AMO – Advanced Modeling and Optimization, Volume 15, Number 2, 2013

Equilibrium Models for Multi-commodity Auction Market Problems[1]

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Abstract

We propose an extension of the auction market model for the case where the market deals with many commodities and participants have binding constraints. Besides, the model includes external economic agents. We show that the formulation is an extended primal-dual system of variational inequalities or a convex-concave saddle point problem, however, cost functions require different treatment. We suggest convergent iterative methods, which can be viewed as dynamic adjustment processes for such markets.

Key words: Equilibrium models, multi-commodity auction markets, primal-dual system; variational inequalities; saddle point problem; iterative processes.

1 Introduction

There exist rather a lot of equilibrium type models which play a central role in mathematical physics, economics, transportation and other sciences. Traditionally, the classical perfectly (Walrasian) and imperfectly (Cournot - Bertrand) competitive models are paid considerable attention in economics in comparison with auction market models whose investigation was usually based on game theory techniques which evaluate strategies of players for capturing a desired lot; see e.g. [Moulin, 1981], [Weber, 1985], [Milgrom, 2004] and references therein. However, the recent development of information and telecommunication technologies together with great changes in several economic sectors such as energy and electronic commerce yield new challenges in creation of adequate mathematical models and derivation of efficient control decisions; see e.g. [Ilic et al, 1998], [Zaccour, 1998], [Stańczak et al, 2006], [Courcoubetis and Weber, 2003]. Observe that rather complex behavior of separate markets (participants) and the presence of binding constraints may lead to very complicated mathematical problems such as global optimization problems with equilibrium constraints or mixed integer programming problems within the traditional approaches.

[1]This work was supported in part by the RFBR grant, project No. 13-01-00029.

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*AMO – Advanced Modeling and Optimization. ISSN: 1841-4311**

The auction market principles may represent one of possible ways in resolving these problems, but they should be incorporated within rather simple and clear models.

A new approach to modeling auction markets was proposed in [Konnov, 2006], [Konnov, 2007a], [Konnov, 2007b], where variational inequality models of separate auctions with price functions of participants were suggested, i.e. they allowed for rather complex behavior of traders and buyers. Note that the multi-commodity model from [Konnov, 2007b] involves separate capacity bounds for each participant and for each particular commodity.

In this paper, following the above approach, we suggest several auction based equilibrium models for more general settings. That is, we propose an extension of the auction market model for the case where the market deals with many commodities and participants have binding constraints. Besides, the model includes external economic agents. We formulate the model as an extended primal-dual system of variational inequalities or a convex-concave saddle point problem. At the same time, the previous treatment of the cost functions as price ones does not match the new environment of the system. In such a way, we give a different treatment of the same mathematical model. Within this approach, we suggest convergent iterative methods and give their description as dynamic adjustment processes for such markets.

2 An auction of a homogeneous commodity

We first describe a single auction market of a homogeneous commodity. Denote by I and J the index sets of traders and buyers at this auction. For each $i \in I$, the i-th trader chooses some offer value x_i in his/her capacity segment $[\alpha'_i, \alpha''_i]$ and has a price function $g_i(x_i)$. Similarly, for each $j \in J$, the j-th buyer chooses some bid value y_j in his/her capacity segment $[\beta'_j, \beta''_j]$ and has a price function $h_j(y_j)$. Therefore, the prices depend on offer/bid values in general. Denote by u the value of external excess demand for this market. That is, u reflects the excess demand of external economic agents who do not participate explicitly in this auction, but agree beforehand with its price. Then we can define the feasible set of offer/bid values

$$D = \left\{ (x,y) \;\middle|\; \begin{array}{c} \sum\limits_{i \in I} x_i - \sum\limits_{j \in J} y_j - u = 0; \\ x_i \in [\alpha'_i, \alpha''_i], i \in I, y_j \in [\beta'_j, \beta''_j], j \in J \end{array} \right\},$$

where $x = (x_i)_{i \in I}, y = (y_j)_{j \in J}$. The solution of the auction problem consists in finding a feasible volume vector $(\bar{x}, \bar{y}) \in D$ and a price $\bar{p}$ such that

$$g_i(\bar{x}_i) \begin{cases} \geq \bar{p} & \text{if} \quad \bar{x}_i = \alpha'_i, \\ = \bar{p} & \text{if} \quad \bar{x}_i \in (\alpha'_i, \alpha''_i), \qquad i \in I, \\ \leq \bar{p} & \text{if} \quad \bar{x}_i = \alpha''_i, \end{cases} \tag{1}$$

and

$$h_j(\bar{y}_j)\begin{cases}\le \bar{p} & \text{if} \quad \bar{y}_j = \beta'_j,\\ = \bar{p} & \text{if} \quad \bar{y}_j \in (\beta'_j, \beta''_j),\\ \ge \bar{p} & \text{if} \quad \bar{y}_j = \beta''_j,\end{cases} \qquad j \in J. \tag{2}$$

In this model, participants report their price functions and capacity bounds to an auction manager (regulator). The latter has to solve problem (1)–(2) and to report the auction clearing price, which yields also the offer/bid values. In this procedure, the impact of the regulator is restricted in information and tools, unlike the fully centralized economy scheme. At the same time, we can easily extend this model to the case when the price functions depend on the whole offer/bid volume vector, which corresponds to imperfect competition schemes; see [Konnov, 2006].

The auction procedure should be accomplished within a limited time period. Note that the participants may not know price functions of the others. In order to derive efficient solution methods, a suitable reformulation of the above auction market problem is necessary.

In [Konnov, 2006] (see also [Konnov, 2007a], [Konnov, 2007b]), the following basic relation between the auction market problem (1)–(2) and a variational inequality (VI for short) was established.

Proposition 2.1 *(a) If $(\bar{x}, \bar{y}, \bar{p})$ satisfies (1)–(2) and $(\bar{x}, \bar{y}) \in D$, then $(\bar{x}, \bar{y})$ solves VI: Find $(\bar{x}, \bar{y}) \in D$ such that*

$$\sum_{i \in I} g_i(\bar{x}_i)(x_i - \bar{x}_i) - \sum_{j \in J} h_j(\bar{y}_j)(y_j - \bar{y}_j) \ge 0 \quad \forall (x, y) \in D. \tag{3}$$

(b) If a pair $(\bar{x}, \bar{y})$ solves VI (3), then there exists $\bar{p}$ such that $(\bar{x}, \bar{y}, \bar{p})$ satisfies (1)–(2).

Moreover, it was also noticed that the set of possible auction prices $\bar{p}$ in (1)–(2) denoted by $P(u)$ coincides with the set of Lagrange multipliers corresponding to the balance constraint

$$\sum_{i \in I} x_i - \sum_{j \in J} y_j - u = 0.$$

It seems natural to utilize monotonicity properties of the functions g_i and $-h_j$. In fact, from (1) we conclude that if the i-th trader announces a price $g_i = g_i(x_i)$ and an offer x_i, then he/she agrees to sell any smaller volume with the same price, hence any smaller volume is not associated with a greater price, and the function g_i is monotone. Similarly, it follows from (2) that if the j-th buyer announces a price $h_j = h_j(y_j)$ and a bid y_j, then he/she agrees to purchase any smaller volume with the same price, hence any smaller volume is not associated with a smaller price, and the function $-h_j$ is monotone.

For this reason we can suppose that all the functions $g_i, i \in I$ and $-h_j, j \in J$ are continuous and monotone. Then we can define convex differentiable functions

$\mu_i : [\alpha'_i, \alpha''_i] \to \mathbb{R}, i \in I$ and concave differentiable functions $\eta_j : [\beta'_j, \beta''_j] \to \mathbb{R}, j \in J$ such that

$$\mu'_i(x_i) = g_i(x_i) \text{ and } \eta'_j(y_j) = h_j(y_j). \tag{4}$$

Therefore, VI (3) is replaced with the convex optimization problem:

$$\begin{array}{ll} \min \to & \sum\limits_{i \in I} \mu_i(x_i) - \sum\limits_{j \in J} \eta_j(y_j) \\ \text{subject to } (x, y) \in D. & \end{array} \tag{5}$$

Since the cost function in (5) is the difference between the sold and paid amounts within the market, which can be treated as the negative profit of the auction manager, problem (5) maximizes this profit subject to the balance and participants' capacity constraints.

Proposition 2.2 *If (4) holds, then under the assumptions made problems (3) and (5) are equivalent.*

It also follows that $P(u)$ is precisely the solution set of the dual optimization problem of (5) and we can utilize the usual perturbation analysis. Let us define the perturbation function $\varphi(u)$, which determines the optimal value in (5) dependent of the perturbation u. Then φ is a convex function, $u \mapsto P(u)$ is a maximal monotone mapping, and $P(u)$ is the subdifferential of φ at u. These properties can be used in creating more general auction market models.

Propositions 2.1 and 2.2 enable us to apply various iterative solution methods for finding a solution of the auction market problem; see [Konnov, 2007a], [Konnov, 2007b], [Konnov, 2007c]. For instance, we describe briefly iterative methods which have a natural interpretation as dynamic adjustment processes and ensure stability (convergence) under rather mild conditions.

In fact, the Frank-Wolfe or conditional gradient method represents sequential solution of the corresponding auction problems with fixed prices. At the k-th iteration, we first find vectors $(\tilde{x}^k, \tilde{y}^k)$ as solutions of the linear programming problem

$$\begin{array}{ll} \min \to & \left\{ \sum\limits_{i \in I} g_i(x_i^k) x_i - \sum\limits_{j \in J} h_j(y_j^k) y_j \right\} \\ \text{subject to } (x, y) \in D. & \end{array} \tag{6}$$

Then we find the next iterate (x^{k+1}, y^{k+1}) as follows:

$$\begin{array}{rcl} x^{k+1} & = & \theta_k \tilde{x}^k + (1 - \theta_k) x^k, \\ y^{k+1} & = & \theta_k \tilde{y}^k + (1 - \theta_k) y^k; \end{array}$$

where $\theta_k \in (0, 1)$ is the stepsize parameter.

Problem (6) can be solved in a finite number of iterations by a simple arrangement type procedure; see [Konnov, 2007c], but it requires additionally the boundedness of the set D. Note that this method has clear interpretation. Indeed, after solution of a current auction with temporarily fixed prices, the participants correct their offer/bid volumes. During this process, the participants do not use price functions of the others, but only their current volumes from the auction manager. Nevertheless, the above assumptions provide convergence; see [Dem'yanov and Rubinov, 1968].

The well-known projection method consists in generating the iteration sequence $\{(x^k, y^k)\}$ in conformity with the formula: Find $(x^{k+1}, y^{k+1}) \in D$ such that

$$\begin{array}{l} \sum\limits_{i \in I} (g_i(x_i^k) + \theta_k^{-1}(x_i^{k+1} - x_i^k))(x_i - x_i^{k+1}) \\ - \sum\limits_{j \in J} (h_j(y_j^k) - \theta_k^{-1}(y_j^{k+1} - y_j^k))(y_j - y_j^{k+1}) \geq 0 \\ \forall (x, y) \in D, \end{array}$$

where $\theta_k > 0$ is a stepsize parameter. The preference of this method is that it always well defined and convergent on an unbounded feasible set; see e.g. [Patriksson, 1999]. However, its implementation may require additional information on participants.

If we are interested in explicit finding the auction clearing price together with offer/bid values, we can apply one of the dual methods. In fact, we can solve the dual problem

$$\max_p \rightarrow \quad \psi(p), \tag{7}$$

where

$$\begin{array}{rl} \psi(p) = & \min\limits_{(x,y) \in X \times Y} \left\{ \left(\sum\limits_{i \in I} \mu_i(x_i) - \sum\limits_{j \in J} \eta_j(y_j) \right) \right. \\ & \left. -p \left(\sum\limits_{i \in I} x_i - \sum\limits_{j \in J} y_j - u \right) \right\}. \end{array} \tag{8}$$

instead of the primal optimization problem (5) with the help of a suitable one-dimensional search method, say, golden section. Given an approximation p_k, calculation of the value of $\psi(p_k)$ in (8) and its gradient decomposes into a set of one-dimensional problems:

$$\min_{x_i \in [\alpha_i', \alpha_i'']} \rightarrow \ (\mu_i(x_i) - p_k x_i), \ i \in I, \tag{9}$$

$$\max_{y_j \in [\beta_j', \beta_j'']} \rightarrow \ (\eta_j(y_j) - p_k y_j), \ j \in J. \tag{10}$$

If these problems have unique solutions x_i^k, $i \in I$, and y_j^k, $j \in J$, then

$$\psi'(p_k) = -\left(\sum_{i \in I} x_i^k - \sum_{j \in J} y_j^k - u \right),$$

and we can even set $p_{k+1} = p_k + \theta_k \psi'(p_k)$ with some $\theta_k > 0$, thus obtaining the Uzawa method; see [Arrow et al, 1958], Ch. 10. These methods also have rather natural interpretation. The auction manager corrects sequentially the current price p_k by using the balance relation, whereas the participants select their offer/bid volumes via independent solution of partial problems with this given price. Note that again the participants do not use price functions of the others.

Nevertheless, together with the model, the iterative processes are applicable to the more general case when the price functions depend on the whole offer/bid volume vector; see [Konnov, 2007a], [Konnov, 2007b], [Konnov, 2007c].

3 A separable multi-commodity auction market model

We now present a separable multi-commodity extension of the model described in Section 2, thus extending those in [Konnov, 2007b], [Konnov, 2007c]. In this intermediate model, the auction market subordinates external ones in the sense that the agents of those markets accept the price decisions of auction markets.

The model is an n-commodity market involving external economic agents (consumers and producers) whose joint behavior is described by the excess demand mapping $p \mapsto E(p)$, where $p = (p_1, \dots, p_n)^\top$ is a given price vector. For the sake of simplicity, it supposed to be single-valued. Denote again by I and J the index sets of inner traders and buyers at this auction. For each l-th commodity, each i-th trader chooses some offer value x_{il} in his/her capacity segment $[\alpha'_{il}, \alpha''_{il}]$ and has a price function $g_{il}(x^{(i)})$ where $x^{(i)} = (x_{i1}, \dots, x_{in})^\top$. Similarly, for each $j \in J$, the j-th buyer chooses some bid value y_{jl} in his/her capacity segment $[\beta'_{jl}, \beta''_{jl}]$ and has a price function $h_{jl}(y^{(j)})$ where $y^{(j)} = (y_{j1}, \dots, y_{jn})^\top$ for $j \in J$. That is, the prices may depend on bid/offer volumes of all the commodities for each participant.

For brevity, set $x_{(l)} = (x_{il})_{i\in I}$, $y_{(l)} = (y_{jl})_{j\in J}$,

$$X_{(l)} = \prod_{i\in I}[\alpha'_{il}, \alpha''_{il}], \quad \text{and} \quad Y_{(l)} = \prod_{j\in J}[\beta'_{jl}, \beta''_{jl}].$$

We say that vectors $(\bar{x}_{(l)}, \bar{y}_{(l)}) \in X_{(l)} \times Y_{(l)}$ for $l = 1, \dots, n$ and $\bar{p} \in P$ constitute the equilibrium if

$$g_{il}(\bar{x}^{(i)}) \begin{cases} \geq \bar{p}_l & \text{if} \quad \bar{x}_{il} = \alpha'_{il}, \\ = \bar{p}_l & \text{if} \quad \bar{x}_{il} \in (\alpha'_{il}, \alpha''_{il}), \\ \leq \bar{p}_l & \text{if} \quad \bar{x}_{il} = \alpha''_{il}, \end{cases} \quad \text{for } i \in I; \tag{11}$$

$$h_{jl}(\bar{y}^{(j)}) \begin{cases} \leq \bar{p}_l & \text{if} \quad \bar{y}_{jl} = \beta'_{jl}, \\ = \bar{p}_l & \text{if} \quad \bar{y}_{jl} \in (\beta'_{jl}, \beta''_{jl}), \\ \geq \bar{p}_l & \text{if} \quad \bar{y}_{jl} = \beta''_{jl}, \end{cases} \quad \text{for } j \in J; \tag{12}$$

$l = 1, \dots, n$; and

$$\sum_{l=1}^{n} \left[\sum_{i \in I} \bar{x}_{il} - \sum_{j \in J} \bar{y}_{jl} - E_l(\bar{p}) \right] (p_l - \bar{p}_l) \geq 0 \quad \forall p \in P, \tag{13}$$

where P denotes the set of feasible prices, which is supposed to be a non-empty and convex subset in $\mathbb{R}^n$. Obviously, (11) and (12) represent the auction price decisions whereas (13) is the usual market price equilibrium condition. In fact, if P is the non-negative orthant

$$\mathbb{R}^n_+ = \{ z \in \mathbb{R}^n \mid z_i \geq 0 \quad i = 1, \dots, n \},$$

it is equivalent to the complementarity conditions

$$\bar{p}_l \geq 0, \sum_{i \in I} \bar{x}_{il} - \sum_{j \in J} \bar{y}_{jl} - E_l(\bar{p}) \geq 0, \bar{p}_l \left[\sum_{i \in I} \bar{x}_{il} - \sum_{j \in J} \bar{y}_{jl} - E_l(\bar{p}) \right] = 0,$$

for $k = 1, \dots, n$, whereas $P = \mathbb{R}^n$ gives the balance condition:

$$\sum_{i \in I} \bar{x}_{il} - \sum_{j \in J} \bar{y}_{jl} - E_l(\bar{p}) = 0, \quad \text{for } k = 1, \dots, n.$$

However, conditions (11) and (12) are equivalent to the system of VIs

$$\sum_{i \in I} (g_{il}(\bar{x}^{(i)}) - \bar{p}_l)(x_{il} - \bar{x}_{il}) \geq 0 \quad \forall x_{il} \in [\alpha'_{il}, \alpha''_{il}], \ l = 1, \dots, n, \ i \in I; \tag{14}$$

$$\sum_{j \in J} (h_{jl}(\bar{y}^{(j)}) - \bar{p}_l)(y_{jl} - \bar{y}_{jl}) \leq 0 \quad \forall y_{jl} \in [\beta'_{jl}, \beta''_{jl}], \ l = 1, \dots, n, \ j \in J. \tag{15}$$

Clearly, they can be rewritten as follows:

$$\begin{aligned} &\sum_{i \in I} g_{il}(\bar{x}^{(i)})(x_{il} - \bar{x}_{il}) - \sum_{j \in J} h_{jl}(\bar{y}^{(j)})(y_{jl} - \bar{y}_{jl}) \\ &-\bar{p}_l \left[\left(\sum_{i \in I} x_{il} - \sum_{j \in J} y_{jl} \right) - \left(\sum_{i \in I} \bar{x}_{il} - \sum_{j=1}^{l_l} \bar{y}_{jl} \right) \right] \geq 0 \\ &\forall (x_{(l)}, y_{(l)}) \in X_{(l)} \times Y_{(l)} \end{aligned} \tag{16}$$

for $l = 1, \dots, n$. The above equilibrium problem (16), (13) can be regarded as an extended primal-dual system of VIs; see e.g. [Konnov, 2002], [Konnov, 2003], [Konnov, 2004], [Konnov, 2007a]. We can then deduce existence and uniqueness results for this model by using the theory of VIs; see [Konnov, 2007a]. For instance, we give the existence results for the case of compact feasible sets.

Proposition 3.1 *Suppose that the sets $X_{(l)}$ and $Y_{(l)}, l = 1, \dots, n$ are nonempty and bounded, the set P is nonempty, convex and compact, the functions g_{il} and h_{jl} are continuous on $X_{(l)} \times Y_{(l)}$ for all i, j, l, and the mapping E is continuous on P. Then problem (16), (13) has a solution.*

In the unbounded case, similar results are usually based upon suitable coercivity conditions. Besides, by using the results e.g. from [Konnov, 2002], [Konnov, 2003], [Konnov, 2004], [Konnov, 2007a], we can suggest a number of iterative methods for finding a solution of this system, which also can be treated as dynamic control processes in this system. Convergence of such methods requires certain monotonicity or/and integrability properties of the mappings $g, -h$, and $-E$.

4 A generalized multi-commodity auction market model

We now intend to present an extension of the previous model in the sense that each participant has joint capacity constraints, i.e. his / her feasible set is not a Cartesian product of segments and the price function depends on his / her whole offer/bid value in general. Markets with joint constraints arise often in telecommunication and energy sectors; see e.g. [Hobbs and Helman, 2004], [Iosifidis and Koutsopoulos, 2010], [Pang et al, 2010].

In order to describe the model, we first write analogues of equilibrium conditions (11)–(13). As above, we set $x^{(i)} = (x_{i1}, \dots, x_{in})^\top$ for $i \in I$ and $y^{(j)} = (y_{j1}, \dots, y_{jn})^\top$ for $j \in J$. Then we define the feasible sets of inner traders X_i for $i \in I$ and buyers Y_j for $j \in J$ which are supposed to be nonempty, convex and closed sets in $\mathbb{R}^n$. For each commodity offer vector $x^{(i)} \in X_i$, the i-th trader has a price vector function value $g^i(x^{(i)}) \in \mathbb{R}^n$. Similarly, for each each commodity bid vector $y^{(j)} \in Y_j$, the j-th buyer has a price vector function value $h^j(y^{(j)})$. As above, let P denote the set of feasible prices and $E(p)$ denote the excess demand of external economic agents at p. We suppose that P is a non-empty and convex subset in $\mathbb{R}^n$.

We say that vectors $\bar{x}^{(i)} \in X_i$ for $i \in I$ and $\bar{y}^{(j)} \in Y_j$ for $j \in J$ and $\bar{p} \in P$ constitute an equilibrium if

$$\langle g^i(\bar{x}^{(i)}) - \bar{p}, x^{(i)} - \bar{x}^{(i)} \rangle \geq 0 \quad \forall x^{(i)} \in X_i \quad \text{for } i \in I, \tag{17}$$

$$\langle h^j(\bar{y}^{(j)}) - \bar{p}, y^{(j)} - \bar{y}^{(j)} \rangle \leq 0 \quad \forall y^{(j)} \in Y_j \quad \text{for } j \in J, \tag{18}$$

(cf. (14)–(15)) and

$$\left\langle \sum_{i \in I} \bar{x}^{(i)} - \sum_{j \in J} \bar{y}^{(j)} - E(\bar{p}), p - \bar{p} \right\rangle \geq 0 \quad \forall p \in P. \tag{19}$$

It is easy to see that conditions (17) and (18) are equivalent to the system of VIs

$$\begin{array}{l}\sum_{i\in I}\langle g^i(\bar{x}^{(i)}), x^{(i)}-\bar{x}^{(i)}\rangle - \sum_{j\in J}\langle h^j(\bar{y}^{(j)}), y^{(j)}-\bar{y}^{(j)}\rangle \\ -\left\langle \bar{p}, \left(\sum_{i\in I} x^{(i)} - \sum_{j\in J} y^{(j)}\right) - \left(\sum_{i\in I}\bar{x}^{(i)} - \sum_{j\in J}\bar{y}^{(j)}\right)\right\rangle \geq 0 \\ \forall x^{(i)}\in X_i, i\in I,\ \forall y^{(j)}\in Y_j, j\in J.\end{array} \tag{20}$$

This equilibrium problem (20), (19) is also an extended primal-dual system of VIs.

We now try to give a suitable treatment of conditions (17)–(19). Clearly, (19) presents the usual equilibrium condition (cf. (13)). That is, the auction manager utilizes this condition to find the auction price vector. However, conditions (17)–(18) seem more complicated in comparison with (1)–(2) or (11)–(12). By definition, (17) means that the auction manager determines a normative auction price vector $\bar{p}\in P$ and the i-th trader chooses the corresponding offer vector $\bar{x}^{(i)}\in X_i$ in order to minimize his/her superfluous sold amount at $\bar{x}^{(i)}$ in comparison with any other offer vector $x^{(i)}\in X_i$, whereas the j-th buyer chooses the corresponding bid vector $\bar{y}^{(j)}\in Y_j$ in order to maximize his/her superfluous bought amount at $\bar{y}^{(j)}$ in comparison with any other bid vector $y^{(j)}\in Y_j$. Of course, this procedure is too cumbersome and not suitable for implementation within the limited time period, especially in the case of many independent participants. Besides, we observe that participants of many contemporary markets give very limited information about their opportunities in order to keep certain advantages over competition agents; see e.g. [Hobbs and Helman, 2004], [Iosifidis and Koutsopoulos, 2010] and references therein. For this reason, we should suggest some other market mechanism, which is still based on model (17)–(19) or (20), (19).

Observe that, given a price vector $\bar{p}\in P$, all the participants find their offer/bid vectors independently from problems (17)–(18), hence they may only give these offer/bid vectors $x^{(i)}$ and $y^{(j)}$ and not report their price functions at all. Knowing their answers, the auction manager finds the market price vector by using the same condition (19). In this model, we have to define the mappings g^i and h^j. For the i-th trader, g^i is now treated as marginal cost mapping, then (17) gives an optimality condition for the profit maximization of this trader over the set X_i. Similarly, for the j-th buyer, h^j is now treated as marginal purchasing income(utility) mapping, then (18) gives also an optimality condition for the profit maximization of this buyer over the set Y_j. As to the excess demand mapping $p\mapsto E(p)$, we suppose that it also describes behavior of external economic agents or/and economic agents (consumers and producers) whose behavior is different from the above ones, but accepted in the Walrasian equilibrium models; see [Nikaido, 1968], [Arrow and Hahn, 1971].

As above, we can replace conditions (17)–(19) with system (20), (19). We can for example deduce existence results for this model by using the theory of VIs; see [Konnov, 2007a].

Proposition 4.1 *Suppose that the sets X_i for $i \in I$ and Y_j for $j \in J$ are bounded, the set P is nonempty, convex and compact, the mappings g^i and h^j are continuous on X_i for $i \in I$ and Y_j for $j \in J$, respectively, and that the mapping E is continuous on P. Then problem (20), (19) has a solution.*

The unbounded case usually requires coercivity conditions; e.g. see [Konnov, 2007a]. Next, we can find a solution of system (20), (19) by using the iterative methods from [Konnov, 2002], [Konnov, 2003], [Konnov, 2004], [Konnov, 2007a]. However, to make this market model more clear, we consider a dual iterative method, which is based on additional monotonicity and integrability assumptions.

5 Reformulation and iterative solution processes

Let us suppose that *there exist convex differentiable functions $\mu_i : X_i \to \mathbb{R}, i \in I$ and concave differentiable functions $\eta_j : Y_j \to \mathbb{R}, j \in J$ such that*

$$\mu_i'(x^{(i)}) = g^i(x^{(i)}) \text{ and } \eta_j'(y^{(j)}) = h_j(y^{(j)});$$

cf. (4). Next, we suppose that there exist a concave differentiable function $\tau : P \to \mathbb{R}$ such that $\tau'(p) = E(p)$. Observe that the integrability and monotonicity of the negative excess demand holds true for several known consumer models such as fixed budget ones; see [Polterovich, 1990]. Then VI's (17) and (18) are replaced by the following partial optimization problems:

$$\text{Find } \bar{x}^{(i)} = \arg \min_{x^{(i)} \in X_i} \left\{ \mu_i(x^{(i)}) - \langle \bar{p}, x^{(i)} \rangle \right\}, \ i \in I, \quad (21)$$

$$\text{Find } \bar{y}^{(j)} = \arg \max_{y^{(j)} \in Y_j} \left\{ \eta_j(y^{(j)}) - \langle \bar{p}, y^{(j)} \rangle \right\}, \ j \in J. \quad (22)$$

Also, VI (19) is replaced by the following optimization problem:

$$\text{Find } \bar{p} = \arg \max_{p \in P} \left\{ \tau(p) - \left\langle \sum_{i \in I} \bar{x}^{(i)} - \sum_{j \in J} \bar{y}^{(j)}, p \right\rangle \right\}. \quad (23)$$

It is clear that system (21)–(23) is equivalent to the saddle point problem: Find a triplet $(\bar{x}, \bar{y}, \bar{p}) \in X \times Y \times P$ such that

$$M(\bar{x}, \bar{y}, p) \le M(\bar{x}, \bar{y}, \bar{p}) \le M(x, y, \bar{p}) \quad \forall x \in X, \forall y \in Y, \forall p \in P; \quad (24)$$

where $x = (x^{(i)})_{i \in I}$, $y = (y^{(j)})_{j \in J}$,

$$M(x, y, p) = \sum_{i \in I} \mu_i(x^{(i)}) - \sum_{j \in J} \eta_j(y^{(j)}) + \tau(p) - \left\langle p, \sum_{i \in I} x^{(i)} - \sum_{j \in J} y^{(j)} \right\rangle,$$

and

$$X = \prod_{i \in I} X_i, \ Y = \prod_{j \in J} X_j.$$

Observe that the function $M(x, y, p)$ is differentiable, convex in (x, y) and concave in p. This means that we can utilize a number of saddle point methods to solve system (20), (19).

For instance, we now describe an extension of the Uzawa method from Section 2. First we write the extended dual optimization problem:

$$\max_{p \in P} \rightarrow \quad \{\psi(p) + \tau(p)\}, \tag{25}$$

where

$$\psi(p) = \min_{(x,y) \in X \times Y} \left\{ \sum_{i \in I} \mu_i(x^{(i)}) - \sum_{j \in J} \eta_j(y^{(j)}) - \left\langle p, \sum_{i \in I} x^{(i)} - \sum_{j \in J} y^{(j)} \right\rangle \right\},$$

cf. (7)–(8), which is solved by the following gradient projection method.

Dual method. Choose an initial price vector $p^0 \in P$. At the k-th iteration, $k = 0, 1, \ldots$, the regulator announces a price vector $p^k \in P$. For each $i \in I$, the i-th trader finds the offer vector $x^{k,(i)}$ by solving the problem

$$\min_{x^{(i)} \in X_i} \rightarrow \quad \left\{ \mu_i(x^{(i)}) - \langle p^k, x^{(i)} \rangle \right\}, \tag{26}$$

cf. (9), and for each $j \in J$, the j-th buyer finds the bid vector $y^{k,(j)}$ by solving the problem

$$\max_{y^{(j)} \in Y_j} \rightarrow \quad \left\{ \eta_j(y^{(j)}) - \langle p^k, y^{(j)} \rangle \right\}, \tag{27}$$

cf. (10). Afterwards the regulator calculates the dis-balance vector

$$F(p^k) = E(p^k) - \sum_{i \in I} x^{k,(i)} + \sum_{j \in J} y^{k,(j)}$$

and corrects the price vector by the formula

$$p^{k+1} = \pi_P[p^k + \theta_k F(p^k)], \theta_k > 0, \tag{28}$$

where $\pi_P[\cdot]$ denotes the projection mapping onto P.

Observe that in this process all the participants find their offer/bid vectors in (26)–(27) independently from each other since they maximize their current profit functions without reporting their cost/income functions, respectively. Also, the dual function ψ is concave, but non-differentiable in general, which requires some special step-size control rules. We now give a convergence result by using e.g. Lemma 2.1 in [Gol'shtein and Tret'yakov, 1989], Ch. 2.

Proposition 5.1 *Suppose that the sets X_i for $i \in I$ and Y_j for $j \in J$ are bounded, the set P is nonempty, convex and compact, the mappings g^i and h^j are continuous on X_i for $i \in I$ and Y_j for $j \in J$, respectively, and that the mapping E is continuous on P. If a sequence $\{p^k\}$ is constructed by the projection method, where*

$$\sum_{k=0}^{\infty} \theta_k = \infty, \ \sum_{k=0}^{\infty} \theta_k^2 < \infty, \tag{29}$$

then it converges to a solution of problem (25).

Remark 5.1 *We observe that the regulator can provide convergence by using rule (28), but is not able to identify the problem under solution since the functions μ_i and η_j are unknown to him/her. Hence we can thus indicate the difficulty in centralized planning schemes when real preferences of economic agents are unknown to the planning center. At the same time, we can not assert that a perfect competition market can provide rule (28) without any regulation, i.e. the price sequence then need not converge just to a market equilibrium point in general; see [Nikaido, 1968].*

This process admits various extensions and modifications adjusted to the basic assumptions. In fact, if the functions μ_i and $-\eta_j$ are strictly convex, then ψ becomes differentiable and the regulator can apply more efficient stepsize rules instead of (29). Next, the procedure can be easily extended to the case where the mappings g^i, $-h_j$ and $-E$ are not integrable, but possess strengthened monotonicity properties; see e.g. [Konnov, 2002], [Konnov, 2007a], besides, all the auxiliary problems can be solved approximately; see e.g. [Konnov, 2005]. On the other hand, if the mappings g^i, $-h_j$ and $-E$ possess only usual monotonicity properties, we can apply the two-level combined proximal point and dual method; see e.g. [Konnov, 2003]. For instance, under the assumptions of this section, we then consider a sequence of perturbed saddle point problems of form (24), where however the bi-function $M(x, y, p)$ is replaced by its regularization

$$M_s(x, y, p) = M(x, y, p) + \frac{\alpha}{2} \left(\|x - x^{s-1}\|^2 + \|y - y^{s-1}\|^2 - \|p - p^{s-1}\|^2 \right), \ \alpha > 0,$$

and $(x^{s-1}, y^{s-1}, p^{s-1})$ is the previous iterate. Each perturbed saddle point problem possesses strengthened convexity-concavity properties and is solved approximately within some tolerances by the proper modification of the above dual method.

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