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Profit Maximization of Fishermen

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Abstract In this work we define a bio-economic equilibrium model for several fishermen who catch two fish species; these species compete with each other for space or food. The natural growth of each species is modeled using a logistic law. The objective of the work is to find the fishing effort that maximizes the profit of each fisherman constrained by the conservation of the biodiversity. The existence of the steady states and its stability are studied using eigenvalue analysis. The problem of determining the equilibrium point that maximizes the profit of each fisherman is then solved by using the generalized Nash equilibrium problem. Finally, some numerical simulations are given to illustrate the results.

Keywords *Fisheries; Bio-economic model; Maximizing profits; Generalized Nash Equilibrium GNE; Linear Complementarity Problem LCP; Biodiversity of renewable resources.*

1. Introduction

A bio-economic model of a fishery, as the name implies, combines two parts, the first one is a biological model and the second one is an economic model; it is intended to give an explanation stock, catch, and effort dynamics under different regimes, and provide guidance on the optimal management of the stock. This is accomplished by specifying the harvest function that is usually based on the value of the total revenue and the total cost, and constraints representing the sustainable management of the resources and the preservation of the biodiversity of the stocks. Therefore, the optimal level of effort is determined, on the one hand, by the biological dynamics of the stock and, on the other, by the cost structure of the fishery and the value of the harvest. It is interesting to note that this solution process is distinguished from financial analysis because it explicitly includes the opportunity costs of harvest, usually in the form of a flow time of the present value of net benefits representing a specific model of crops and stocks.

Bio-economic theory was pioneered by Gordon [12], and Schaefer [19] static model of a single species. In the present paper, we propose to define a bio-economic model of two fish species. The evolution of these fish species is described by a density dependent model taking into account the competition between species which compete with each other for space or food (see the model of Verhulst [20]). In this model, we assume that we have 'n' fishermen who catch two fish species.

More specifically, the bio-economic model includes three parts: a biological part that connects the catch to the biomass stock, an exploitation part that connects the catch to fishing effort at equilibrium, and an economic part that connects the fishing effort to profit.

The objective of each fisherman is to maximize his income without any consultation of the other fishermen. However, all of them have to respect two constraints, the first one is the sustainable management of the resources and the second one is the preservation of the biodiversity. With all these considerations, our problem leads to a generalized Nash equilibrium problem, to solve this problem we transform it into a linear complementarity problem.

The paper is organized as follows. In section 2 we define a mathematical model of two fish species that compete with each other for space or food. In section 3 we compute the Linear Complementarity Problem. In section 4 we give numerical simulations of the mathematical model and discussion of the results. Finally, we give conclusions in section 5.

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2. Mathematical model

The logistic equation describes population growth based on the following mathematical expression (see G. F. Gause [10])

$$\begin{cases} \dot{B}_1 = r_1 B_1 (1 - \dfrac{B_1}{K_1}) - c_{12} B_1 B_2 \\ \dot{B}_2 = r_2 B_2 (1 - \dfrac{B_2}{K_2}) - c_{21} B_1 B_2 \end{cases} \tag{1}$$

where B_1 and B_2 are the densities of populations 1 and 2 respectively; $(r_j)_{j=1,2}$ are the intrinsic growth rates; $(K_j)_{j=1,2}$ are the carrying capacities for the respective species; and $(c_{jk})_{1 \le j \ne k \le 2}$ are the coefficients of the competition between species k and species j.

2.1. The steady states of the system

The steady states of the system of equations (1) are obtained by solving the equations

$$\begin{cases} r_1 B_1^* (1 - \dfrac{B_1^*}{K_1}) - c_{12} B_1^* B_2^* = 0 \\ r_2 B_2^* (1 - \dfrac{B_2^*}{K_2}) - c_{21} B_1^* B_2^* = 0 \end{cases} \tag{2}$$

A qualitative study of system (1) shows that there are three equilibrium on the axes of coordinates $P_1(0,0)$, $P_2(K_1,0)$, $P_3(0,K_2)$ and a fourth equilibrium $P_4(B_1^*, B_2^*)$ given by

$$\begin{cases} B_1^* = r_2 K_1 (r_1 - c_{12} K_2) / (r_1 r_2 - c_{12} c_{21} K_1 K_2) \\ B_2^* = r_1 K_2 (r_2 - c_{21} K_1) / (r_1 r_2 - c_{12} c_{21} K_1 K_2) \end{cases} \tag{3}$$

This solution can give coexistence of the two fish species; in this case the biomasses of the two fish species are positive.

On the four figures below, we observe, according to the values of bio-economic parameters, in the first one, the extinction of fish specie 2, in the second one, the extinction of fish specie 1, in the third one, the coexistence of both species and in the fourth one, the extinction of one species

Evolution of two fish species in competition
according to the values of bio-economic parameters

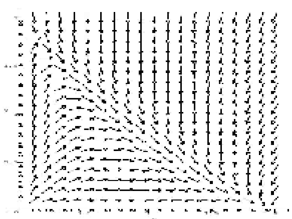

Fig.1: Extinction of fish specie 2

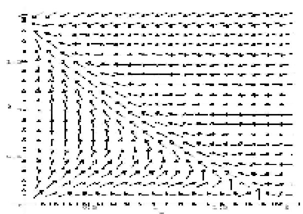

Fig.2: Extinction of fish specie 1

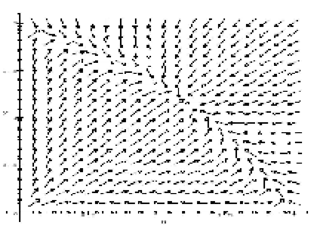

Fig.3: Coexistence of both species

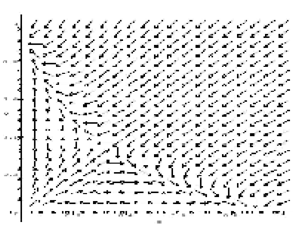

Fig.4: Extinction of one species

Now we will prove a result which gives the stability of the point $P_4(B_1^*, B_2^*)$ given by (3).

Theorem 1 *The steady state* $P_4(B_1^*, B_2^*)$ *is locally asymptotically stable if the hypotheses*

$$\begin{cases} r_1 - c_{12}K_2 > 0 \\ r_2 - c_{21}K_1 > 0 \end{cases} \tag{4}$$

hold simultaneously.

Proof. : The variational matrix of the system (1) at $P_4(B_1^*, B_2^*)$ is

$$J = \begin{bmatrix} r_1(1 - \frac{2B_1^*}{K_1}) - c_{12}B_2^* & -c_{12}B_1^* \\ -c_{21}B_2^* & r_2(1 - \frac{2B_2^*}{K_2}) - c_{21}B_1^* \end{bmatrix}$$

Using the fact that by (2) we have

$$\begin{cases} r_1(1 - \frac{2B_1^*}{K_1}) - c_{12}B_2^* = -r_1\frac{B_1^*}{K_1} \\ r_2(1 - \frac{2B_2^*}{K_2}) - c_{21}B_1^* = -r_2\frac{B_2^*}{K_2} \end{cases}$$

then

$$J = \begin{bmatrix} -r_1\frac{B_1^*}{K_1} & -c_{12}B_1^* \\ -c_{21}B_2^* & -r_2\frac{B_2^*}{K_2} \end{bmatrix}$$

The trace of the Jacobian matrix is negative: $trace(J) = -r_1\frac{B_1^*}{K_1} - r_2\frac{B_2^*}{K_2} < 0.$

The determinant of the Jacobian matrix is given by: $det(J) = \frac{(r_1r_2 - c_{12}c_{21}K_1K_2)}{K_1K_2}B_1^*B_2^*$

Now, if $r_1 - c_{12}K_2 > 0$ and $r_2 - c_{21}K_1 > 0$ then $r_1r_2 - c_{12}c_{21}K_1K_2 > 0$ therefore $P_4(B_1^*, B_2^*)$ is locally asymptotically stable.

2.2. Bio-economic model

In this work we define a bio-economic equilibrium model for 'n' fishermen who catch two fish species. In order to simplify the model and gain a better understanding of the rest of this work, we consider as a first step two fishermen who catch two fish species. In a second step we generalize this result by considering several fishermen who catch two fish species.
Now, we introduce the fishing by reducing the rate of fish population growth by the amount. Under exploitation, Schaefer [19] introduced the catch rate $(H_j)_{j=1,2}$ as $H_j = q_jE_jB_j$ where E_j is the fishing effort to exploit a fish species j and q_j is the catchability coefficient of fish species j, defined as the fraction of the population fished by an effort unit (see Gulland [13]). Biomass changes through time can be expressed as

$$\begin{cases} \dot{B}_1 = r_1 B_1 (1 - \frac{B_1}{K_1}) - c_{12} B_1 B_2 - q_1 E_1 B_1 \\ \dot{B}_2 = r_2 B_2 (1 - \frac{B_2}{K_2}) - c_{21} B_1 B_2 - q_2 E_2 B_2 \end{cases} \quad (5)$$

In cases where fish species are non interacting, the parameters of competition are zero and we find the models proposed by Gordon [11].

It is interesting to note that according to the literature, the effort depends on several variables, namely for example: number of hours spent fishing; search time; number of hours since the last fishing; number of days spent fishing; number of operations; number of sorties flown; ship, technology, fishing gear, crew, etc. However, in this paper, the fishing effort is treated as a unidimensional variable which includes a combination of all these factors.

On the other hand, it is clear that the fishing effort E_j to exploit a fish species j is the sum of the fishing effort E_{1j} of the first fisherman to exploit a fish species j and the fishing effort E_{2j} of the second fisherman to exploit a fish species j; mathematically: $E_j = E_{1j} + E_{2j}$ for all $j = 1,2$.

Now we give the expression of biomass as a function of fishing effort.

The biomasses at biological equilibrium (i.e., the variation of the biomass of each species is zero), are the solutions of the system

$$\begin{cases} r_1 (1 - \frac{B_1}{K_1}) = c_{12} B_2 + q_1 E_1 \\ r_2 (1 - \frac{B_2}{K_2}) = c_{21} B_1 + q_2 E_2 \end{cases} \quad (6)$$

The solutions of this system are given by

$$\begin{cases} B_1 = \frac{K_1}{(r_1 r_2 - c_{12} c_{21} K_1 K_2)} (r_1 r_2 - q_1 E_1 r_2 - c_{12} K_2 (r_2 - q_2 E_2)) \\ B_2 = \frac{K_2}{(r_1 r_2 - c_{12} c_{21} K_1 K_2)} (r_1 r_2 - q_2 E_2 r_1 - c_{21} K_1 (r_1 - q_1 E_1)) \end{cases} \quad (7)$$

To make the formulas more readable, we'll use the following notations

$$\alpha_1 = -r_2 q_1 K_1 / (r_1 r_2 - c_{12} c_{21} K_1 K_2)$$
$$\alpha_2 = -r_1 q_2 K_2 / (r_1 r_2 - c_{12} c_{21} K_1 K_2)$$
$$\beta_1 = c_{12} q_2 K_1 K_2 / (r_1 r_2 - c_{12} c_{21} K_1 K_2)$$
$$\beta_2 = c_{21} q_1 K_1 K_2 / (r_1 r_2 - c_{12} c_{21} K_1 K_2)$$
$$\gamma_1 = (r_1 r_2 K_1 - c_{12} r_2 K_1 K_2) / (r_1 r_2 - c_{12} c_{21} K_1 K_2)$$
$$\gamma_2 = (r_1 r_2 K_2 - c_{21} r_1 K_1 K_2) / (r_1 r_2 - c_{12} c_{21} K_1 K_2)$$

With these notations the equilibrium biomass as a function of fishing effort can be defined as

$$\begin{cases} B_1 = \alpha_1 E_1 + \beta_1 E_2 + \gamma_1 \\ B_2 = \beta_2 E_1 + \alpha_2 E_2 + \gamma_2 \end{cases} \quad (8)$$

or in matrix form $B=-AE+\gamma$ where $A=\begin{bmatrix}-\alpha_1 & -\beta_1\\ -\beta_2 & -\alpha_2\end{bmatrix}$, $E=(E_1,E_2)^T$ and $\gamma=(\gamma_1,\gamma_2)^T$.

Now we give the expression of profit as a function of fishing effort. We use, as usual in the bio-economic models, the fact that the total revenue (TR) depends linearly on the catch, that is *Total revenue = Price* $\times$ *Catches*.

On the other hand, we shall assume, in keeping with many standard fisheries models (e.g., the model of Clark [2], Clark [3], Crutchfield [5], Gordon [11] and Gordon [12]), that the total effort cost of the fisherman i is given by $(TC)_i=\langle c,E^i\rangle$, where c_j is a constant cost per unit of harvesting effort of the species j.

As mentioned previously, we note that the $H_{ij}=q_jE_{ij}B_j$ catches of species j by the fisherman i, where E_{ij} is the effort of the fisherman i to exploit the species j.

It is clear that $H_j=\sum_{i=1}^{2}H_{ij}$ is the total catches of species j by all fishermen.

On the other hand, we denote by $E^i=(E_{i1},E_{i2})^T$ the vector fishing effort must provide by the fisherman i to catch the two fish species.

The profit (net revenues) for each fisherman $\pi_i(E)$ is equal to total revenue $(TR)_i$ minus total cost $(TC)_i$, in other words, the profit for each fisherman is represented by the following function $\pi_i(E)=(TR)_i-(TC)_i$, so that the profit of fisherman i is given by

$$\begin{aligned}\pi_i(E)&=(TR)_i-(TC)_i\\ &=p_1H_{i1}+p_2H_{i2}-c_1E_{i1}-c_2E_{i2}\\ &=p_1q_1E_{i1}B_1+p_2q_2E_{i2}B_2-c_1E_{i1}-c_2E_{i2}\\ &=p_1q_1E_{i1}(\alpha_1E_1+\beta_1E_2+\gamma_1)-c_1E_{i1}+p_2q_2E_{i2}(\beta_2E_1+\alpha_2E_2+\gamma_2)-c_2E_{i2}\\ &=p_1q_1E_{i1}(\alpha_1\sum_{i=1}^{2}E_{i1}+\beta_1\sum_{i=1}^{2}E_{i2}+\gamma_1)-c_1E_{i1}+p_2q_2E_{i2}(\beta_2\sum_{i=1}^{2}E_{i1}+\alpha_2\sum_{i=1}^{2}E_{i2}+\gamma_2)-c_2E_{i2}\end{aligned}$$

then

$$\pi_i(E)=\langle E^i,-pqAE^i\rangle+\langle E^i,pq\gamma-c-\sum_{j=1,\,j\neq i}^{2}pqAE^j\rangle \tag{9}$$

where p_1 (resp. p_2) is the price of the fish species 1 (resp. 2).

To maintain the biodiversity of species, it is natural to assume that all biomasses remain positive, therefore

$$B=-AE+\gamma\geq 0. \tag{10}$$

In other word, for the fisherman i

$$AE^i\leq-\sum_{j=1,\,j\neq i}^{2}AE^j+\gamma. \tag{11}$$

Now we will trace the total revenue and total cost as a function of fishing effort as in the diagram Gordon. A bio-economic model of a fishery with fishing costs linearly proportional to fishing effort. Note that MEY (maximum economic yield, i.e., the maximum difference between the gross value of catch and cost of fishing) achieves at a level of fishing effort lower than that needed to obtain MSY (maximum sustainable yield).

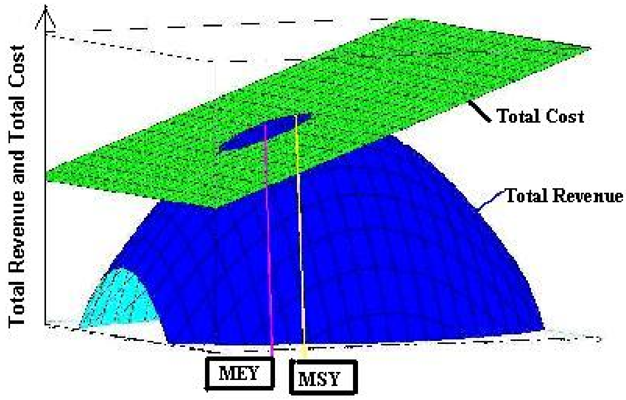

Fig.5: Total revenue and Total Cost as a function of Fishing Effort

3. Linear Complementarity Problem

Each fisherman trying to maximize his profit and achieve a fishing effort that it is an optimal response to the fishing effort of the other fishermen. We have a generalized Nash equilibrium where each fisherman's strategy is optimal taking into consideration the strategies of all the other fishermen. A Nash Equilibrium exists when there is no unilateral profitable deviation from any of the fishermen involved. In other words, no fisherman would take a different action as long as every other fisherman remains the same. This problem can be translated into the following two mathematical problems:
The first fisherman must solve the problem (P_1)

$$(P_1)\begin{cases} \max \pi_1(E) = < E^1, -pqAE^1 > + < E^1, pq\gamma - c - pqAE^2 > \\ \text{sc} \\ \qquad AE^1 \le -AE^2 + \gamma \\ \qquad E^1 \ge 0 \\ \qquad E^2 \text{ is given.} \end{cases}$$

The second fisherman must solve the problem (P_2)

$$(P_2)\begin{cases} \max \pi_2(E) = < E^2, -pqAE^2 > + < E^2, pq\gamma - c - pqAE^1 > \\ \text{sc} \\ \qquad AE^2 \le -AE^1 + \gamma \\ \qquad E^2 \ge 0 \\ \qquad E^1 \text{ is given.} \end{cases}$$

We recall that (E^1, E^2) is called Generalized Nash Equilibrium Point if and only if E^1 is a solution of problem $(P1)$ for E^2 given, and E^2 is a solution of problem $(P2)$ for E^1 given.
Now we give two lemmas and two theorems that give this Generalized Nash Equilibrium Point.

Lemma 1 *The Generalized Nash Equilibrium Point* (E^1, E^2) *is a solution of the* ***L****inear* ***C****omplementarity* ***P****roblem* $LCP(M,b)$*:*
Find vectors $z, w \in \mathrm{IR}^6$ such that $z \geq 0$, $w = Mz + b \geq 0$ and $z^T w = 0$
Where

$$z = (E^1, E^2, 0)^T;\ \ w = (u^1, u^2, v)^T, M = \begin{bmatrix} 2pqA & pqA & A^T \\ pqA & 2pqA & 0 \\ -A & -A & 0 \end{bmatrix} \text{ and } b = \begin{pmatrix} c - pq\gamma \\ c - pq\gamma \\ \gamma \end{pmatrix}$$

Proof.: The essential conditions of Karush-Kuhn-Tucker applied to the problem (P_1) require that if E^1 is a solution of the problem (P_1) then there exist constants $u^1 \in \mathrm{IR}^2_+$, $v^1 \in \mathrm{IR}^2_+$, et $\lambda^1 \in \mathrm{IR}^2_+$ such that

$$\begin{cases} 2pqAE^1 + c - pq\gamma + pqAE^2 - u^1 + A^T\lambda^1 = 0 \\ AE^1 + v^1 = -AE^2 + \gamma \\ < u^1, E^1 > = < \lambda^1, v^1 > = 0 \end{cases} \qquad \text{(KKT1)}$$

In the same way, the conditions of Karush-Kuhn-Tucker applied to the problem (P_2), require that if E^2 is a solution of the problem (P_2) then there exist constants $u^2 \in \mathrm{IR}^2_+$, $v^2 \in \mathrm{IR}^2_+$ et $\lambda^2 \in \mathrm{IR}^2_+$ such that

$$\begin{cases} 2pqAE^2 + c - pq\gamma + pqAE^1 - u^2 + A^T\lambda^2 = 0 \\ AE^2 + v^2 = -AE^1 + \gamma \\ < u^2, E^2 > = < \lambda^2, v^2 > = 0 \end{cases} \qquad \text{(KKT2)}$$

It is immediately seen from $(KKT1)$ and $(KKT2)$ that

$$\begin{cases} u^1 = 2pqAE^1 + c - pq\gamma + pqAE^2 + A^T\lambda^1 \\ u^2 = 2pqAE^2 + c - pq\gamma + pqAE^1 + A^T\lambda^2 \\ v^1 = -AE^1 - AE^2 + \gamma & (*^1) \\ v^2 = -AE^1 - AE^2 + \gamma & (*^2) \\ < u^i, E^i > = < \lambda^i, v^i > = 0 \quad \text{for all } i = 1,2 \\ E^i, u^i, \lambda^i, v^i \geq 0 \qquad \text{for all } i = 1,2 \end{cases}$$

It is clear from equation $(*^1)$ and from equation $(*^2)$ that $v^1 = v^2$.
To maintain the biodiversity of species, it is natural to assume that all biomasses remain strictly positive, that is $B_j > 0$ for all $j = 1,2$; therefore $v^1 = v^2 > 0$.
As the scalar product of $(\lambda^i)_{i=1,2}$ and $(v^i)_{i=1,2}$ is zero, so $\lambda^i = 0$ for all $i = 1,2$. In what follows of this paper, we denote by $v = v^1 = v^2$. So that we have the following expressions

$$\begin{cases} u^1 = 2pqAE^1 + pqAE^2 + c - pq\gamma \\ u^2 = pqAE^1 + 2pqAE^2 + c - pq\gamma \\ v = -AE^1 - AE^2 + \gamma \\ < u^i, E^i > = 0 \qquad \text{for all } i = 1,2 \\ E^i, u^i, v^i \geq 0 \qquad \text{for all } i = 1,2 \end{cases}$$

thus

$$\begin{pmatrix} u^1 \\ u^2 \\ v \end{pmatrix} = \begin{bmatrix} 2pqA & pqA & A^T \\ pqA & 2pqA & 0 \\ -A & -A & 0 \end{bmatrix} \begin{pmatrix} E^1 \\ E^2 \\ 0 \end{pmatrix} + \begin{pmatrix} c-pq\gamma \\ c-pq\gamma \\ \gamma \end{pmatrix}. \tag{12}$$

Let us denote by

$$z = \begin{pmatrix} E^1 \\ E^2 \\ 0 \end{pmatrix}, w = \begin{pmatrix} u^1 \\ u^2 \\ v \end{pmatrix}, M = \begin{bmatrix} 2pqA & pqA & A^T \\ pqA & 2pqA & 0 \\ -A & -A & 0 \end{bmatrix}, b = \begin{pmatrix} c-pq\gamma \\ c-pq\gamma \\ \gamma \end{pmatrix}$$

then z is a solution of the **L**inear **C**omplementarity **P**roblem $LCP(M,b)$.
To show that $LCP(M,b)$ has a unique solution, we will use the following results:

Theorem 2 *(see [4] and [17]): $LCP(M,b)$ has a unique solution for every b if and only if M is a P-matrix.*

Recall that a matrix M is called $P-matrix$ if the determinant of every principal submatrix of M is positive (see Fiedler [9], Murty [16]).
The class of P-matrices generalizes many important classes of matrices, such as positive definite matrices, M-matrices, and inverse M-matrices, and arises in applications.
Note that each matrix symmetric positive definite is $P-matrix$, but the reverse is not always true.
Now we show that the matrix M of our problem is $P-matrix$; which is equivalent to the existence and uniqueness of a solution of $LCP(M,b)$, therefore, the existence and uniqueness of a generalized Nash equilibrium.

Lemma 2 *The matrix*

$$M = \begin{bmatrix} 2pqA & pqA & A^T \\ pqA & 2pqA & 0 \\ -A & -A & 0 \end{bmatrix}$$

is $P-matrix$.

Proof. : if we note by $(M_i)_{i=1,..,6}$ the submatrix of M, we obtain

$$\det(M_1) = -2p_1q_1\alpha_1 > 0$$

$$\det(M_2) = 4p_1q_1p_2q_2q_1(\alpha_1\alpha_2 - \beta_1\beta_2) > 0$$

$$\det(M_3) = -6p_1^2q_1^2p_2q_2\alpha_1(\alpha_1\alpha_2 - \beta_1\beta_2) > 0$$

$$\det(M_4) = 9p_1^2q_1^2p_2^2q_2^2(\alpha_1\alpha_2 - \beta_1\beta_2)^2 > 0$$

$$\det(M_5) = -3p_1q_1p_2^2q_2^2\alpha_1(\alpha_1\alpha_2 - \beta_1\beta_2)^2 > 0$$

$$\det(M_6) = p_1q_1p_2q_2(\alpha_1\alpha_2 - \beta_1\beta_2)^3 > 0.$$

where

$$\alpha_1\alpha_2-\beta_1\beta_2=\frac{r_2q_1K_1}{r_1r_2-c_{12}c_{21}K_1K_2}\frac{r_1q_2K_2}{r_1r_2-c_{12}c_{21}K_1K_2}-\frac{c_{12}q_2K_1K_2}{r_1r_2-c_{12}c_{21}K_1K_2}\frac{c_{21}q_1K_1K_2}{r_1r_2-c_{12}c_{21}K_1K_2}$$
$$=\frac{q_1q_2K_1K_2}{r_1r_2-c_{12}c_{21}K_1K_2}>0.$$

So the matrix M is $P-matrix$.
It is not difficult to see that the previous lemma shows that the linear complementarity problem $LCP(M,b)$ admits one and only one solution. This solution is given by the following theorem

Theorem 3 *The fishing effort that maximizes the profit of each fisherman is given by*

$$E^*=\frac{1}{3}A^{-1}(\gamma-\frac{c}{pq}) \tag{13}$$

where

$$A^{-1}=\begin{bmatrix}\frac{r_1}{K_1q_1} & \frac{c_{12}}{q_1}\\ \frac{c_{21}}{q_2} & \frac{r_2}{K_2q_2}\end{bmatrix}.$$

Proof. : It is natural to assume that the two fish species must be catched by two fishermen, this leads to write $E^1,E^2>0$, and therefore: $(u^i)_{i=1,2}=0$.
Take these results and previous results, the solution of $LCP(M,b)$ is $z(E^1,E^2,0)$ where

$$\begin{cases}E^1=\frac{1}{3}A^{-1}(\gamma-\frac{c}{pq})\\ E^2=\frac{1}{3}A^{-1}(\gamma-\frac{c}{pq})\end{cases}. \tag{14}$$

where A^{-1} is the inverse of A, this matrix is given by

$$A^{-1}=\begin{bmatrix}\frac{r_1}{K_1q_1} & \frac{c_{12}}{q_1}\\ \frac{c_{21}}{q_2} & \frac{r_2}{K_2q_2}\end{bmatrix}.$$

It is clear that the fishing efforts E^1 and E^2 are positive since they are the solutions of $LCP(M,b)$; it remains to verify the positivity of biomass of two fish species, as expressions of biomass of both fish species are given by

$$\begin{cases}B_1=\alpha_1E_1+\beta_1E_2+\gamma_1\\ B_2=\beta_2E_1+\alpha_2E_2+\gamma_2\end{cases}$$

which gives

$$\begin{cases}B_1=\alpha_1(E_{11}+E_{21})+\beta_1(E_{12}+E_{22})+\gamma_1\\ B_2=\beta_2(E_{11}+E_{21})+\alpha_2(E_{12}+E_{22})+\gamma_2\end{cases}$$

and finally $B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = -AE + \gamma \; = v > 0.$

Now we give the expression of the profit function of fisherman i at E^*

$$\begin{aligned} \pi_i(E) &= < E^i, -pqAE^i > + < E^i, pq\gamma - c - \sum_{j=1,\, j\neq i}^{2} pqAE^j > \\ &= < E^i, pqAE^i > \\ &> 0. \end{aligned}$$

4. Discussion of results and numerical simulations

The comparison between the two cases shows that there is a difference between them.
In the first one when we consider only one fisherman who catches the two fish species (which are competing for space or food), then the fishing effort that maximizes the benefit of this fisherman is given by

$$E^* = (r_1 r_2 - c_{12}c_{21}K_1K_2)[(\beta_1 - \alpha_1)(\gamma_{21} - c_2/p_2q_2) + (\beta_2 - \alpha_2)(\gamma_1 - c_1/p_1q_1)]/(2q_1q_2K_1K_2).$$

In the second one when we consider two fishermen who catches the two fish species (which are competing for space or food), then the fishing effort that maximizes the profits of the two fishermen are given by relations (13).
Now we deal with the general case by considering 'n' fishermen who catch two fish species that compete for space or food, this leads to the following generalized Nash equilibrium problem
The fisherman $i = 1,..,n$ must solve the following problem $(P_i)_{1\le i\le n}$

$$(P_i)\begin{cases} \max \pi_i(E) = < E^i, -pqAE^i + pq\gamma - c - \sum\limits_{k=1,k\neq i}^{n} pqAE^k > \\ \text{subject to} \\ \qquad AE^i \le -\sum\limits_{k=1,k\neq i}^{n} AE^k + \gamma \\ \qquad E^i \ge 0 \\ \qquad (E^k)_{1\le k\neq i\le n} \text{ is given.} \end{cases}$$

To solve this problem we transform it into a linear complementarity problem of finding the two vectors $z = (E^1,..,E^n,0)^T$ and $w = (u^1,..,u^n,v)^T$ satisfying $z \ge 0,$ $w = Mz + b \;\; 0$ and $< z, w >= 0$, where

$$M = \begin{bmatrix} 2pqA & pqA & ... & pqA & A^T \\ pqA & 2pqA & ... & pqA & 0 \\ ... & ... & ... & ... & ... \\ pqA & ... & pqA & 2pqA & 0 \\ -A & -A & ... & -A & 0 \end{bmatrix} \quad and \quad b = \begin{pmatrix} c - pq\gamma \\ c - pq\gamma \\ ... \\ c - pq\gamma \\ \gamma \end{pmatrix}.$$

It is very complicated to solve such a linear complementarity problem (LCP) for a large n even numerically. Many algorithms exist in the literature for solving this kind of problems (see for instance, Borigi [1], Cryer [6], Kojima [14], Lemke [15], Murty [18]), but for (LCP) with a large scale matrix these methods need very powerful machines to be implemented. That is why we developed algorithms ([7] and [8]) more efficient for solving this problem.
We take as a case study ten fishermen who catch two fish species where the first and the second fish species have the following characteristics

Table 1. Characteristics of the two fish species

	Fish species1	Fish species 2
Intrinsic growth rate	2.00	1.00
Catchability coefficient	0.004	0.02
Carrying capacities	5000	1000
Coefficient of competition	0.0002	0.00001
Cost per unit of effort.	20.00	10.00

We find the same results (see following table) as in a model of Gordon-Schaefer mono-specific: an increase in price leads to an increase in fishing effort and reduced catch levels.

Table 2. The influence of the price on the fishing effort and reduced catch levels

p_1	p_2	E_1	E_2	H_1	H_2	H_1+H_2
2.00	4.00	22.16	3.86	2378.89	154.88	2533.77
5.00	13.00	36.19	4.33	1911.11	105.33	2016.44
7.00	19.00	38.84	4.39	1647.65	97.29	1744.94
10.00	28.00	40.83	4.44	1413.78	91.52	1505.30
14.00	40.00	42.15	4.47	1240.62	87.80	1328.43
19.00	55.00	43.02	4.49	1119.20	85.41	1204.61
24.00	70.00	43.53	4.50	1045.62	84.03	1129.64

We add that since the number of fishermen is increasing, the catch level is getting lower as shown in figure 6.
As mentioned above, it is interesting to note that if we consider only one fisherman who catches the two fish species, then the fishing effort that maximizes the profit of this fisherman is equal to 228.85 to catch the first species and 24.58 to catch the second; his profit in this case is equal to 31817.60 (see fig. 7).
On the contrary, when we consider two fishermen who catch the two species, then for each fisherman to maximize his profit, he must provide a fishing effort which is equal to 152.57 for catch the first species and 16.39 to catch the second species, the profit of each fisherman in this case is equal to 28446.20 (see fig. 7).

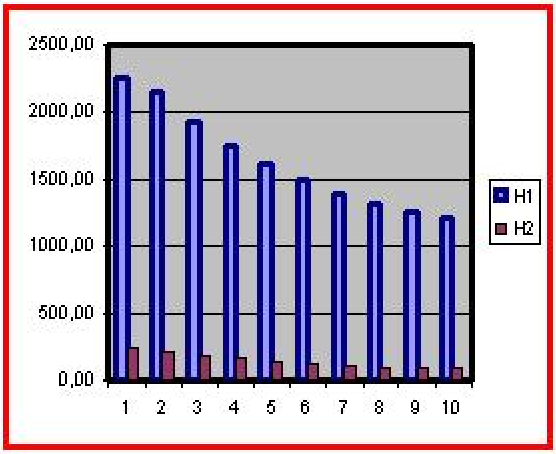

Fig. 6: Influence of the fishermen number on the catch levels

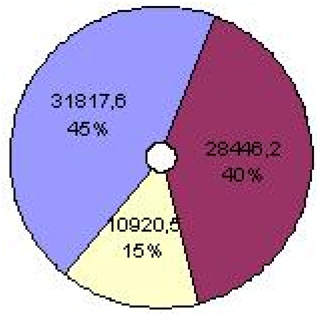

Fig. 7: Influence of the fishermen number on the fishermen profits

In the end when we consider ten fishermen who catch the two fish species, then for each fisherman to maximize his profit, he must provide a fishing effort which is equal to 41.61 to catch the first species and 4.47 to catch the second species, the profit of each fisherman in this case is equal to 10920.49 (see fig. 7).
We add that since the number of fishermen is increasing, the fishermen profits are getting lower.

5. Conclusion and perspectives

In this work we have defined a bio-economic equilibrium model for 'n' fishermen who catch two fish species, these species compete with each other for space or food. The natural growth of each species is modeled using a logistic law. We have calculated the fishing effort that maximizes the profit of each fisherman at biological equilibrium by using the generalized Nash equilibrium problem. The existence of the steady states and its stability are studied using eigenvalue analysis. Finally, some numerical examples are given to illustrate the results.
In this work, we have considered that the prices of fish species are constants, we consider in a future work to define functions of providing long term, where price is no longer a constant but depends on the level of effort and biomass stock of each species remaining.

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