POSITION VECTORS OF HELICES IN THE UNIVERSAL COVERING GROUP $E(2)$ WITH RIEMANNIAN METRIC

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Abstract. In this paper, we study position vectors helices in the universal covering group of $E(2)$ with Riemannian metric. We characterize helices in terms of its curvature and torsion in the universal covering group of $E(2)$.

1. Introduction

Helices arise in nanosprings, carbon nanotubes, α-helices, DNA double and collagen triple helix, the double helix shape is commonly associated with DNA, since the double helix is structure of DNA. They constructed a molecular model of DNA in which there were two complementary, antiparallel (side-by-side in opposite directions) strands of the bases guanine, adenine, thymine and cytosine, covalently linked through phosphodiester bonds. Each strand forms a helix and two helices are held together through hydrogen bonds, ionic forces, hydrophobic interactions and van der Waals forces forming a double helix, lipid bilayers, bacterial flagella in Salmonella and E. coli, aerial myshae in actinomycetes, bacterial shape in spirilicetes, horns, tendrils, vines, screws, springs, helical staircases and sea shells, [3,15].

In this paper, we study position vectors helices in the universal covering group of $E(2)$ with Riemannian metric. We characterize helices in terms of its curvature and torsion in the universal covering group of $E(2)$.

2. The Universal Covering Group of $E(2)$

The Euclidean motion group $E(2)$ is given explicitly by the following matrix group:

$$E(2) = \left\{ \begin{pmatrix} \cos \theta & -\sin \theta & x \\ \sin \theta & \cos \theta & y \\ 0 & 0 & 1 \end{pmatrix} : x, y \in \mathbb{R}, \theta \in S^1 \right\}.$$

Let $\widetilde{E(2)}$ denote the universal covering group of $E(2)$. Then, $\widetilde{E(2)}$ is $\mathbb{R}^3$ with multiplication

$$(x, y, z) \circ (x', y', z') = (x + x' \cos z - y' \sin z, y + x' \sin z + y' \cos z, z + z').$$
A left-invariant frame

\begin{equation}
(2.1) \quad \mathbf{e}_1 = -\sin z \frac{\partial}{\partial x} + \cos z \frac{\partial}{\partial y}, \quad \mathbf{e}_2 = \frac{\partial}{\partial z}, \quad \mathbf{e}_3 = \cos z \frac{\partial}{\partial x} + \sin z \frac{\partial}{\partial y}.
\end{equation}

Then this frame satisfies the following commutation relations [4]:

\begin{equation}
[\mathbf{e}_1, \mathbf{e}_2] = \mathbf{e}_3, \quad [\mathbf{e}_2, \mathbf{e}_3] = \mathbf{e}_1, \quad [\mathbf{e}_3, \mathbf{e}_1] = 0.
\end{equation}

The left-invariant Riemannian metric determined by the condition that \(\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}\) is orthonormal, is given by

\begin{equation}
g = (\cos z dx + \sin z dy)^2 + (-\sin z dx + \cos z dy)^2 + dz^2.
\end{equation}

The Levi Civita connection is given by

\begin{equation}
\nabla_{\mathbf{e}_1} \mathbf{e}_1 = 0, \quad \nabla_{\mathbf{e}_2} \mathbf{e}_1 = 0, \quad \nabla_{\mathbf{e}_3} \mathbf{e}_1 = 0,
\end{equation}

\begin{equation}
\nabla_{\mathbf{e}_2} \mathbf{e}_2 = -\mathbf{e}_3, \quad \nabla_{\mathbf{e}_2} \mathbf{e}_2 = 0, \quad \nabla_{\mathbf{e}_2} \mathbf{e}_3 = \mathbf{e}_1,
\end{equation}

\begin{equation}
\nabla_{\mathbf{e}_3} \mathbf{e}_1 = 0, \quad \nabla_{\mathbf{e}_3} \mathbf{e}_2 = 0, \quad \nabla_{\mathbf{e}_3} \mathbf{e}_3 = 0.
\end{equation}

The curvature of the space is determined by

\begin{equation}
R_{1212} = R_{1313} = R_{2323} = 0.
\end{equation}

3. Helices in Universal Covering Group of \(E(2)\)

Let \(\gamma : I \longrightarrow \tilde{E}(2)\) be a non geodesic curve in the group of rigid motions \(\tilde{E}(2)\) parametrized by arc length. Let \(\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}\) be the Frenet frame fields tangent to the group of rigid motions \(E(2)\) along \(\gamma\) defined as follows:

\(\mathbf{T}\) is the unit vector field \(\gamma'\) tangent to \(\gamma\), \(\mathbf{N}\) is the unit vector field in the direction of \(\nabla_{\mathbf{T}} \mathbf{T}\) (normal to \(\gamma\)) and \(\mathbf{B}\) is chosen so that \(\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}\) is a positively oriented orthonormal basis. Then, we have the following Frenet formulas:

\begin{equation}
\nabla_{\mathbf{T}} \mathbf{T} = \kappa \mathbf{N},
\end{equation}

\begin{equation}
\nabla_{\mathbf{T}} \mathbf{N} = -\kappa \mathbf{T} + \tau \mathbf{B},
\end{equation}

\begin{equation}
\nabla_{\mathbf{T}} \mathbf{B} = -\tau \mathbf{N},
\end{equation}

where \(\kappa\) is the curvature of \(\gamma\), \(\tau\) is its torsion and

\begin{equation}
(3.2) \quad g(\mathbf{T}, \mathbf{T}) = g(\mathbf{N}, \mathbf{N}) = g(\mathbf{B}, \mathbf{B}) = 1,
\end{equation}

\begin{equation}
(3.2) \quad g(\mathbf{T}, \mathbf{N}) = g(\mathbf{T}, \mathbf{B}) = g(\mathbf{N}, \mathbf{B}) = 0.
\end{equation}

With respect to the orthonormal basis \(\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}\) we can write

\begin{equation}
(3.3) \quad \mathbf{T} = T_1 \mathbf{e}_1 + T_2 \mathbf{e}_2 + T_3 \mathbf{e}_3,
\end{equation}

\begin{equation}
(3.3) \quad \mathbf{N} = N_1 \mathbf{e}_1 + N_2 \mathbf{e}_2 + N_3 \mathbf{e}_3,
\end{equation}

\begin{equation}
(3.3) \quad \mathbf{B} = B_1 \mathbf{e}_1 + B_2 \mathbf{e}_2 + B_3 \mathbf{e}_3.
\end{equation}
Theorem 3.1. Let \( \gamma : I \rightarrow \widehat{E(2)} \) be a helix in the universal covering group of \( E(2) \). Then, the parametric equations of \( \gamma \) are

\[
\begin{align*}
  x(s) &= -\frac{1}{\kappa} \sin^2 \beta \cos[(\frac{\kappa}{\sin \beta})s + C \theta] + \varepsilon_1, \\
y(s) &= \frac{1}{\kappa} \sin^2 \beta \sin[(\frac{\kappa}{\sin \beta})s + C \theta] + \varepsilon_2, \\
z(s) &= \cos \beta s + \theta,
\end{align*}
\]

where \( \varepsilon_1, \varepsilon_2, \theta, C \) are constants of integration.

Proof. Assume that \( \gamma \) is a helix in \( \widehat{E(2)} \). Then,

\[
\mathbf{T} = \sin \beta \cos \varpi(s) \mathbf{e}_1 + \cos \beta \mathbf{e}_2 + \sin \beta \sin \varpi(s) \mathbf{e}_3.
\]

From covariant derivative of \( \mathbf{T} \), we have

\[
\nabla_T \mathbf{T} = (T_1^\prime + T_2 T_3) \mathbf{e}_1 + T_2^\prime \mathbf{e}_2 + (T_3^\prime - T_1 T_2) \mathbf{e}_3.
\]

Applying the Frenet formulas of \( \gamma \), we get

\[
\varpi(s) = \left( \frac{\kappa}{\sin \beta} + \cos \beta \right) s + C,
\]

where \( C \) constant of integration.

The last equation gives us

\[
\mathbf{T} = \sin \beta \cos[\left( \frac{\kappa}{\sin \beta} + \cos \beta \right)s + C] \mathbf{e}_1 + \cos \beta \mathbf{e}_2 \\
+ \sin \beta \sin[\left( \frac{\kappa}{\sin \beta} + \cos \beta \right)s + C] \mathbf{e}_3.
\]

It follows that

\[
\mathbf{T} = (- \sin z \sin \beta \cos[\left( \frac{\kappa}{\sin \beta} + \cos \beta \right)s + C] \\
+ \cos z \sin \beta \sin[\left( \frac{\kappa}{\sin \beta} + \cos \beta \right)s + C],
\]

\[
\cos z \sin \beta \cos[\left( \frac{\kappa}{\sin \beta} + \cos \beta \right)s + C] \\
+ \sin z \sin \beta \sin[\left( \frac{\kappa}{\sin \beta} + \cos \beta \right)s + C], \cos \beta).
\]

Then

\[
\frac{dx}{ds} = -\sin z \sin \beta \cos[\left( \frac{\kappa}{\sin \beta} + \cos \beta \right)s + C] \\
+ \cos z \sin \beta \sin[\left( \frac{\kappa}{\sin \beta} + \cos \beta \right)s + C],
\]

\[
\frac{dy}{ds} = \cos z \sin \beta \cos[\left( \frac{\kappa}{\sin \beta} + \cos \beta \right)s + C] \\
+ \sin z \sin \beta \sin[\left( \frac{\kappa}{\sin \beta} + \cos \beta \right)s + C],
\]

\[
\frac{dz}{ds} = \cos \beta.
\]

Integrating the last equation gives the result.

We draw a picture of this curve.
By this theorem we immediately have

**Theorem 3.2.** Let $\gamma : I \rightarrow \hat{E}(2)$ be a helix in the universal covering group of $E(2)$. Then, the position vector of $\gamma$ is

$$\gamma (s) = [-\sin[\cos{\beta s + \vartheta}][-\frac{1}{\kappa} \sin^{\beta} \beta \cos[(\frac{1}{\sin{\beta}})s + C - \vartheta] + \varepsilon_1]$$

$$+ \cos[\cos{\beta s + \vartheta}][\frac{1}{\kappa} \sin^{\beta} \beta \sin[(\frac{1}{\sin{\beta}})s + C - \vartheta] + \varepsilon_2)]e_1$$

$$+ [\cos{\beta s + \vartheta}]e_2$$

$$+ [\cos[\cos{\beta s + \vartheta}][\frac{1}{\kappa} \sin^{\beta} \beta \cos[(\frac{1}{\sin{\beta}})s + C - \vartheta] + \varepsilon_1]$$

$$+ \sin[\cos{\beta s + \vartheta}][\frac{1}{\kappa} \sin^{\beta} \beta \sin[(\frac{1}{\sin{\beta}})s + C - \vartheta] + \varepsilon_2)]e_3,$$

(3.6)

where $\varepsilon_1, \varepsilon_2, \vartheta, C$ are constants of integration.

**Proof.** By a direct computation, we have

$$\frac{\partial}{\partial x} = -\sin \vartheta e_1 + \cos \vartheta e_3,$$

$$\frac{\partial}{\partial y} = \cos \vartheta e_1 + \sin \vartheta e_3,$$

$$\frac{\partial}{\partial z} = e_2.$$

(3.7)

Combining (2.1) and (3.4), we have (3.6). So, the proof is completed.

We can use Mathematica to draw the picture of projections of $\gamma$. 
Figure 2: Projections of $\gamma$ to $yz$, $xz$, $xy$ planes are illustrated color purple, red, cyan, respectively.

REFERENCES


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