix if for each non zero $x \in R(A)$ there is an $i \in \{1, 2, \cdots, n\}$ such that $x_i(Ax_i) > 0$.

In other words, for any $x \in R(A)$ the inequality $x_i(Ax_i) \leq 0$ for $i \in \{1, 2, \cdots, n\}$ imply $x = 0$.

Example 3.4. *Let* $A = \begin{pmatrix} 3 & 1 & 0 \\ 3 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. *Here* $ind(A) = 1$ *and* $R(A) =$ *span of* $\left\{ \begin{pmatrix} 3 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} \right\}$. *Then* $x_i(Ax_i) \leq 0$ *for any* $x \in R(A)$. *Hence* A *is a* $P_{\#}$*-matrix.*

We discuss below some useful properties of P_D-matrices. The first one is the Drazin inverse analogue of Theorem 2.3, [9].

Theorem 3.5. *A is a P_D-matrix if and only if A^D is a P_D-matrix.*

Proof. Suppose that A is a P_D-matrix. So for each $0 \neq y \in R((A^D)^k) = R(A^k)$, there is an $i \in \{1, 2, \cdots n\}$ such that $y_i(Ay)_i > 0$. Let $y \in R((A^D)^k) = R(A^k)$ then $y = AA^Dx$, for some $x \in \mathbb{R}^n$, so $y_i(A^Dy)_i = (AA^Dx)_i(A^DA^DAx)_i = (A^DAx)_i(A^DAA^Dx)_i = (A^DAx)_i(A^Dx)_i = (A^Dx)_i(AA^Dx)_i = u_i(Au)_i > 0$. Set $u = A^Dx \in R(A^D) = R(A^k)$. So A^D is a P_D-matrix.

Conversely: Let A^D be a P_D-matrix. In order to show that A is a P_D-matrix, we have to prove that $y_i(Ay)_i > 0$ for $y \in R(A^k)$, $i \in \{1, 2, \cdots n\}$. Since A^D is a P_D-matrix, $x_i(A^Dx)_i > 0$, $0 \neq x \in R((A^D)^k) = R(A^k)$. Therefore $y_i(Ay)_i = (A^Dx)_i(AA^Dx)_i = (A^DAA^Dx)_i(AA^Dx)_i = (A^Du)_iu_i = u_i(A^Du)_i > 0$, where $u = AA^Dx$. Again A^D is a P_D-matrix, $y_i(Ay)_i > 0$, $y \in R(A^k)$. Hence A is a P_D-matrix. □

When $ind(A) = 1$, we then have Theorem 5.1, [9] as a corollary. However, we give a different proof for the existence of the group inverse.

Corollary 3.6. *A is a $P_{\#}$-matrix if and only if $A^{\#}$ is a $P_{\#}$-matrix.*

Proof. The proof is same as the proof for P_D-matrices for $k = 1$, but here only to show $A^{\#}$ exists. For this, suppose that A is a $P_{\#}$-matrix. Let $x \in R(A)$. Then, $x_i(Ax)_i = 0$ for each $i \in \{1, 2, \cdots n\}$, so $R(A) = 0 \Rightarrow r(A) = 0$. Again $R(A) = 0 \Rightarrow R(A^2) = 0$. So $r(A^2) = 0$. Hence $ind(A) = 1$. Therefore $A^{\#}$ exists. □

Next theorem says that a P_D-matrix has a nonnegative eigenvalue decomposition under a given condition.

Theorem 3.7. *Let A be a P_D-matrix. Suppose $Ax = \lambda x$, $0 \neq x \in R(A^k)$ and $\lambda \in \mathbb{R}$. Then $\lambda > 0$.*

Proof. Assume that $Ax = \lambda x$, $0 \neq x \in R(A^k)$ and $\lambda \in \mathbb{R}$ and A is a P_D-matrix. Then $\lambda x_i^2 = \lambda x_i x_i = x_i(Ax)_i > 0$, for some $i \in \{1, 2, \cdots n\}$. Hence $\lambda > 0$. □

The above theorem admits the following corollary.

Corollary 3.8. *Let A be a $P_\#$-matrix. Suppose $Ax = \lambda x$, $0 \neq x \in R(A)$ and $\lambda \in \mathbb{R}$. Then $\lambda > 0$.*

A characterization of a P_D-matrix is presented next.

Theorem 3.9. *Let $A \in \mathbb{R}^{n\times n}$. Then A be a P_D-matrix if and only if for each $x \in R(A^k)$ there is a positive diagonal matrix $D_x \in \mathbb{R}^{n\times n}$ such that $x^T(D_x Ax) > 0$.*

Proof. Necessity: Let A be a P_D-matrix. So for each $0 \neq x \in R(A^k)$, there is an $i_0 \in \{1, 2, \cdots n\}$ such that $x_{i_0}(Ax)_{i_0} > 0$. Then there exists $\epsilon > 0$ such that $x_{i_0}(Ax)_{i_0} + \epsilon \sum_{j=1, i_0\neq j}^{n} x_j(Ax)_j > 0$. Let $D_x = diag(d_1, d_2, \cdots d_n)$ with $d_{i_0} = 1$ and $d_j = \epsilon$ for all $j \neq i_0$. Hence $x^T(D_x Ax) > 0$.

Sufficiency: Suppose for each $x \in R(A^k)$ there is a positive diagonal matrix $D_x \in \mathbb{R}^{n\times n}$ such that $x^T(D_x Ax) > 0$. So, $D_x Ax = (d_1 \sum_{j=1}^{n} a_{nj}x_j, \cdots d_n \sum_{j=1}^{n} a_{nj}x_j)^T$. Since $x^T(D_x Ax) > 0$ and $d_i > 0$, $x_i(Ax)_i > 0$ for each i. Hence A is a P_D-matrix. □

For $ind(A) = 1$, the above theorem yields a characterization of a $P_\#$-matrix. With this, we proceed to present the definition of a sign-change matrix.

Definition 3.10. *A diagonal matrix S is called a sign-change matrix if the diagonals of S are 1 or -1.*

A relationship between a P-matrix and a block P_D-matrix is shown next.

Theorem 3.11. *Let* $A = \begin{pmatrix} L & O \\ O & O \end{pmatrix} \in \mathbb{R}^{m\times m}$ *be a partition matrix such that* $L \in \mathbb{R}^{n\times n}$ *with* $m \geq n$.

(a) If L *is a* P*-matrix, then* A *is a* P_D*-matrix.*

(b) A *is a* P_D*-matrix and* L *is invertible, then* L *is a* P*-matrix. In this case,* A^T *and* SAS *also are* P_D*-matrices, where* S *is a sign-change matrix.*

Proof. (a): Let $0 \neq x = (x_1, x_2, \cdots x_m)^T \in R(A^k)$. Define $u = (x_1, x_2, \cdots x_m)^T$. Then $u \in R(A^k)$, L is a P-matrix. Hence there exists at least one $i \in \{1, 2, \cdots n\}$ such that $u_i(Lu)_i > 0$. $x_i(Ax)_i = u_i(Lu)_i$ for each $1 \leq i \leq n$, then it follows that A is a P_D-matrix.

(b) Let $0 \neq x = (x_1, x_2, \cdots x_n)^T$. Define $v = (x_1, x_2, \cdots x_n, 0, 0, \cdots 0)^T \in R(A^k)$. Hence there exists at least one $i \in \{1, 2, \cdots n\}$ such that $v_i(Av)_i > 0$, since for $n+1 \leq i \leq m$, $x_i = 0$. As $v_i(Av)_i = x_i(Lx)_i$ for each $1 \leq i \leq n$, it then follows that L is a P-matrix. Also L^T and SLS are P-matrices. □

Then the above theorem produces the following corollary.

Corollary 3.12. *Let* $A = \begin{pmatrix} L & O \\ O & O \end{pmatrix} \in \mathbb{R}^{m\times m}$ *be a partition matrix such that* $L \in \mathbb{R}^{n\times n}$ *with* $m \geq n$.

(a) If L *is a* P*-matrix, then* A *is a* $P_\#$*-matrix.*

(b) A *is a* $P_\#$*-matrix and* L *is invertible, then* L *is a* P*-matrix. In this case,* A^T *and* SAS *also are* $P_\#$*-matrices, where* S *is a sign-change matrix.*

Let us recall the definition of a Z-matrix and an M-matrix. A square matrix whose off-diagonal elements are non-positive is called a Z-matrix. It follows that a Z-matrix A can be written as $A = sI - B$, where $B \geq 0$, $s \geq \rho(B)$. A Z-matrix A is called a M-matrix if $s \geq \rho(B)$. A Z-matrix A is called a nonsingular M-matrix if A is monotone. It is well known that if A is a Z-matrix then A is a P-matrix if and only if A is an invertible M-matrix. The matrix in Example 3.2 is a Z-matrix and is also a P_D-matrix.

However, it is not always true that $A^D \geq 0$. In order to study this property we have to apply the well-known result stated in [6] and is recalled below.

Theorem 3.13. *(Theorem 3.9,* [6]*)*
Let A be a Z-matrix having all principal minors are nonnegative. Then $A^\dagger \geq 0$ if and only if there exists a permutation matrix S such that $SAS^T = \begin{pmatrix} L & O \\ O & O \end{pmatrix}$ where L is an invertible M-matrix.

In Theorem 3.13, if L is an invertible M-matrix, then L is an P-matrix. Next theorem says about a relation between a Z-matrix and a P_D-matrix.

Theorem 3.14. *Let $A \in \mathbb{R}^{n\times n}$ be a Z-matrix having all principal minors are nonnegative and $A^D \geq 0$. Then there exists a permutation matrix S such that SAS^T is a P_D-matrix.*

Proof. Let $B = SAS^T = \begin{pmatrix} L & O \\ O & O \end{pmatrix}$ where $L \in \mathbb{R}^{n\times n}$ is an invertible M-matrix, i.e., P-matrix. We will show that B is a P_D-matrix. Let $0 \neq x = (x_1, x_2, \cdots x_m)^T \in R(B^k)$. Taking $v = (v_1, v_2, \cdots v_n)^T$. Since every P-matrix is a P_D-matrix. So $v \in R(L^k)$. Hence there exists at least one $i \in \{1, 2, \cdots n\}$ such that $v_i(Lv)_i > 0$ and $x_i(Bx)_i = v_i(Lv)_i$ for each $1 \leq i \leq n$ which follows that B is a P_D-matrix. $\square$

The corollary of the above theorem comes when we take $x \in R(A)$.

Corollary 3.15. *Let $A \in \mathbb{R}^{n\times n}$ be a Z-matrix having all principal minors are nonnegative, $A^\#$ exists and $A^\# \geq 0$. Then there exists a permutation matrix S such that SAS^T is a $P_\#$-matrix.*

Next theorem relates Drazin monotonicity and P_D-matrices.

Theorem 3.16. *Let $A \in \mathbb{R}^{n\times n}$ be any matrix. Then the following statements are equivalent.*
(i) A is a P_D-matrix.
(ii) A^D is a P_D-matrix and $A^D \geq 0$.

Proof. $(i) \Rightarrow (ii)$: Suppose that A is a P_D-matrix. Then by Theorem 3.5 A^D is a P_D-matrix. Next, to show $A^D \geq 0$. Let $u \in R(A^k)$, $u > 0$ and $y = A^D u$. Then $y \in R(A^k)$ and $Ay = AA^D u = u > 0$. Hence $Ay > 0$. Since A is a P_D-matrix, so for $0 \neq y \in R(A^k)$ there is an $i \in \{1, 2, \cdots n\}$ such that $y_i(Ay)_i > 0$. So $y_i > 0$ where y_i is the ith component of y, as $Ay > 0$. Therefore $A^D \geq 0$.

$(ii) \Rightarrow (i)$: The proof is same as in Theorem 3.5. □

We then have the follwing corollary for $P_\#$-matrices.

Corollary 3.17. *Let $A \in \mathbb{R}^{n\times n}$ be any matrix. Then the following statements are equivalent.*
(i) A is a $P_\#$-matrix.
(ii) $A^\#$ is a $P_\#$-matrix, $A^\#$ exists and $A^\# \geq 0$.

4. A Connection with Linear Complementarity Problems

It is well-known that a P-matrix A is characterized by the condition that the standard linear complementarity problem $LCP(q, A)$ has a unique solution for all $q \in \mathbb{R}^n$ in [4]. A relation between them is shown next.

Theorem 4.1. *$LCP(q, A)$ has unique solution for each $q \in \mathbb{R}^n$ if and only if A is a P matrix.*

Motivated by the work of Kannan and Sivakumar [9], we are now going to prove the existence of solution of LCP with the help of P_D-matrices.

Theorem 4.2. *$LCP(q, A)$ has unique solution for each $q \in \mathbb{R}^n$ if and only if A is a P_D-matrix.*

Proof. The proof of this theorem is similar to the proof for P-matrix (see page: 274-275, [2]). □

The following result is well-known in the theory of linear complementarity problems.

Theorem 4.3. *(Theorem 3.4.4,* [4]*)*
Let $A \in \mathbb{R}^{n\times n}$. Then the following statements are equivalent.
(a) For all $q \in S(A)$, if $x^1,\ x^2 \in SOL(q, A)$, then $Ax^1 = Ax^2$.
(b) Every vector whose sign is reversed by A must belong to $N(A^k)$, i.e., if $x_i(Ax)_i \leq 0$ for all i, then $x \in N(A)$.

Using this result we show a sufficient condition for a matrix to be a P_D-matrix.

Theorem 4.4. *Let $A \in \mathbb{R}^{n\times n}$. Suppose that For all $q \in S(A)$, and for every $x^1,\ x^2 \in SOL(q, A)$, it follows that $Ax^1 = Ax^2$. Then A is a P_D-matrix.*

Proof. Let $x \in R(A^k)$ be such that $x_i(Ax)_i \leq 0$ for all i. Then by the condition (b) of the Theorem 4.3, it follows that $x \in N(A^k)$. Hence $x = 0$. So A is a P_D-matrix. □

We conclude this section with the remark that all the theorems mentioned in this section are also true for $P_\#$-matrices.

5. Characterization of P_D-matrices with $i(A, B)$

In this section, first we discuss the inclusion $r(A, B) \subseteq K$ with P_D-matrices. We also state a result of Johnson and Tsatsomero, [7] for P-matrices, and then extend it for index-range symmetric matrices.

Theorem 5.1. *(Theorem 3.3,* [7]*)*
Let $A, B \in \mathbb{R}^{n\times n}$ be such that A and B are invertible. Then $r(A, B) \subseteq K$ if and only if BA^{-1} is P-matrix.

The extension of the above result is proposed next.

Theorem 5.2. *Let $A, B \in \mathbb{R}^{n\times n}$ be such that $R(A^k) = R(B^k)$ and $N(A^k) = N(B^k)$ and A, B are commutative. Then $r(A, B) \subseteq K$ if and only if $BA^D(AB^D)$ is a P_D-matrix.*

Proof. Necessity: Let $r(A, B) \subseteq K$ and suppose BA^D is not a P_D-matrix. Then, there exists $0 \neq x \in R((BA^D)^k) = R(B^k(A^D)^k) \subseteq R(B^k) = R(A^k)$ such that $x_i(BA^Dx)_i \leq 0$

for all i. For $1 \leq i \leq n$, consider the function $f_i : [0,1] \to \mathbb{R}$ defined by $f_i(t) = tx_i + (1-t)(BA^D)x_i$. Then by intermediate value theorem, there exists $t_i \in [0,1]$ such that $tx_i + (1-t)(BA^D)x_i = 0$. Let $L = diag(t_1, t_2 \cdots t_n)$. Then, $Lx + (I-L)(BA^D)x = 0$. Since $x \in R(A^k)$, then $x = AA^Dx$ for some $x \in \mathbb{R}^n$. Hence, $0 = Lx + (I-L)(BA^D)x = LAA^Dx + (I-L)(BA^D)AA^Dx = (LAA^D + (I-L)(BA^DAA^D))x = (LAA^D + (I-L)BA^D)x = (LA + (I-L)B)A^Dx$. This implies $A^Dx = 0$ implies $x \in N(A^D) = N(A^k)$ (as $LA + (I-L)B \in r(A,B)$), a contradiction. So BA^D is a P_D-matrix.

Sufficiency: Let $t_i \in [0,1]$, $i = \{1, 2, \cdots n\}$, $L = diag(t_1, t_2 \cdots t_n)$ and $(LA + (I-L)B)x = 0$ for some $x \in R(A^k)$. Since $x \in R(A^D) = R(A^k)$, we have $x = A^Dy$ for some $y \in R(A^k)$. Therefore $(LA + (I-L)B)x = 0$ implies $(LA + (I-L)B)A^Dy = 0$ which again yields $LAA^Dy + (I-L)BA^Dy = 0$. If $y \in R(A^k)$, then $LAA^Dy + (I-L)BA^Dy = Ly + (I-L)BA^Dy = 0$. Also, $(BA^D)^D(BA^D)y = y$, since $R((BA^D)^k) \subseteq R(A^k)$. Thus, $y \in R((BA^D)^k)$. The fact $L \geq 0$ and $(I-L) \geq 0$, so y_i and $(BA^Dy)_i$ are opposite in signs for each i, i.e., $y_i(BA^Dy)_i \leq 0$. So BA^D is not a P_D-matrix, a contradiction. Hence $r(A,B) \subseteq K$. □

Corollary 5.3. *Let $A, B \in \mathbb{R}^{n\times n}$ be such that $R(A) = R(B)$ and $N(A) = N(B)$. Then $r(A,B) \subseteq K$ if and only if $A^\#(B^\#)$ exists and $BA^\#(AB^\#)$ is a $P_\#$-matrix.*

Theorem 5.4. *Let $A, B \in \mathbb{R}^{n\times n}$ be such that $R(A^k) = R(B^k)$ and $N(A^k) = N(B^k)$. Then $c(A,B) \subseteq K$ if and only if $B^DA(A^DB)$ is a P_D-matrix.*

The proof is similar to proof of Theorem 5.2 and this theorem carries a corollary, given next.

Corollary 5.5. *Let $A, B \in \mathbb{R}^{n\times n}$ be such that $R(A) = R(B)$ and $N(A) = N(B)$. Then $c(A,B) \subseteq K$ if and only if $A^\#(B^\#)$ exists and $B^\#A(A^\#B)$ is a $P_\#$-matrix.*

Combining Theorem 5.2 and 5.4, we have the following theorem.

Corollary 5.6. *Let $A, B \in \mathbb{R}^{n\times n}$ be such that $R(A^k) = R(B^k)$ and $N(A^k) = N(B^k)$. Then $r(A,B) \subseteq K$ and $c(A,B) \subseteq K$ if and only if BA^D, AB^D, A^DB and B^DA are P_D-matrices.*

Now, we produce the following result which is proved by the authors Rohn [12], and Johnson and Tsatsomeros [7] for the matrices whose interval hull contains no singular matrices.

Theorem 5.7. *Let $A, B \in \mathbb{R}^{n\times n}$ such that each matrix in $i(A,B)$ is invertible. Then BA^{-1}, $A^{-1}B$, $B^{-1}A$ and AB^{-1} are P-matrices.*

Now, we present the generalization of above theorem to singular case.

Theorem 5.8. *Let $A, B \in \mathbb{R}^{n\times n}$ be such that $R(A^k) = R(B^k)$ and $N(A^k) = N(B^k)$. Further, let $i(A,B) = K$. Then BA^D, AB^D, A^DB and B^DA are P_D-matrices.*

Proof. Suppose BA^D is not a P_D-matrix. Then, there exists $0 \neq x \in R((BA^D)^k$ such that $x_i(BA^Dx)_i \leq 0$ for all i. Let C_i denotes the *ith* row of $C \in \mathbb{R}^{n\times n}$ defined by $C_i = B_i + t_i(A_i - B_i)$, where A_i, B_i are the *ith* rows of A, B respectively. Let $t_i = 1$ if $x_i = 0$ and if $x_i \neq 0$, then t_i be an arbitrary root of the continuous function $\phi_i(t) = x_i(B + t(A - B))_iA^Dx$ in $[0,1]$; such a root exists, since $\phi(0) = x_i(BA^Dx)_i \leq 0$ and $\phi(1) = x_i(AA^Dx)_i = x_i^2 \geq 0$. So C_i is a convex combination of A_i and B_i for each $i = \{1, 2, \cdots n\}$, hence $C \in i(A,B)$. Now, we will show $C \in K$. Let $A^Dx \in N(C) \subseteq N(C^k)$. If $x_i = 0$, then $(CA^Dx)_i = C_iA^Dx = A_i(A^Dx) = (AA^Dx)_i = x_i = 0$, and if $x_i \neq 0$, then $(CA^Dx)_i = (C_iA^Dx) = \frac{\phi(t_i)}{x_i} = 0$. Hence, $A^Dx \in N(C) \subseteq N(C^k)$. If $A^Dx \in N(A) \subseteq N(A^k)$, then $x_i = 0$, a contradiction. So, $N(C^k) \neq N(A^k)$, again a contradiction. Hence BA^D is a P_D-matrix. $\square$

Corollary 5.9. *Let $A,\ B \in \mathbb{R}^{n\times n}$ be such that $R(A) = R(B)$ and $N(A) = N(B)$. Further, let $i(A,B) = K$. Then $BA^{\#}$, $AB^{\#}$, $A^{\#}B$ and $B^{\#}A$ are P_D-matrices provided $A^{\#}$ and $B^{\#}$ exists.*

6. A Characterization for the Inclusion $h(A, B) \subseteq K$

In this section, we present a theorem which creates a relation between the inclusion $h(A, B) \subseteq K$ and a constrained eigenvalue condition of the matrix $A^D B$. Then the result obtained by taking $J = [A, B]$ and $h(A, B) \subseteq K$ is discussed. Finally, a new result is proved which is based on the inclusion $h(A, B) \subseteq K$, and nonnegativity of Drazin inverse of certain element in $J = [A, B]$.

Theorem 6.1. *Let $A, B \in \mathbb{R}^{n\times n}$ be such that $R(A^k) = R(B^k)$ and $N(A^k) = N(B^k)$. Then the following conditions are equivalent:*
(a) $h(A, B) \subseteq K$.
(b) $A^D Bx = \lambda x$, $0 \neq \lambda \in \mathbb{R}$. Then $\lambda > 0$.

Proof. $(a) \Rightarrow (b)$: Suppose that (a) holds. Assume that $A^D Bx = \lambda x$ holds for some $\lambda < 0$. Then $Bx = P_{R(A^k),N(A^k)} Bx = AA^D Bx = \lambda Ax$. If $Bx = 0$, then $x = 0$. So $Bx \neq 0$ and $Ax \neq 0$. Set $u = \frac{-\lambda}{1-\lambda}$ and $C^k = (uA + (1 - u)B) \in h(A, B)$. Then, $u \in (0, 1)$ and $A^D C^k x = A^D (uA + (1 - u)B)x = \frac{-1}{1-\lambda}(-\lambda I + A^D B)x = 0$. So $C^k x \in N(A^D) = N(A^k)$ and $C^k x \in R(A^k)$. Therefore $C^k x = 0$. Thus $N(C^k) \nsubseteq N(A^k)$ and then $h(A, B) \nsubseteq K$, a contradiction. So $\lambda > 0$.

$(b) \Rightarrow (a)$: Suppose that (b) holds and assume that $h(A, B) \nsubseteq K$. Then, $(uA + (1 - u)B) \notin h(A, B)$ for some $u \in (0, 1)$. As $N(A^k) \subseteq N(uA + (1 - u)B)$. Suppose that $N(A^k) \neq N(uA + (1 - u)B)$. Then, $(uA + (1 - u)B)x = 0$ for some $x \notin N(A^k)$. Let $x = x^1 + x^2$, where $x^1 \in N(A^k)$ and $0 \neq x^2 \in R(A^k)$. Then, $(uA + (1 - u)B)x^2 = 0$. Pre-multiplying by A^D, we get $(uA^D A + (1 - u)A^D B)x^2 = 0$. By setting $\lambda = \frac{-u}{1-u} < 0$, it follows that $A^D Bx^2 = \lambda x^2$, a contradiction. □

For $k = 1$, the property index range-symmetric reduces to range-symmetric and we have the follwing corollary.

Corollary 6.2. *Let $A, B \in \mathbb{R}^{n\times n}$ be such that $R(A) = R(B)$ and $N(A) = N(B)$. Then The following conditions are equivalent:*

(a) $h(A,B) \subseteq K$.

(b) $A^{\#}Bx = \lambda x$, $0 \neq \lambda \in \mathbb{R}$. *Then* $\lambda > 0$.

An square interval matrix is defined as the set of matrices of the form $J = [A, B] = \{C : A \leq C \leq B\}$ for $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times n}$ and $A \leq B$. We shall often use the center matrix $J_C = \frac{1}{2}(B + A)$ and the radius matrix $\Delta = \frac{1}{2}(B - A)$. Thus, $A = J_C - \Delta$, $B = J_C + \Delta$ and $\Delta \geq 0$ which yields $J = [J_C - \Delta, J_C + \Delta]$.

An interval matrix $J = [A, B]$ where $A, B \in \mathbb{R}^{n\times n}$ is called *index range-kernel regular* if for all $C \in J$, $R(C^k) = R(A^k)$ and $N(C^k) = N(A^k)$. When $k = 1$, it coincides with range-kernel regular, i.e., $R(C) = R(A)$ and $N(C) = N(A)$.

Theorem 6.3. *Let* $J = [A, B]$. *If* $h(A, B) \subseteq K$, *then* $\rho(J_c^D\Delta) < 1$.

Proof. Since $J_c \in K$, then it follows that $R(A^k) = R(J_c)$ and $N(A^k) = N(J_c)$. Suppose on contrary, $\beta = \rho(J_c^D\Delta) \geq 1$. Then, there exists $0 \neq x \in \mathbb{R}^n$ such that $J_c^D\Delta = \beta x$. Then, $x \in R(J_c^D) = R(J_c^k) = R(A^k)$. Also, $P_{R(J_c^k)}, N(J_c^k)(J_c - A)x = J_cJ_c^D\Delta x = \beta J_c x$ so that $(J_c - A - \beta J_c)x = 0$. Dividing it by $\frac{-1}{\beta}$ and taking $\eta = 1 + \frac{-1}{\beta} \geq 0$, then we have $(\eta J_c + (1-\eta)A)x = 0$. Let $P = (\eta J_c + (1-\eta)A)$. Then, $P \in K$ and $x \in N(P^k) = N(A^k)$. As $x \in R(A^k)$, thus $x = 0$, a contradiction. □

Corollary 6.4. *Let* $J = [A, B]$. *If* $h(A, B) \subseteq K$, *then* $\rho(J_c^{\#}\Delta) < 1$.

We next present an analogous result to Theorem 3.5, [8] for square singular matrices using the Drazin inverse.

Theorem 6.5. *Let* J *be index range-kernel regular. Then the following are equivalent.*

(i) $C^D \geq 0$ *whenever* $C \in K$,

(ii) $B^D \geq 0$ *and* $A^D \geq 0$,

(iii) $B^D \geq 0$ *and* $\rho(B^D(B - A)) < 1$.

Proof. (i) ⇒(ii) Follows from definition of J.

(ii) ⇒(iii) $A = B - (B - A)$ is an index-proper splitting of A. Then $B^D(B - A) \geq 0$. So by Theorem 2.1, $\rho(B^D(B - A)) < 1$.

(iii) $\Rightarrow$ (i) Let $C = B(I - B^D(B - C))$. Now to show $(I - B^D(B - C))$ is invertible. Let $(I - B^D(B - C))x = 0$ then, $x = B^D(B - C))x \in R(B^D) = R(B^k)$. So $x = BB^Dx$ and hence $x = BB^Dx - B^DCx = x - B^DCx$. Therefore $B^DCx = 0$. Thus, $Cx \in N(B^D) = N(C^D)$ it implies that $x = CC^Dx = 0$ implies $x = 0$. Hence $(I - B^D(B - C))$ is invertible. As $C = B(I - B^D(B - C))$. Next to show, $C^D = (I - B^D(B - C))^{-1}B^D$. For this, let $X = B$, $Y = (I - B^D(B - C))$. Then, $(XY)^D = Y^{-1}X^D$ if and only if $YY^kX^k = X^DXYY^kX^k$. We have $R(YY^kX^k) = R(Y(XY)^k) = R((I-B^D(B-C))C^k) = R(C^k - B^D(B-C)C^k) \subseteq R(C^k) = R(B^k) \subseteq R(B) = R(X)$ (since $P_{L,M}A = A$ if and only if $R(A) \subseteq L$). Therefore $C^D = (I - B^D(B - C))^{-1}B^D = \sum_{k=0}^{\infty}(B^D(B - C))^kB^D \geq 0$ $\square$

Using the above result, the next result follows.

Theorem 6.6. *Let $J = [A, B]$ be index range-kernel regular. Then the following are equivalent.*
(a) $B^D \geq 0$ and $A^D \geq 0$.
(b) $h(A, B) \subseteq K$ and $C^D \geq 0$ for all $C \in h(A, B)$.

Proof. $(a) \Rightarrow (b)$: Let $C = \lambda A + (1 - \lambda)B$ for some $\lambda \in [0, 1]$. Then, $N(A^k) \subseteq N(C^k)$ and $R(A^k) \subseteq R(C^k)$. Also, we have $0 \leq B^D(B - C) \leq B^D(B - A)$ and hence $\rho(B^D(B - A)) < 1$. Thus, $\rho(B^D(B - C)) < 1$ and $(I - B^D(B - C))$ is invertible. Now, $(I - B^D(B - C)) = I - P_{R(B^k),N(B^k)} + B^DC = -P_{R(B^k),N(B^k)} + B^DC = E$. Then, $BE = BP_{R(B^k),N(B^k)} + BB^DC = BB^DC = C$. So, $B = CE^{-1}$ and hence $R(B^k) = R(C^k)$. By rank-nullity dimension theorem, it follows that $N(A^k) = N(C^k)$. So, $C \in K$. Then, by Theorem 6.3, we have $C^D \geq 0$.
$(a) \Rightarrow (b)$: Since $h(A, B) \subseteq K$ and $C^D \geq 0$ for all $C \in h(A, B)$, then $C = \lambda A + (1 - \lambda)B$ for some $\lambda \in [0, 1]$. Again as $C^D \geq 0$ for all $C \in h(A, B)$, it is obviously true that $B^D \geq 0$ and $A^D \geq 0$. $\square$

Corollary 6.7. *Let $J = [A, B]$ be range-kernel regular. Then the following are equivalent.*
(a) $B^\# \geq 0$ and $A^\# \geq 0$.
(b) $h(A, B) \subseteq K$ and $C^\# \geq 0$ for all $C \in h(A, B)$.

References

[1] Ben-Israel, A.; Greville, T. N. E., *Generalized Inverses. Theory and applications*, Springer-Verlag, New York, 2003.

[2] Berman, A.; Plemmons, R. J., *Nonnegative matrices in the mathematical sciences*, Academic Press, New York, 1979.

[3] Chen. G; Chen, X., *A new splitting for singular linear system and Drazin inverse*, J. East China Norm. Univ. Natur. sci. Ed., 3 (1996) 12-18.

[4] Cottle, R. W., Pang, J. S. and Stone, R. E., *The linear complementarity problem*, Academic, New York, 1992.

[5] Fiedler, M. and Ptak, V., *On matrices with non-positive off-diagonal elements and positive principal minors*, Czech. J. Math., 12 (1962), 382-400.

[6] Kuo, I-wen, *The Moore-Penrose inverses of singular M-matrices* Lin. Alg. Appl., 17 (1977), 1-14.

[7] Johnson, C. R. and Tsatsomeros, M. J., *Convex sets of nonsingular and P-matrices*, Lin. Mult. Alg., 38 (1995), 233-239.

[8] Kannan, R. and Sivakumar, K. C., *Moore-Penrose inverse positivity of interval matrices*, Lin. Alg. Appl., 436 (2012), 571-578.

[9] Kannan, R. and Sivakumar, K. C., *$P_\dagger$-matrices: A generalization of P-matrices*, Lin. Mult. Alg., 2013, doi:10.1080/03081087.2012.754438

[10] Murty, K. G., *On a characterization of P-matrices*, SIAM Journal of Applied Mathematics, 20(3) (1971), 378-384.

[11] Rohn, J., *A theorem on P-matrices*, Lin. Mult. Alg., 30 (1991), 209-211.

[12] Rohn, J., *Systems of linear interval equations*, Lin. Alg. Appl., 126 (1989), 39-78.

[13] Samelson, H., Thrall, R. M., Welsler, D., *A partison theorem for Euclidean n-space*, Proc. Amer. Math. Soc. 9 (1958), 805-807.

[14] Tamir, A., *Short Communication On a Characterization of P-Matrices*, Mathematical Programming, 4 (1973), 110-112.

[15] Varga, R. S., *Matrix Iterative Analysis*, Springer-Verlag, Berlin, 2000.

[16] Wei, Y., *Index splitting for the Drazin inverse and the singular linear system*, Appl. Math. Comput. 95 (1998), 115-124.