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POSITION VECTORS OF HELICES IN THE UNIVERSAL COVERING GROUP $\widetilde{\mathbb{E}(2)}$ WITH RIEMANNIAN METRIC

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Abstract. In this paper, we study position vectors helices in the universal covering group of $E(2)$ with Riemannian metric. We characterize helices in terms of its curvature and torsion in the universal covering group of $E(2)$.

1. Introduction

Helices arise in nanosprings, carbon nanotubes, α-helices, DNA double and collagen triple helix, the double helix shape is commonly associated with DNA, since the double helix is structure of DNA. They constructed a molecular model of DNA in which there were two complementary, antiparallel (side-by-side in opposite directions) strands of the bases guanine, adenine, thymine and cytosine, covalently linked through phosphodiester bonds. Each strand forms a helix and two helices are held together through hydrogen bonds, ionic forces, hydrophobic interactions and van der Waals forces forming a double helix, lipid bilayers, bacterial flagella in Salmonella and E. coli, aerial hyphae in actynomycetes, bacterial shape in spirochetes, horns, tendrils, vines, screws, springs, helical staircases and sea shells, [3,15].

In this paper, we study position vectors helices in the universal covering group of $E(2)$ with Riemannian metric. We characterize helices in terms of its curvature and torsion in the universal covering group of $E(2)$.

2. The Universal Covering Group of $E(2)$

The Euclidean motion group $E(2)$is given explictly by the following matrix group:

$$E(2)=\left\{\begin{pmatrix}\cos\theta & -\sin\theta & x\\ \sin\theta & \cos\theta & y\\ 0 & 0 & 1\end{pmatrix} : x,y\in\mathbb{R},\theta\in\mathbb{S}^1\right\}.$$

Let $\widetilde{E(2)}$ denote the universal covering group of $E(2)$. Then, $\widetilde{E(2)}$ is $\mathbb{R}^3$ with multiplication

$$(x,y,z)\circ(x',y',z')=(x+x'\cos z-y'\sin z, y+x'\sin z+y'\cos z, z+z').$$

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A left-invariant frame

$$\mathbf{e}_1 = -\sin z\frac{\partial}{\partial x} + \cos z\frac{\partial}{\partial y},\ \mathbf{e}_2 = \frac{\partial}{\partial z},\ \mathbf{e}_3 = \cos z\frac{\partial}{\partial x} + \sin z\frac{\partial}{\partial y}. \tag{2.1}$$

Then this frame satisfies the following commutation relations [4]:

$$[\mathbf{e}_1, \mathbf{e}_2] = \mathbf{e}_3,\ \ [\mathbf{e}_2, \mathbf{e}_3] = \mathbf{e}_1,\ \ [\mathbf{e}_3, \mathbf{e}_1] = 0.$$

The left-invariant Riemannian metric determined by the condition that $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is orthonormal, is given by

$$g = (\cos z dx + \sin z dy)^2 + (-\sin z dx + \cos z dy)^2 + dz^2.$$

The Levi Civita connection is given by

$$\begin{aligned} &\nabla_{\mathbf{e}_1}\mathbf{e}_1 = 0, && \nabla_{\mathbf{e}_1}\mathbf{e}_2 = 0, && \nabla_{\mathbf{e}_1}\mathbf{e}_3 = 0,\\ &\nabla_{\mathbf{e}_2}\mathbf{e}_1 = -\mathbf{e}_3, && \nabla_{\mathbf{e}_2}\mathbf{e}_2 = 0, && \nabla_{\mathbf{e}_2}\mathbf{e}_3 = \mathbf{e}_1,\\ &\nabla_{\mathbf{e}_3}\mathbf{e}_1 = 0, && \nabla_{\mathbf{e}_3}\mathbf{e}_2 = 0, && \nabla_{\mathbf{e}_3}\mathbf{e}_3 = 0, \end{aligned}$$

The curvature of the space is determined by

$$R_{1212} = R_{1313} = R_{2323} = 0.$$

3. Helices in Universal Covering Group of $E(2)$

Let $\gamma : I \longrightarrow \widetilde{E(2)}$ be a non geodesic curve in the group of rigid motions $\widetilde{\mathbb{E}(2)}$ parametrized by arc length. Let $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ be the Frenet frame fields tangent to the group of rigid motions $\widetilde{\mathbb{E}(2)}$ along γ defined as follows:

$\mathbf{T}$ is the unit vector field γ' tangent to γ, $\mathbf{N}$ is the unit vector field in the direction of $\nabla_{\mathbf{T}}\mathbf{T}$ (normal to γ) and $\mathbf{B}$ is chosen so that $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ is a positively oriented orthonormal basis. Then, we have the following Frenet formulas:

$$\begin{aligned} \nabla_{\mathbf{T}}\mathbf{T} &= \kappa\mathbf{N},\\ \nabla_{\mathbf{T}}\mathbf{N} &= -\kappa\mathbf{T} + \tau\mathbf{B},\\ \nabla_{\mathbf{T}}\mathbf{B} &= -\tau\mathbf{N}, \end{aligned} \tag{3.1}$$

where κ is the curvature of γ, τ is its torsion and

$$\begin{aligned} g(\mathbf{T}, \mathbf{T}) &= g(\mathbf{N}, \mathbf{N}) = g(\mathbf{B}, \mathbf{B}) = 1,\\ g(\mathbf{T}, \mathbf{N}) &= g(\mathbf{T}, \mathbf{B}) = g(\mathbf{N}, \mathbf{B}) = 0. \end{aligned} \tag{3.2}$$

With respect to the orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ we can write

$$\begin{aligned} \mathbf{T} &= T_1\mathbf{e}_1 + T_2\mathbf{e}_2 + T_3\mathbf{e}_3,\\ \mathbf{N} &= N_1\mathbf{e}_1 + N_2\mathbf{e}_2 + N_3\mathbf{e}_3,\\ \mathbf{B} &= B_1\mathbf{e}_1 + B_2\mathbf{e}_2 + B_3\mathbf{e}_3. \end{aligned} \tag{3.3}$$

Theorem 3.1. *Let* $\gamma : I \longrightarrow \widetilde{E(2)}$ *be a helix in the universal covering group of* $E(2)$. *Then, the parametric equations of* γ *are*

$$
\begin{aligned}
x(s) &= -\frac{1}{\kappa}\sin^2\beta\cos[(\frac{\kappa}{\sin\beta})s + C - \vartheta] + \varepsilon_1, \\
y(s) &= \frac{1}{\kappa}\sin^2\beta\sin[(\frac{\kappa}{\sin\beta})s + C - \vartheta] + \varepsilon_2, \\
z(s) &= \cos\beta s + \vartheta,
\end{aligned} \tag{3.4}
$$

where $\varepsilon_1, \varepsilon_2, \vartheta, C$ *are constants of integration.*

Proof. Assume that γ is a helix in $\widetilde{E(2)}$. Then,

$$\mathbf{T} = \sin\beta\cos\varpi(s)\,\mathbf{e}_1 + \cos\beta\mathbf{e}_2 + \sin\beta\sin\varpi(s)\,\mathbf{e}_3. \tag{3.5}$$

From covariant derivative of $\mathbf{T}$, we have

$$\nabla_{\mathbf{T}}\mathbf{T} = (T_1' + T_2T_3)\,\mathbf{e}_1 + T_2'\mathbf{e}_2 + (T_3' - T_1T_2)\,\mathbf{e}_3.$$

Applying the Frenet formulas of γ, we get

$$\varpi(s) = (\frac{\kappa}{\sin\beta} + \cos\beta)s + C,$$

where C constant of integration.

The last equation gives us

$$
\begin{aligned}
\mathbf{T} &= \sin\beta\cos[(\frac{\kappa}{\sin\beta} + \cos\beta)s + C]\mathbf{e}_1 + \cos\beta\mathbf{e}_2 \\
&\quad + \sin\beta\sin[(\frac{\kappa}{\sin\beta} + \cos\beta)s + C]\mathbf{e}_3.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\mathbf{T} &= (-\sin z\sin\beta\cos[(\frac{\kappa}{\sin\beta} + \cos\beta)s + C] \\
&\quad + \cos z\sin\beta\sin[(\frac{\kappa}{\sin\beta} + \cos\beta)s + C], \\
&\quad \cos z\sin\beta\cos[(\frac{\kappa}{\sin\beta} + \cos\beta)s + C] \\
&\quad + \sin z\sin\beta\sin[(\frac{\kappa}{\sin\beta} + \cos\beta)s + C], \cos\beta).
\end{aligned}
$$

Then

$$
\begin{aligned}
\frac{dx}{ds} &= -\sin z\sin\beta\cos[(\frac{\kappa}{\sin\beta} + \cos\beta)s + C] \\
&\quad + \cos z\sin\beta\sin[(\frac{\kappa}{\sin\beta} + \cos\beta)s + C], \\
\frac{dy}{ds} &= \cos z\sin\beta\cos[(\frac{\kappa}{\sin\beta} + \cos\beta)s + C] \\
&\quad + \sin z\sin\beta\sin[(\frac{\kappa}{\sin\beta} + \cos\beta)s + C], \\
\frac{dz}{ds} &= \cos\beta.
\end{aligned}
$$

Integrating the last equation gives the result.

We draw a picture of this curve.

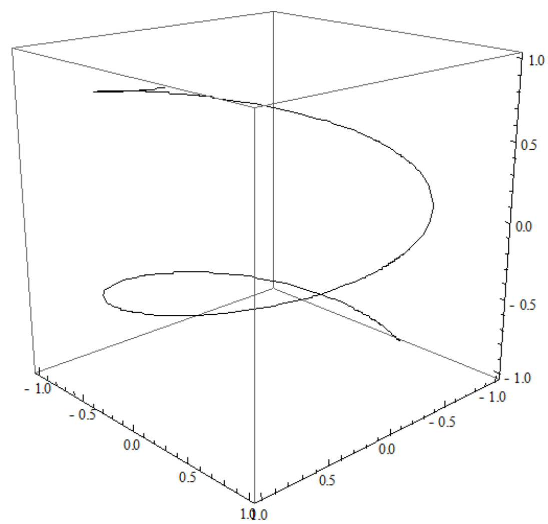

Figure 1

Figure 1: *A helix in* $\widetilde{E(2)}$

By this theorem we immediately have

Theorem 3.2. *Let* $\gamma : I \longrightarrow \widetilde{E(2)}$ *be a helix in the universal covering group of* $E(2)$*. Then, the position vector of* γ *is*

$$
\begin{aligned}
\gamma(s) = {} & [-\sin[\cos\beta s+\vartheta][-\frac{1}{\kappa}\sin^2\beta\cos[(\frac{\kappa}{\sin\beta})s+C-\vartheta]+\varepsilon_1] \\
& +\cos[\cos\beta s+\vartheta][\frac{1}{\kappa}\sin^2\beta\sin[(\frac{\kappa}{\sin\beta})s+C-\vartheta]+\varepsilon_2]]\mathbf{e}_1 \\
& +[\cos\beta s+\vartheta]\mathbf{e}_2 \\
& +[\cos[\cos\beta s+\vartheta][-\frac{1}{\kappa}\sin^2\beta\cos[(\frac{\kappa}{\sin\beta})s+C-\vartheta]+\varepsilon_1] \\
& +\sin[\cos\beta s+\vartheta][\frac{1}{\kappa}\sin^2\beta\sin[(\frac{\kappa}{\sin\beta})s+C-\vartheta]+\varepsilon_2]]\mathbf{e}_3,
\end{aligned} \tag{3.6}
$$

where $\varepsilon_1, \varepsilon_2, \vartheta, C$ *are constants of integration.*

Proof. By a direct computation, we have

$$
\begin{aligned}
\frac{\partial}{\partial x} &= -\sin z\mathbf{e}_1+\cos z\mathbf{e}_3, \\
\frac{\partial}{\partial y} &= \cos z\mathbf{e}_1+\sin z\mathbf{e}_3, \\
\frac{\partial}{\partial z} &= \mathbf{e}_2.
\end{aligned} \tag{3.7}
$$

Combining (2.1) and (3.4), we have (3.6). So, the proof is completed.

We can use Mathematica to draw the picture of projections of γ.

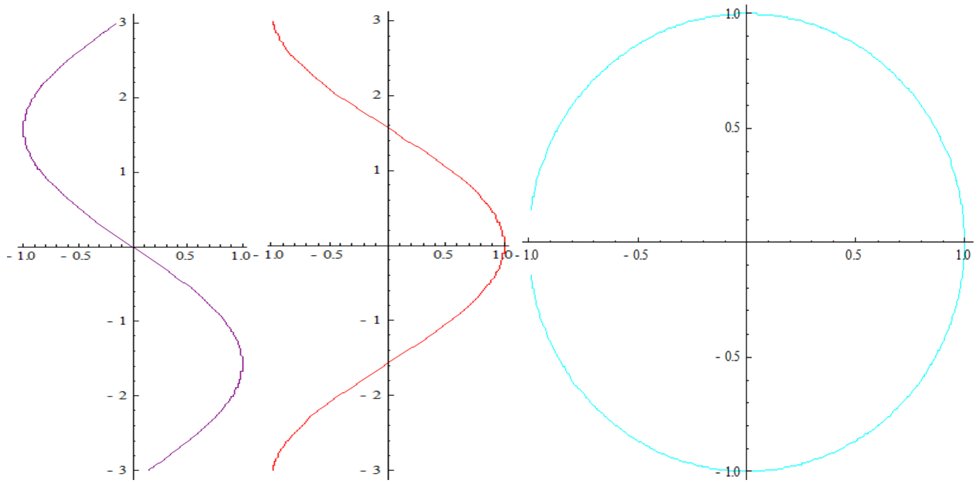

Figure 2

Figure 2: *Projections of γ to yz, xz, xy planes are illustrated colour purple, red, cyan, respectively.*

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