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SPECTRAL APPROXIMATIONS OF DIRICHLET PROBLEM FOR HARMONIC OPERATOR

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Abstract. In this work, we study a Dirichlet problem for harmonic operator. Some theoretic spectral approaches are given. Numerical solutions and illustrations are established to prove our theoretic study.

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1. Introduction

Spectral methods are classical techniques to resolve theoretically and numerically differential equations, partial differential equations and integral equations. These methods appear competitive with finite differences and finite element methods. Moreover, it is possible to verify a solution of these problems easily by these methods. Physically, they are based on quest of a solution as well-known charges series. Test functions in the case of spectral methods are infinitely differentiable functions. They appear as tensorial products of proper functions. The choice of test functions arrange according to three spectral schemes: Galerkin, collocation and tau.

Galerkin approach consists to replace test functions space by a finite dimensional linear subspace $V_N(\Omega)$. Thus approach solution is in the form

$$u_N = \sum_{n=0}^{N} a_n \varphi_n$$

where a_n are reals and $V_N = Span\{\varphi_0, \varphi_1, ..., \varphi_N\}$.

A most disadvantage of this approach requires integral calculus, which is not always easy to do and at the same time very expensive.

Hence idea was then to introduce collocation approach. This approach restates again on variational formulation but computing integrals by adapted quadrature formulae. Collocation approach has been used first by Slater in 1934 and by Kantorovic in 1934 in some specific applications ([2]).

In 1937, Frazer, Jones and Skan evolved this approach as a global approach to resolve ordinary differential equations ([2]). This approach is particularly attractive because it is easy to be applied to nonlinear problems.

Tau approach has been discovered by Lanczos in 1938 ([5]). An approach solution is given by

$$u_N = \sum_{n=0}^{N+k} a_n \varphi_n,$$

where k is the number of independent constraints of the form $Bu_N = 0$ and B is a linear differential operator. An important difference between tau approach and Galerkin approach is in the first, test functions do not verify boundary conditions.

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Let Ω be a bounded open set of $\mathbb{R}^d$ $(d = 1, 2, 3)$ and of regular boundary $Fr(\Omega)$. Consider the following problem

$$(P_0) \qquad \begin{cases} -\Delta u = f \ \text{ in } \ \Omega \\ \\ u = 0 \ \text{ on } \ Fr(\Omega). \end{cases}$$

In this work, we study a Dirichlet problem for harmonic operator. Some theoretic spectral approaches are given. Numerical solutions and illustrations are established to prove our theoretic study.

To study a problem (P_0), we shall require some definitions and preliminary results.

2. Generalities

Let V be a real Hilbert space equipped with a scalar product $(.,.)$ and associated norm $\|.\|_V$. Denote V' a dual of Hilbert space V.

Definition 1. Let $\mathfrak{a}(.,.)$ be a bilinear form from $V \times V$ into $\mathbb{R}$. We say that $\mathfrak{a}(.,.)$ is

(1) continuous if there exists a constant $c > 0$ such that

$$|\mathfrak{a}(u, v)| \leq c \|u\|_V \|v\|_V , \ \ \forall u, \ v \in V;$$

(2) V-elliptic if there exists an $\alpha > 0$ such that

$$|a(v, v)| \geq \alpha \|v\|_V^2 , \ \ \forall v \in V.$$

Lemma 1. *(Lax-Milgram) Let us given*

1) *a Hilbert space V equipped with the norm $\|.\|_V$;*

2) *a continuous bilinear form $a(.,.)$ on $V \times V$ and verifies a V-ellipticity condition:*

$$\exists \alpha > 0, \ a(v, v) \geq \alpha \|v\|_V^2 , \ \ \forall v \in V \ ;$$

3) *a continuous linear form $l(.)$ on V.*

Then a problem

$$(P_1) \qquad \begin{cases} \textit{Find } u \in V \textit{ such that} \\ \\ a(u, v) = l(v), \ \forall v \in V \end{cases}$$

has one and only one solution.

Definition 2. Define

- $C_0^\infty(\Omega)$: space of infinitely differentiable functions with compact support in Ω, namely
$$C_0^\infty(\Omega) := \{\varphi : \varphi \in C^\infty(\Omega), \ \text{supp}\varphi \subset \Omega\} = \mathfrak{D}(\Omega);$$
- $\mathfrak{D}(\overline{\Omega})$: restrictions space to $\overline{\Omega}$ of infinitely differentiable functions with compact support in $\mathbb{R}^d$;
- $\mathfrak{D}'(\Omega)$: distributions space in Ω as a dual space of $C_0^\infty(\Omega)$, namely, continuous linear forms space on $\mathfrak{D}(\Omega)$.

Definition 3. We call $H^m(\Omega)$ Sobolev space of functions whose generalized derivatives up to order $m \in \mathbb{N}$ belong to $L^2(\Omega)$, namely

$$H^m(\Omega) := \left\{v : v \in L^2(\Omega), \partial^\alpha v \in L^2(\Omega); |\alpha| \leq m\right\}.$$

We equip this space with the following scalar product:

$$(u, v)_{H^m(\Omega)} = \int\limits_\Omega \sum_{|\alpha| \leq m} (\partial^\alpha u)(\partial^\alpha v) dx$$

and associated norm

$$\|u\|_{H^m(\Omega)} := (u, u)_{H^m(\Omega)}^{\frac{1}{2}}.$$

Denote $H_0^m(\Omega)$ a closure in $H^m(\Omega)$ of $\mathfrak{D}(\Omega)$ in comparison with a norm $\|u\|_{H^m(\Omega)}$ and $H^{-m}(\Omega)$ the dual of $H_0^m(\Omega)$.

A characterization of spaces $H_0^m(\Omega)$ is made by trace theorems.

Definition 4. Let $v \in \mathfrak{D}(\overline{\Omega})$ and let η be outward normal to $Fr(\Omega)$. We call trace up to order j ($j \in \mathbb{N}$) of u on the boundary $Fr(\Omega)$, a linear mapping γ_j defined by

$$\begin{cases} \gamma_j : v \longrightarrow \gamma_j v \\ \\ \gamma_j v = \frac{\partial^j u}{\partial \eta^j}_{|Fr(\Omega)} \end{cases}$$

where $\gamma_0 v = v_{|Fr(\Omega)}$ and $\frac{\partial^j u}{\partial \eta^j}$ is a normal derivative up to order j on $Fr(\Omega)$ facing outward of $Fr(\Omega)$.

Theorem 1. *([4])Let m be a positive integer. Then a mapping $\overrightarrow{\gamma}_m : v \longrightarrow \overrightarrow{\gamma}_m v = (\gamma_0 v, \gamma_1 v, ..., \gamma_{m-1} v)$ defined on $\mathfrak{D}(\overline{\Omega})$ into $(\mathfrak{D}(Fr(\Omega)))^m$ is prolonged by density to a continuous surjective linear mapping from $H^m(\Omega)$ into $\prod\limits_{j=0}^{m-1} H^{m-j-\frac{1}{2}}(Fr(\Omega))$.*

Thus $H_0^m(\Omega)$ is characterized by

$$H_0^m(\Omega) = \left\{ v \in H^m(\Omega) : \gamma_j v = \frac{\partial^j u}{\partial \eta^j} = 0 \ \text{ on } Fr(\Omega),\ j = 0, 1, ..., m-1 \right\},$$

where $\gamma_0 v = v_{|Fr(\Omega)}$.

As a consequence, we have the following result:

Corollary 1. *For $m \geq k$, one has*

$$\mathfrak{D}(\Omega) \subset H_0^m(\Omega) \subset H_0^k(\Omega) \subset L^2(\Omega) \subset H_0^{-k}(\Omega) \subset H_0^{-m}(\Omega) \subset \mathfrak{D}^{'}(\Omega)$$

with continuous and density injections.

Remark 1. The mapping $|.|_{H^m(\Omega)}$ defined in $H^m(\Omega)$ by

$$|u|_{H^m(\Omega)} = (\int\limits_{\Omega} \sum_{|\alpha|=m} |\partial^\alpha u|^2 \, dx)^{\frac{1}{2}}$$

is a seminorm in $H^m(\Omega)$.

Moreover we have the following theorem:

Theorem 2. *A seminorm $|.|_{H^m(\Omega)}$ is a norm in $H_0^m(\Omega)$ equivalent to an usual norm induced by those of $H^m(\Omega)$.*

We can define, also Sobolev spaces using Fourier transform. Indeed, $u \in H^m(\mathbb{R}^d)$ is equivalent to say $D^\alpha u \in L^2(\mathbb{R}^d)$, $\forall |\alpha| \leq m$ and consequently $\mathfrak{F}(D^\alpha u) \in L^2(\mathbb{R}^d)$, where $\mathfrak{F}(.)$ is Fourier transform. This is comes to say that

$$|\xi|^\alpha |\mathfrak{F}(u)| \in L^2(\mathbb{R}^d) \ \text{ or } \ (1 + |\xi|^2)^{\frac{m}{2}} |\mathfrak{F}(u)| \in L^2(\mathbb{R}^d).$$

So, we will have

$$\|u\|^2_{H^m(\mathbb{R}^d)} = \sum_{|\alpha| \leq m} \|D^\alpha u\|^2_{L^2(\mathbb{R}^d)} = \sum_{|\alpha| \leq m} \|\mathfrak{F}(D^\alpha u)\|^2_{L^2(\mathbb{R}^d)}$$

$$= \sum_{|\alpha| \leq m} \||\xi|^\alpha \mathfrak{F}(u)\|^2_{L^2(\mathbb{R}^d)} \leq \int\limits_{\mathbb{R}^d} (1 + |\xi|^2)^m |\mathfrak{F}(u)|^2 \, d\xi.$$

The second inequality is obvious.

This last defines an equivalent norm to induced norm by the space $H^m(\mathbb{R}^d)$. By interpolation, we can introduce the space $H^s(\mathbb{R}^d)$ for all real s.

Definition 5. Let $s \geq 0$ be real, denote $H^s(\mathbb{R}^d)$ the following Sobolev space:

$$H^s(\mathbb{R}^d) := \left\{ u \in L^2(\mathbb{R}^d) : (1 + |\xi|^2)^{\frac{s}{2}} |\mathfrak{F}(u)| \in L^2(\mathbb{R}^d) \right\}$$

equipped with the norm

$$\|u\|_{H^s(\mathbb{R}^d)} = (\int\limits_{\mathbb{R}^d} (1+|\xi|^2)^s \, |\mathfrak{F}(u)|^2 \, d\xi)^{\frac{1}{2}}.$$

In this work, we will also need the following Sobolev spaces:

$$H^s(\Omega) := \left\{ u_{|\Omega} : u \in H^s(\mathbb{R}^d), \ s \in \mathbb{R} \right\}$$

and

$$H^s(Fr(\Omega)) := \begin{cases} \left\{ u_{|Fr(\Omega)} : u \in H^{s+\frac{1}{2}}(\mathbb{R}^d) \right\} & \text{if } s > 0, \\ L^2(Fr(\Omega)) & \text{if } s = 0, \\ (H^{-s}(Fr(\Omega)))' & \text{if } s < 0. \end{cases}$$

Lemma 2. *(Poincaré-Friedrichs inequality)([7]) There exists a constant $c > 0$ dependent of Ω such that*

$$\|v\|_{L^2(\Omega)} \leq c(\Omega) \, \|v\|_{H_0^m(\Omega)}, \ \forall v \in H_0^m(\Omega).$$

Proposition 1. ***a)*** *All sufficiently regular solution u of problem (P_0) is a solution of problem (P_1);*

b) *a solution in $H_0^1(\Omega)$ of problem (P_1) is a weak solution of problem (P_0).*

Denote I the open interval $]-1,+1[$ in $\mathbb{R}$ and Ω the product $]-1,+1[^d$ in $\mathbb{R}^d$.

Definition 6. We call orthogonal system in $L^2(I)$, all family $(\varphi_i)_{i\in J}$ (J is finite or countable set) of non zeros elements of $L^2(I)$ and two by two orthogonal. Namely

$$\int\limits_{-1}^{+1} \varphi_i(x)\varphi_j(x)dx = \begin{cases} c \ \text{ if } i = j, \ \text{ with } c > 0, \\ 0 \ \text{ if } i \neq j. \end{cases}$$

Definition 7. A system $(\varphi_i)_{i\in J}$ is said linearly independent if all finite subset of this system is linearly independent.

Proposition 2. *If elements $\varphi_1, \varphi_2, \ ..., \ \varphi_N$ form an orthogonal system, they are inevitably linearly independents.*

Definition 8. We call Legendre family of polynomials, a family $(L_n)_{n\in\mathbb{N}}$ of polynomials in I two by two orthogonal in $L^2(I)$.

Theorem 3. *([9]) All function u of $L^2(\Omega)$ can be approximate by polynomials serie which converges uniformly to u.*

Proposition 3. ***a)*** *For all integer $n \geq 0$, a polynomial L_n verifies the differential equation*

$$\frac{d}{dx}((1-x^2)L_n') + n(n+1)L_n = 0;$$

b) *for all integer $n \geq 0$, a polynomial L_n is given by*

$$L_n(x) = \frac{(-1)^n}{2^n n!} \frac{d^n}{dx^n}((1-x^2)^n),$$

called Rodrigues formula.

Corollary 2. ***a)*** *For all integers $m \geq 0$ and $n \geq 0$, one has*

$$\int\limits_{-1}^{+1} L_m'(x)L_n'(x)(1-x^2)dx = n(n+1)\int\limits_{-1}^{+1} L_m(x)L_n(x)dx;$$

b) *for all integer $n \geq 0$, a polynomial L_n verifies*

$$\int_{-1}^{+1} L_n^2(x)dx = \tfrac{2}{2n+1}, \quad L_n(\pm 1) = (\pm 1)^n,$$

$$L_n'(\pm 1) = (\pm 1)^{n-1}\tfrac{1}{2}n(n+1);$$

c) *Legendre polynomials L_n form a system of orthogonal polynomials in $L^2(I)$. They verify relations:*

$$L_0(x) = 1, \quad L_1(x) = x,$$

$$(n+1)L_{n+1}(x) - (2n+1)xL_n(x) + nL_{n-1}(x) = 0$$

and

$$(x^2-1)L_n'(x) = nxL_{n-1}(x) - nL_{n-1}(x)$$

$$= (n+1)L_{n+1}(x) - (n+1)xL_n(x),$$

$$(1-x^2)L_n''(x) - 2xL_n'(x) + n(n+1)L_n(x) = 0;$$

d) *for all integer $n \geq 0$, one has an integral equation:*

$$\int_{-1}^{x} L_n(t)dt = \frac{1}{2n+1}(L_{n+1}(x) - L_{n-1}(x)).$$

3. Main Results

3.1. Approximation of the Problem (P_0) by Spectral Method.

Consider the following problem:

$$(P_0) \qquad \begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } Fr(\Omega). \end{cases}$$

We introduce a variational formulation of (P_0) as follows:

$$(P_1) \qquad \begin{cases} \text{Find } u \in V = H_0^1(\Omega) \text{ such that} \\ \mathfrak{a}(u,v) = l(v), \quad \forall v \in V, \end{cases}$$

where

$$\mathfrak{a}(u,v) = \int_\Omega \nabla u \nabla v dx \text{ and } l(v) = \int_\Omega fv dx.$$

Galerkin method consists to replace test functions space by a finite dimensional linear subspace $V_N^d(\Omega)$, thus Galerkin approximatation of (P_1) comes down to study the following problem:

$$(P_N) \qquad \begin{cases} \text{Find } u_N \in V_N^d(\Omega) \text{ such that} \\ \mathfrak{a}(u_N,v) = l(v), \quad \forall v \in V_N^d(\Omega). \end{cases}$$

Put

$$V_N^d(\Omega) = Span\left\{L_K, \ K \in \mathbb{N}^d, \ |K|_\infty \leq N\right\},$$

where

$$L_K(x) = \prod_{j=1}^{d} L_{k_j}(x_j), \quad x = (x_1, ..., x_d), \quad |K|_\infty = \max_{1\leq j\leq d}(|k_j|).$$

Denote P_N the orthogonal projection operator of $L^2(\Omega)$ in $V_N^d(\Omega)$. This means that

$$(u - P_N u, \phi_N)_{L^2(\Omega)} = 0, \quad \forall \phi_N \in V_N^d(\Omega),$$

where

$$(u,v)_{L^2(\Omega)} = \int_\Omega uv dx.$$

Theorem 4. *([3]) For all real $s \geq 0$, there exists a positive constant c independent of N such that for all function $u \in H^s(\Omega)$, one has*

$$\|u - P_N u\|_{L^2(\Omega)} \leq cN^{-s} \|u\|_{H^s(\Omega)}.$$

Proof. • If s is even, i.e., $s = 2p,\ p \geq 1$, we define an operator

$$A_j = D_j(1 - x_j^2)D_j, \quad D_j = \frac{\partial}{\partial x_j}.$$

Consider sets

$$K(N) = \left\{K \in \mathbb{N}^d : \ |K|_\infty > N\right\}; \quad K^{(1)}(N) = \{K \in K(N) : \ k_1 > N\}$$

and

$$K^{(j)}(N) = \left\{K \in K(N) \setminus \underset{l<j}{\cup} K^{(l)}(N) : \ k_j > N\right\}, \quad j = 2, ..., d.$$

If

$$u = \sum_{K \in \mathbb{N}^d} \widehat{u}_K L_K$$

then,

$$\widehat{u}_K = \frac{1}{\|L_K\|^2_{L^2(\Omega)}} \int_\Omega u(x) L_K(x) dx = \frac{1}{\|L_K\|^2_{L^2(\Omega)}} \int_{\Omega'} L_{K'}(x') dx' \int_{-1}^{+1} u(x_1, x') L_{k_1}(x_1) dx_1$$

where $x' = (x_2, ..., x_d),\ \ K' = (k_2, ..., k_d)$ and $\Omega' =]-1, +1[^{d-1}$. But

$$L_{k_1}(x_1) = -\frac{1}{k_1(k_1+1)} \frac{d}{dx_1}((1 - x_1^2) L'_{k_1}(x_1))$$

then

$$\widehat{u}_K = -\tfrac{1}{\|L_K\|^2_{L^2(\Omega)}} \tfrac{1}{k_1(k_1+1)} \int_{\Omega'} L_{K'}(x') dx' \int_{-1}^{+1} u(x_1, x') \tfrac{d}{dx_1}((1 - x_1^2) L'_{k_1}(x_1) dx_1$$

$$= -\tfrac{1}{\|L_K\|^2_{L^2(\Omega)}} \tfrac{1}{k_1(k_1+1)} \int_{\Omega'} L_{K'}(x') dx' \int_{-1}^{+1} u(x_1, x') A_1 L_{k_1}(x_1)) dx_1$$

and from twice integration by parts we obtain

$$\widehat{u}_K = -\tfrac{1}{\|L_K\|^2_{L^2(\Omega)}} \tfrac{1}{k_1(k_1+1)} \int_{\Omega'} L_{K'}(x') dx' \int_{-1}^{+1} A_1 u(x_1, x') L_{k_1}(x_1) dx_1$$

$$= -\tfrac{1}{\|L_K\|^2_{L^2(\Omega)}} \tfrac{1}{k_1(k_1+1)} \int_\Omega A_1 u(x) L_K(x) dx.$$

Iterating p-time this result, we obtain

$$\widehat{u}_K = \frac{1}{\|L_K\|^2_{L^2(\Omega)}} (-\frac{1}{k_1(k_1+1)})^p \int_\Omega A_1^p u(x) L_K(x) dx.$$

For $1 \leq j \leq d$ one has

$$\widehat{u}_K = \frac{1}{\|L_K\|^2_{L^2(\Omega)}} (-\frac{1}{k_j(k_j+1)})^p \int_\Omega A_j^p u(x) L_K(x) dx$$

and

$$|\widehat{u}_K|^2 = \frac{1}{\|L_K\|^2_{L^2(\Omega)}}\left(\frac{1}{k_j(k_j+1)}\right)^{2p}\left(\frac{\int\limits_\Omega A_j^p u(x)L_K(x)dx}{\|L_K\|^2_{L^2(\Omega)}}\right)^2$$

implies

$$\sum_{K\in K^{(j)}(N)} |\widehat{u}_K|^2 \|L_K\|^2_{L^2(\Omega)} = \left(\frac{1}{k_j(k_j+1)}\right)^{2p} \sum_{K\in K^{(j)}(N)} \frac{(\int\limits_\Omega A_j^p u(x)L_K(x)dx)^2}{\|L_K\|^2_{L^2(\Omega)}}.$$

Lower $k_j(k_j+1)$ by N^2, we obtain

$$\sum_{K\in K^{(j)}(N)} |\widehat{u}_K|^2 \|L_K\|^2_{L^2(\Omega)} \le N^{-4p} \sum_{K\in K^{(j)}(N)} \frac{\left|\int\limits_\Omega A_j^p u(x)L_K(x)dx\right|^2}{\|L_K\|^2_{L^2(\Omega)}}$$

$$\le N^{-4p} \sum_{K\in K^{(j)}(N)} \left\|A_j^p u\right\|^2_{L^2(\Omega)} \le cN^{-4p}\left\|A_j^p u\right\|^2_{L^2(\Omega)}.$$

By ([1]) operator A_j^p is continuous from $H^{2p}(\Omega)$ into $L^2(\Omega)$, hence one has

$$\sum_{K\in K^{(j)}(N)} |\widehat{u}_K|^2 \|L_K\|^2_{L^2(\Omega)} \le N^{-4p}\|u\|^2_{H^{2p}(\Omega)}.$$

But

$$\|u-P_Nu\|^2_{L^2(\Omega)} = \sum_{j=1}^d \sum_{K\in K^{(j)}(N)} |\widehat{u}_K|^2 \|L_K\|^2_{L^2(\Omega)}$$

therefore

$$\|u-P_Nu\|^2_{L^2(\Omega)} \le cN^{-2p}\|u\|_{H^{2p}(\Omega)} = cN^{-s}\|u\|_{H^s(\Omega)}.$$

- If s is odd, i.e., $s = 2p+1$:
  in the same way for s odd, by interpolation we obtain the result.

□

Proposition 4. *([5]) Let*

$$u(x_i) = \sum_{n=0}^{\infty} \widehat{u}_n L_n(x_i)$$

then

$$u'(x_i) = \sum_{n=1}^{\infty} \widehat{u}_n L_n'(x_i) = \sum_{n=0}^{\infty} \widehat{z}_n L_n(x_i),$$

where

$$\widehat{z}_n = (2n+1)\sum_{p=n+1,\, p+n \text{ odd}}^{\infty} \widehat{u}_p.$$

Lemma 3. *Let $u \in H^1(\Omega)$ and $D_1u = \sum_{K\in\mathbb{N}^d} \widehat{z}_K L_K$. Then*

$$P_ND_1u - D_1P_Nu = \begin{cases} Z^{(N)}L_0^{(N)} + Z^{(N+1)}L_1^{(N)} & \text{if } N \text{ is even} \\ Z^{(N)}L_1^{(N)} + Z^{(N+1)}L_0^{(N)} & \text{if } N \text{ is odd,} \end{cases}$$

where

$$Z^{(N)} = \sum_{K'\in\mathbb{N}^{d-1},\, |K'|_\infty \le N}\ \sum_{m=N+1,\, m+N \text{ odd}}^{\infty} \widehat{u}_{(m,K')}L_{K'}$$

$$Z^{(N+1)} = \sum_{K'\in\mathbb{N}^{d-1},\, |K'|_\infty \le N}\ \sum_{m=N+2,\, m+N+1 \text{ odd}}^{\infty} \widehat{u}_{(m,K')}L_{K'}$$

and

$$L_0^{(N)} = \sum_{l_1=0,\, l_1\ even}^{N} (2l_1+1)L_{l_1}, \quad L_1^{(N)} = \sum_{l_1=1,\, l_1\ odd}^{N} (2l_1+1)L_{l_1}.$$

Proof. By Proposition 4 one has

$$\widehat{z}_K = (2k_1+1) \sum_{m=k_1+1,\ m+k_1 \text{ odd}}^{\infty} \widehat{u}_{(m,K')}$$

therefore, one has

$$D_1 u = \sum_{K\in\mathbb{N}^d} ((2k_1+1) \sum_{m=k_1+1,\ m+k_1 \text{ odd}}^{\infty} \widehat{u}_{(m,K')})L_K$$

and

$$P_N D_1 u = \sum_{K'\in\mathbb{N}^{d-1},\ |K'|_\infty \le N} L_{K'} \ (\sum_{l_1=0}^{N}(2l_1+1)(\sum_{k_1=l_1+1,\ k_1+l_1 \text{ odd}}^{\infty} \widehat{u}_{(k_1,K')})L_{l_1})$$

$$D_1 P_N u = \sum_{K'\in\mathbb{N}^{d-1},\ |K'|_\infty \le N} L_{K'} \ (\sum_{l_1=0}^{N-1}(2l_1+1)(\sum_{k_1=l_1+1,\ k_1+l_1 \text{ odd}}^{N} \widehat{u}_{(k_1,K')})L_{l_1}).$$

Then

$$P_N D_1 u - D_1 P_N u = \sum_{K'\in\mathbb{N}^{d-1},\ |K'|_\infty \le N} L_{K'}(\sum_{l_1=0}^{N}(2l_1+1)(\sum_{k_1=l_1+1,\ k_1+l_1 \text{ odd}}^{\infty} \widehat{u}_{(k_1,K')})L_{l_1})$$

$$- \sum_{K'\in\mathbb{N}^{d-1},\ |K'|_\infty \le N} L_{K'} \ (\sum_{l_1=0}^{N-1}(2l_1+1)(\sum_{k_1=l_1+1,\ k_1+l_1 \text{ odd}}^{N} \widehat{u}_{(k_1,K')})L_{l_1}).$$

But

$$\sum_{l_1=0}^{N} = \sum_{l_1=0,\ l_1 \text{ even}}^{N} + \sum_{l_1=1,\ l_1 \text{ odd}}^{N}.$$

So

$$P_N D_1 u - D_1 P_N u = \sum_{K'\in\mathbb{N}^{d-1},\ |K'|_\infty \le N} L_{K'}((\sum_{l_1=0,\ l_1 \text{ even}}^{N}(2l_1+1)(\sum_{k_1=l_1+1,\ k_1+l_1 \text{ odd}}^{\infty} \widehat{u}_{(k_1,K')})L_{l_1})+$$

$$\sum_{K'\in\mathbb{N}^{d-1},\ |K'|_\infty \le N} L_{K'}((\sum_{l_1=1,\ l_1 \text{ odd}}^{N}(2l_1+1)(\sum_{k_1=l_1+1,\ k_1+l_1 \text{ odd}}^{\infty} \widehat{u}_{(k_1,K')})L_{l_1})-$$

$$\sum_{K'\in\mathbb{N}^{d-1},\ |K'|_\infty \le N} L_{K'}((\sum_{l_1=0,\ l_1 \text{ even}}^{N-1}(2l_1+1)(\sum_{k_1=l_1+1,\ k_1+l_1 \text{ odd}}^{N} \widehat{u}_{(k_1,K')})L_{l_1})-$$

$$\sum_{K'\in\mathbb{N}^{d-1},\ |K'|_\infty \le N} L_{K'}((\sum_{l_1=1,\ l_1 \text{ odd}}^{N-1}(2l_1+1)(\sum_{k_1=l_1+1,\ k_1+l_1 \text{ odd}}^{N} \widehat{u}_{(k_1,K')})L_{l_1}).$$

If N is even, one has

$$P_N D_1 u - D_1 P_N u = \sum_{K'\in\mathbb{N}^{d-1},\ |K'|_\infty\le N} L_{K'}((\sum_{l_1=0,\ l_1\text{ even}}^{N}(2l_1+1)(\sum_{k_1=l_1+1,\ k_1+l_1\text{ odd}}^{N}\widehat{u}_{(k_1,K')})L_{l_1})+$$

$$\sum_{K'\in\mathbb{N}^{d-1},\ |K'|_\infty\le N} L_{K'}((\sum_{l_1=0,\ l_1\text{ odd}}^{N}(2l_1+1)(\sum_{k_1=N+1,\ k_1+N\text{ odd}}^{\infty}\widehat{u}_{(k_1,K')})L_{l_1})+$$

$$\sum_{K'\in\mathbb{N}^{d-1},\ |K'|_\infty\le N} L_{K'}((\sum_{l_1=0,\ l_1\text{ odd}}^{N}(2l_1+1)(\sum_{k_1=l_1+1,\ k_1+l_1\text{ odd}}^{N+1}\widehat{u}_{(k_1,K')})L_{l_1})+$$

$$\sum_{K'\in\mathbb{N}^{d-1},\ |K'|_\infty\le N} L_{K'}((\sum_{l_1=0,\ l_1\text{ odd}}^{N}(2l_1+1)(\sum_{k_1=N+2,\ k_1+N+1\text{ odd}}^{\infty}\widehat{u}_{(k_1,K')})L_{l_1})-$$

$$\sum_{K'\in\mathbb{N}^{d-1},\ |K'|_\infty\le N} L_{K'}((\sum_{l_1=0,\ l_1\text{ even}}^{N-1}(2l_1+1)(\sum_{k_1=l_1+1,\ k_1+l_1\text{ odd}}^{N}\widehat{u}_{(k_1,K')})L_{l_1})-$$

$$\sum_{K'\in\mathbb{N}^{d-1},\ |K'|_\infty\le N} L_{K'}((\sum_{l_1=1,\ l_1\text{ odd}}^{N-1}(2l_1+1)(\sum_{k_1=l_1+1,\ k_1+l_1\text{ odd}}^{N}\widehat{u}_{(k_1,K')})L_{l_1}).$$

Therefore

$$P_N D_1 u - D_1 P_N u = \sum_{K'\in\mathbb{N}^{d-1},\ |K'|_\infty\le N} L_{K'}((\sum_{l_1=0,\ l_1\text{ even}}^{N}(2l_1+1)(\sum_{k_1=N+1,\ k_1+N\text{ odd}}^{\infty}\widehat{u}_{(k_1,K')})L_{l_1})+$$

$$\sum_{K'\in\mathbb{N}^{d-1},\ |K'|_\infty\le N} L_{K'}((\sum_{l_1=1,\ l_1\text{ odd}}^{N}(2l_1+1)(\sum_{k_1=N+2,\ k_1+N+1\text{ odd}}^{\infty}\widehat{u}_{(k_1,K')})L_{l_1})$$

$$= Z^{(N)}L_0^{(N)} + Z^{(N+1)}L_1^{(N)}.$$

In the same way for N odd. □

Theorem 5. *([9]) For all reals $0 \le \nu \le \mu$ one has*

$$\|u\|_{H^\mu(\Omega)} \le cN^{2(\mu-\nu)}\|u\|_{H^\nu(\Omega)},\quad \forall u\in V_N^d(\Omega).$$

Lemma 4. *For all reals s and t, $0\le s\le t-1$, there exists a constant $c>0$ such that for any $j=1,...,d$, one has*

$$\|(P_N D_j - D_j P_N)u\|_{H^s(\Omega)} \le cN^{2s-t+\frac{3}{2}}\|u\|_{H^t(\Omega)},\quad \forall u\in H^t(\Omega).$$

Proof. For $j=1$, we remark that $Z^{(N)}, Z^{(N+1)}$ dependent from $x' = (x_2,...,x_d)$ and $Z^{(N)},\ Z^{(N+1)}\in V_N^{d-1}(\Omega);\ L_0^{(N)},\ L_1^{(N)}$ dependent from x_1, and orthogonals in $L^2(I)$.

- For N even, one has

$$\|(P_N D_1 - D_1 P_N)u\|^2_{L^2(\Omega)} = (Z^{(N)}L_0^{(N)} + Z^{(N+1)}L_1^{(N)},\ Z^{(N)}L_0^{(N)} + Z^{(N+1)}L_1^{(N)})$$

$$= (Z^{(N)}L_0^{(N)},\ Z^{(N)}L_0^{(N)}) + 2(Z^{(N)}L_0^{(N)},\ Z^{(N+1)}L_1^{(N)}) + (Z^{(N+1)}L_1^{(N)},\ Z^{(N+1)}L_1^{(N)})$$

$$= \|Z^{(N)}\|^2_{L^2(\Omega)}\left\|L_0^{(N)}\right\|^2_{L^2(\Omega)} + \|Z^{(N+1)}\|^2_{L^2(\Omega)}\left\|L_1^{(N)}\right\|^2_{L^2(\Omega)}.$$

And

$$\left\|Z^{(N)}\right\|_{L^2(\Omega)}^2=\left\|\sum_{K'\in\mathbb{N}^{d-1},\ \left|K'\right|_\infty\le N}\ \sum_{m=N+1,\ m+N \text{ odd}}^{\infty}\widehat{u}_{(m,K')})L_{K'}\right\|_{L^2(\Omega)}^2\le\|D_1u-P_{N-1}D_1u\|_{L^2(\Omega)}^2.$$

By Theorem 4, one has

$$\left\|Z^{(N)}\right\|_{L^2(\Omega)}^2\le cN^{2(1-t)}\|D_1u\|_{H^{t-1}(\Omega)}^2\le cN^{2(1-t)}\|u\|_{H^t(\Omega)}^2$$

and

$$\left\|Z^{(N+1)}\right\|_{L^2(\Omega)}^2=\left\|\sum_{K'\in\mathbb{N}^{d-1},\ \left|K'\right|_\infty\le N}\ \sum_{m=N+2,\ m+N+1 \text{ odd}}^{\infty}\widehat{u}_{(m,K')}L_{K'}\right\|_{L^2(\Omega)}^2$$

$$\le c\|D_1u-P_ND_1u\|_{L^2(\Omega)}^2\le cN^{2(1-t)}\|D_1u\|_{H^{t-1}(\Omega)}^2\le cN^{2(1-t)}\|u\|_{H^t(\Omega)}^2,$$

and

$$\left\|L_0^{(N)}\right\|_{L^2(\Omega)}^2=\left\|\sum_{l_1=0,\ l_1 \text{ odd}}^{N}(2l_1+1)L_{l_1}\right\|_{L^2(\Omega)}^2.$$

Bounded $(2l_1+1)$ by $(2N+1)$, hence $\left\|L_0^{(N)}\right\|_{L^2(\Omega)}^2\le cN$. In the same way for $\left\|L_1^{(N)}\right\|_{L^2(\Omega)}^2\le cN$, therefore

$$\|(P_ND_1-D_1P_N)u\|_{L^2(\Omega)}^2\le cNN^{2(1-t)}\|u\|_{H^t(\Omega)}^2=cN^{3-2t}\|u\|_{H^t(\Omega)}^2.$$

Using Theorem 5, one has

$$\|(P_ND_1-D_1P_N)u\|_{H^s(\Omega)}\le cN^{2s}\|(P_ND_1-D_1P_N)u\|_{L^2(\Omega)}$$

$$\le cN^{2s}N^{\frac{3}{2}-t}\|u\|_{H^t(\Omega)}=cN^{2s-t+\frac{3}{2}}\|u\|_{H^t(\Omega)}.$$

- In the same way for N odd.

□

Theorem 6. *([3]) For all reals $s,t,0\le s\le t$, there exists a constant $c>0$ such that*

$$\|u-P_Nu\|_{H^s(\Omega)}\le cN^{e(s,t)}\|u\|_{H^t(\Omega)},\ \ \forall u\in H^t(\Omega),$$

where

$$e(s,t)=\begin{cases}2s-t-\frac{1}{2} & \text{if } s\ge 1,\\[2ex] \frac{3}{2}s-t & \text{if } 0\le s\le 1.\end{cases}$$

Notation: Denote $P_N^{1,0}$ the orthogonal projection operator from $H_0^1(\Omega)$ on $V_N^{d,0}(\Omega)$ with

$$V_N^{d,0}(\Omega)=\left\{v\in V_N^d(\Omega);\ v=0\ \text{ on }\ Fr(\Omega)\right\}.$$

This means that

$$(u-P_N^{1,0}u,\ \phi_N)_{1,0,\Omega}=0,\ \ \forall\phi_N\in V_N^{d,0}(\Omega),$$

where

$$(u,v)_{1,0,\Omega}=\int_\Omega\nabla u\nabla v dx.$$

Theorem 7. *([1]) For all integer $m\ge 1$, there exists a constant $c>0$ dependent of m such that for all function $u\in H^m(\Omega)\cap H_0^1(\Omega)$, one has*

$$\left|u-P_N^{1,0}u\right|_{H^1(\Omega)}\le cN^{1-m}\|u\|_{H^m(\Omega)}.$$

Theorem 8. *([1]) For all integer $m \geq 1$, there exists a constant $c > 0$ dependent of m such that for all function $u \in H^m(\Omega) \cap H_0^1(\Omega)$, one has*

$$\left\| u - P_N^{1,0} u \right\|_{L^2(\Omega)} \leq cN^{-m} \left\| u \right\|_{H^m(\Omega)} .$$

Theorem 9. *([6]) For all reals $\nu > 1$, one has*

$$\left| u - P_N^{1,0} u \right|_{H^\mu(\Omega)} \leq cN^{\mu-\nu} \left\| u \right\|_{H^\nu(\Omega)} , \ \ 0 < \mu < 1.$$

Proof. By interpolation between Theorem 7 and Theorem 8. □

Theorem 10. *For all integer $m \geq 1$ and for all function $u \in H^m(\Omega) \cap H_0^1(\Omega)$, one has*

$$|u - u_N|_{H^1(\Omega)} + N \left\| u - u_N \right\|_{L^2(\Omega)} \leq cN^{1-m} \left\| u \right\|_{H^m(\Omega)} .$$

3.2. **Numerical Solutions.**

3.2.1. *Choice of Space $V_N^{d,0}(\Omega)$.*

Galerkin approximation method of (P_0) gives

$$(P_N) \qquad \begin{cases} \text{Find } \ u_N \in V_N^{d,0}(\Omega) \text{ such that} \\ \\ (\nabla u_N, \ \nabla v_N) = (f, v_N), \ \ \forall v_N \in V_N^{d,0}(\Omega), \end{cases}$$

where

$$(u, v) = \int_\Omega uv dx$$

is a scalar product in $L^2(\Omega)$.

Galerkin approach consists to replace test functions space by high degree polynomials space. A most disadvantage of this approach requires integral calculus, which is not always easy to do and at the same time very expensive.

The effectiveness of numerical method which has been given in the abstract form will be subordinate to:

(i) the way from which space $V_N^{d,0}(\Omega)$ approaches the space V;

(ii) steepness and simpleness calculus of coefficients a_{ij} and F_j;

(iii) steepness to resolve a linear system $Au = F$.

To satisfy the $1st$ criterion (i), we will consider the space $V_N^{d,0}(\Omega)$ of enough large dimension.

To satisfy the $2nd$ and $3rd$ criteria (ii) and (iii), it will required obtain one sufficiently deep matrix A such that the linear system $Au = F$ does not enough at cost (in time and required space machine), and such that there are not coefficients a_{ij} to compute.

What is to be done? Select a basis of $V_N^{d,0}(\Omega)$ such that a linear system to resolve being easy and possible. To answer to this question, we need the following lemma:

Lemma 5. *([8]) Put*

$$c_k = \tfrac{1}{\sqrt{4k+6}}, \qquad \phi_k(x) = c_k(L_k(x) - L_{k+2}(x))$$

$$a_{jk} = (\phi_k^{'}(x), \ \phi_j^{'}(x)), \qquad b_{jk} = (\phi_k(x), \ \phi_j(x)).$$

Then

$$a_{jk} = \begin{cases} 1 \ \ \textit{if} \ \ k = j \\ \\ 0 \ \ \textit{if} \ \ k \neq j, \end{cases}$$

$$b_{kj} = b_{jk} = \begin{cases} c_k c_j (\frac{2}{2j+1} + \frac{2}{2j+5}), & \textit{if} \ \ k = j, \\ \\ -c_k c_j & \textit{if} \ \ k = j+2, \\ \\ 0 & \textit{else}, \end{cases}$$

and

$$V_N^{1,0}(I) = Span\left\{\phi_0(x),\ \phi_1(x),\ ...,\ \phi_{N-2}(x)\right\}.$$

Distinguish three cases:

- **Case 1: d=1**, in this case the problem (P_N) amounts to

$$(u_N^{'},\ \phi_k^{'}(x)) = (f,\ \phi_k(x)), \quad k = 0, 1,\ ...,\ N-2.$$

Denote

$$f_k = (f,\ \phi_k(x)), \quad F = (f_0, f_1,\ ...,\ f_{N-2})^t, \quad u_N = \sum_{k=0}^{N-2} u_k \phi_k, \quad u = (u_0, u_1,\ ..., u_{N-2})^t,$$

hence, one has $u_k = f_k$.

- **Case 2: d=2**, in this case, one has

$$V_N^{2,0}(\Omega) = Span\left\{\phi_i(x)\phi_j(y),\ i,\ j = 0, 1,\ ...,\ N-2\right\}.$$

Denote

$$u_N = \sum_{k,j=0}^{N-2} u_{k,j}\phi_k(x)\phi_j(y), \quad f_{kj} = (f,\ \phi_k(x)\phi_j(y))$$

and

$$U = (u_{kj})_{k,j=0,\dots,N-2}, \quad F = (f_{kj})_{k,j=0,1,\dots,N-2}, \quad B = (b_{kj})_{k,j=0,1,\dots,N-2}.$$

Put

$$v = \phi_l(x)\phi_m(y), \quad l, m = 0, 1, ..., N-2,$$

therefore the problem (P_N) amounts to

(1) $$UB + BU = F.$$

Now, let Λ be a diagonal matrix consists of eigenvalues of B. Let E be an orthonormal matrix consists of eigenvectors of B such that $E^tBE = \Lambda$.

Put $U = EV$, where V is a variable matrix, therefore equation (1) amounts to

$$EVB + E\Lambda V = F.$$

Multiplying both two sides by E^t and using a relation $E^tE = EE^t = I$, then

(2) $$BV^t + V^t\Lambda = (E^tF)^t = G^t$$

where $G = E^tF$. Now let,

$$v_p = (v_{p_0},\ v_{p_1},\ ...,\ v_{p_{N-2}})^t \quad \text{and} \quad g_p = (g_{p_0},\ g_{p_1},\ ...,\ g_{p_{N-2}})^t,\ p = 0, 1,\ ...,\ N-2,$$

hence equation (2) amounts to

(3) $$(B + \lambda_p I)v_p = g_p, \quad p = 0, 1, ..., N-2,$$

where λ_p are eigenvalues of a matrix B.

Therefore, to resolve (P_0) in the case $d = 2$, we should to:
(i) compute eigenvalues and eigenvectors of a matrix B;
(ii) compute $G = E^tF$;
(iii) resolve system (3) to obtain V, and put $U = EV$.

- **Case 3: d=3**, in this case, one has

$$V_N^{3,0}(\Omega) = Span\left\{\phi_i(x)\phi_j(y)\phi_k(z),\ i,\ j, k = 0, 1,\ ...,\ N-2\right\}.$$

Denote

$$u_N = \sum_{n,m,l=0}^{N-2} u_{n,m,l}\phi_n(x)\phi_m(y)\phi_l(z), \quad f_{ijk} = (f,\ \phi_i(x)\phi_j(y)\phi_k(z))$$

and

$$U = (u_{kj})_{k,j=0,\dots,N-2}, \quad F = (f_{kj})_{k,j=0,1,\dots,N-2}, \quad B = (b_{kj})_{k,j=0,1,\dots,N-2}.$$

Put

$$v = \phi_i(x)\phi_j(y)\phi_k(z), \quad i, j, k = 0, 1, ..., N-2,$$

hence the problem (P_N) amounts to

$$u_{iml}b_{jm}b_{kl} + b_{in}u_{njl}b_{kl} + b_{in}u_{nmk}b_{jm} = f_{ijk}, \quad i, j, k = 0, 1, ..., \ N-2 \tag{4}$$

By definition of E and Λ, one has

$$b_{in}e_{nq} = \lambda_q e_{iq}, \qquad e_{iq}e_{ip} = \delta_{qp}.$$

Put $u_{nml} = e_{nq}v_{qml}$, hence (4) can be written in the form

$$e_{iq}v_{qml}b_{jm}b_{kl} + \lambda_q e_{iq}v_{qjl}b_{kl} + \lambda_q e_{iq}v_{nmk}b_{jm} = f_{ijk}, \quad i, j, k = 0, 1, ..., \ N-2$$

Multiplying both two sides by e_{ip}, we obtain

$$u_{pml}b_{jm}b_{kl} + \lambda_p(v_{pjl}b_{kl} + v_{pmk}b_{jm}) = g_{pjk}, \quad p, j, k = 0, 1, \ ..., \ N-2 \tag{5}$$

Let $V^p = (v_{pml})_{0\leq m,l\leq N-2}$ and $G^p = (g_{pml})_{0\leq m,l\leq N-2}$, hence a system (5) can be written in the form

$$BV^pB + \lambda_p(V^pB + BV^p) = G^p, \quad p = 0, 1, \ ..., \ N-2. \tag{6}$$

Therefore, to resolve (P_0) in the case $d = 3$, we should to:
(i) compute eigenvalues and eigenvectors of the matrix B;
(ii) compute $g_{pjk} = e_{ip}f_{ijk}$;
(iii) resolve system (6) to obtain V^p, and put $u_{nml} = e_{ml}v_{qml}$.

3.2.2. *Numerical Results.*

Example 1. *Consider the following problem*

$$\begin{cases} -\Delta u = 16\pi^2 \sin(4\pi x) & in \ \ \Omega =]-1, +1[\\ \\ u = 0 & on \ \ Fr(\Omega). \end{cases}$$

This problem has one and only one solution: $u(x) = sin(4\pi x)$

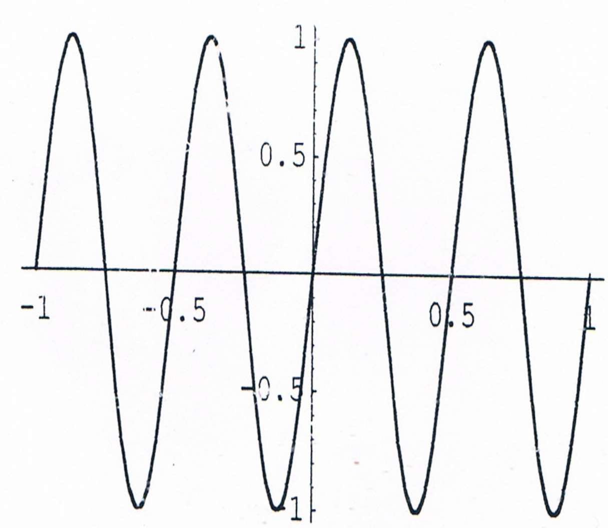

FIGURE 1. Exact solution

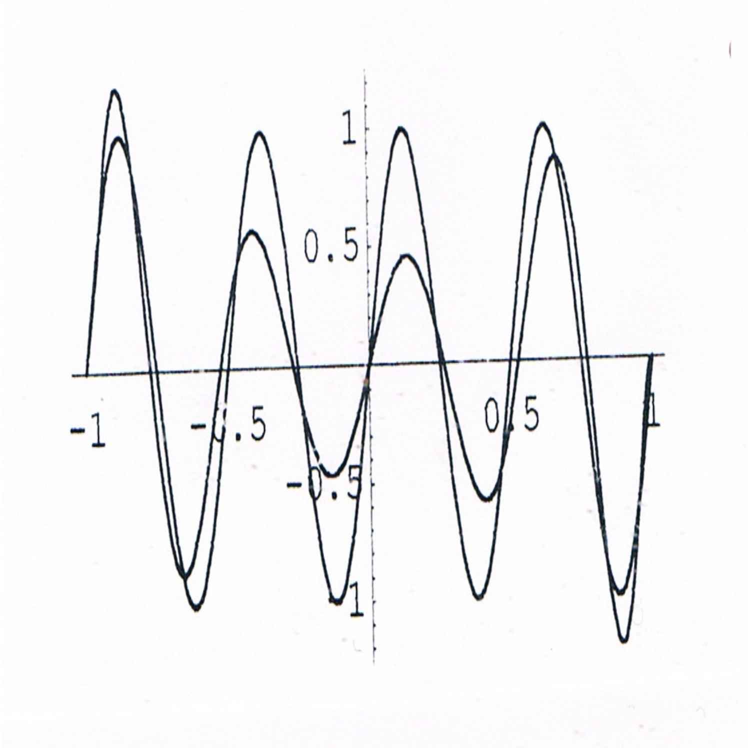

FIGURE 2. Comparison between exact and approach solutions for N = 11.

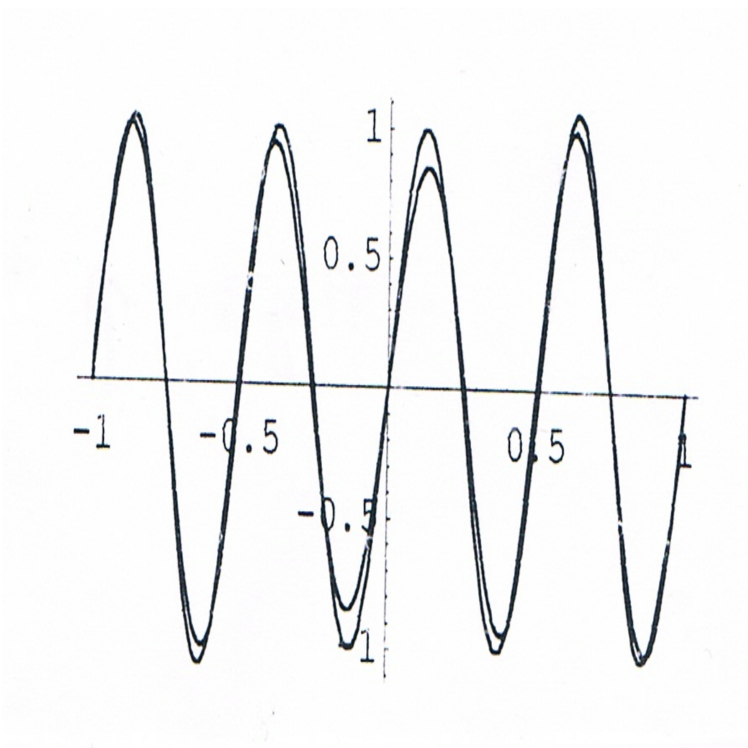

FIGURE 3. Comparison between exact and approach solutions for N = 13.

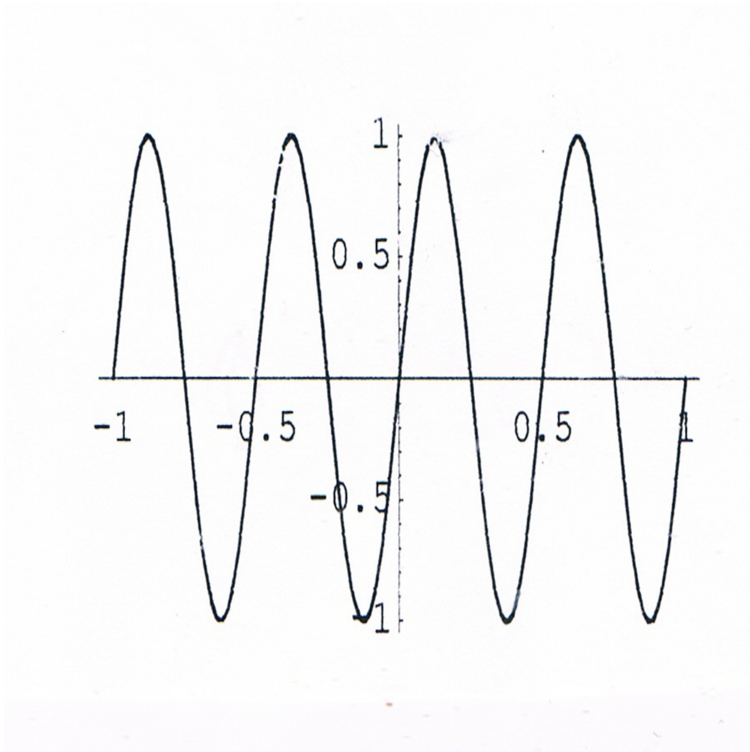

FIGURE 4. Comparison between exact and approach solutions for N = 16.

Example 2. *Consider the following problem*

$$\begin{cases} -\Delta u = 8\pi^2 \sin(2\pi x)\sin(2\pi y) \;\; in \;\; \Omega =]-1,+1[^2 \\ u = 0 \qquad\qquad\qquad\qquad\qquad on \;\; Fr(\Omega). \end{cases}$$

This problem has one and only one solution: $u(x,y) = sin(2\pi x)sin(2\pi y)$.

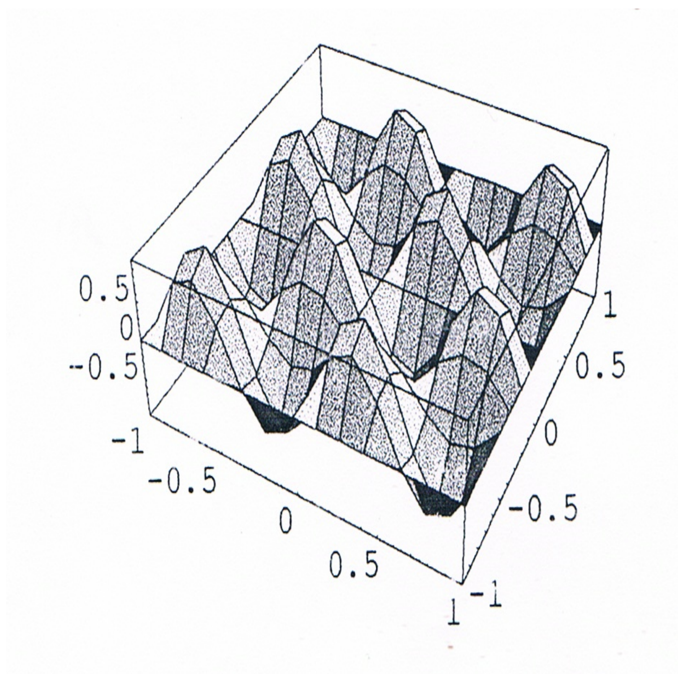

FIGURE 5. Exact solution

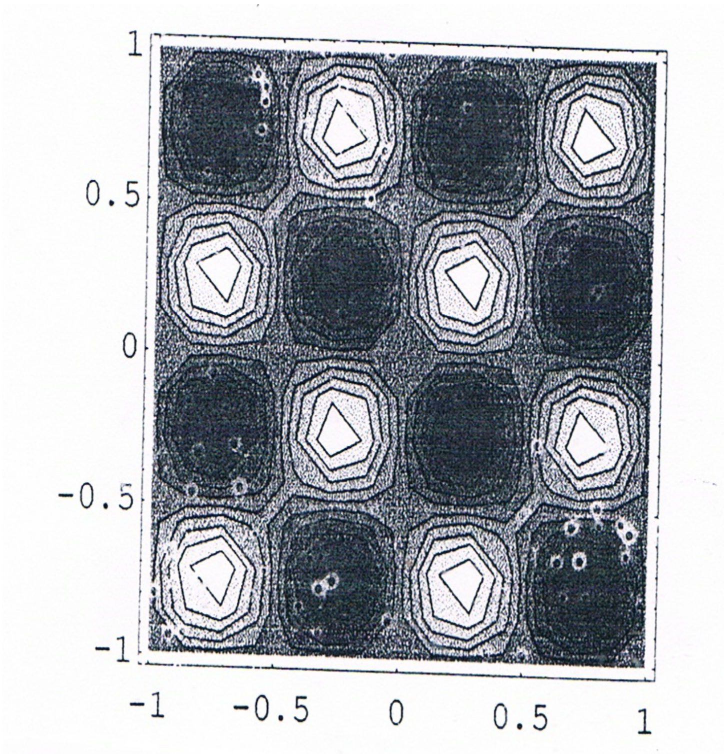

FIGURE 6. Exact solution (with line of contour)

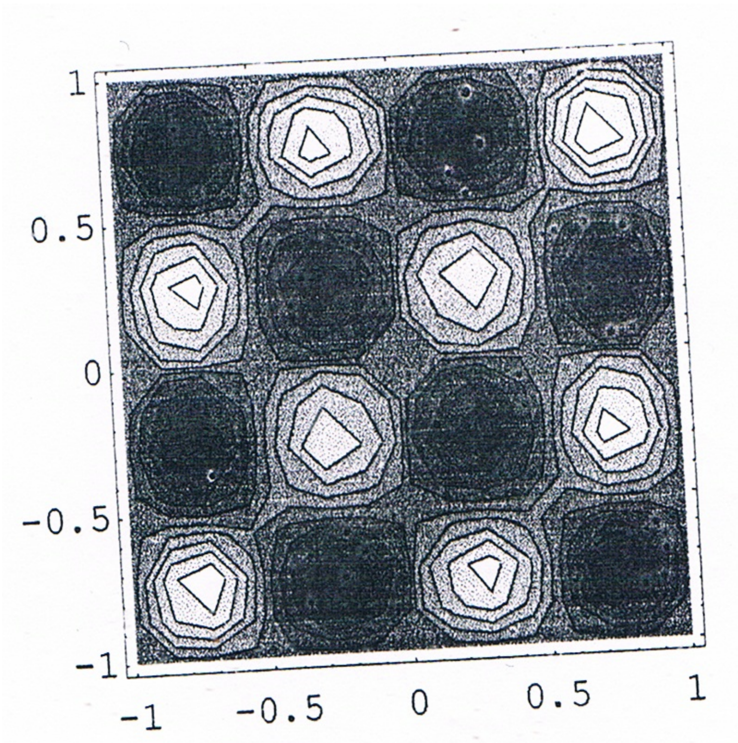

FIGURE 7. Approach solution for N=7

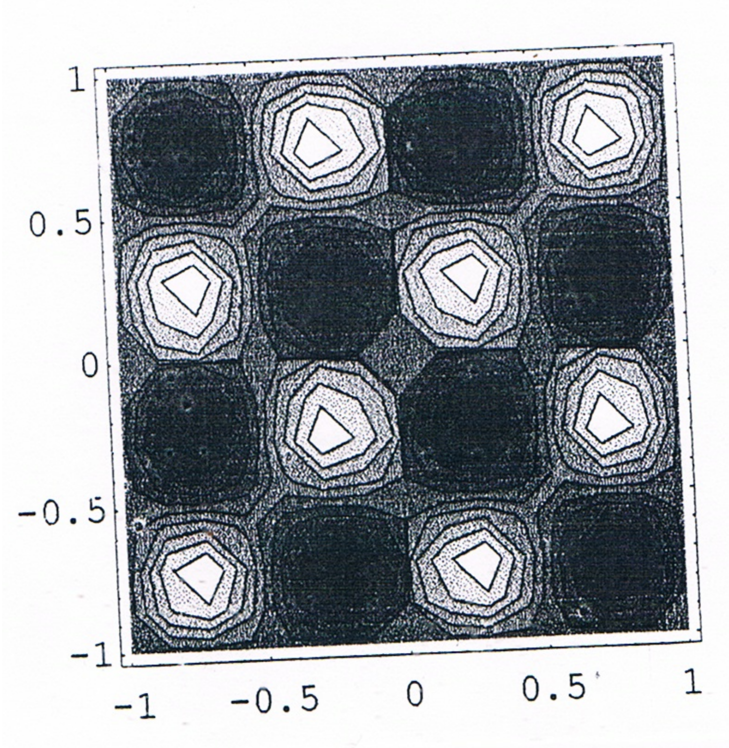

FIGURE 8. Approach solution for N=9

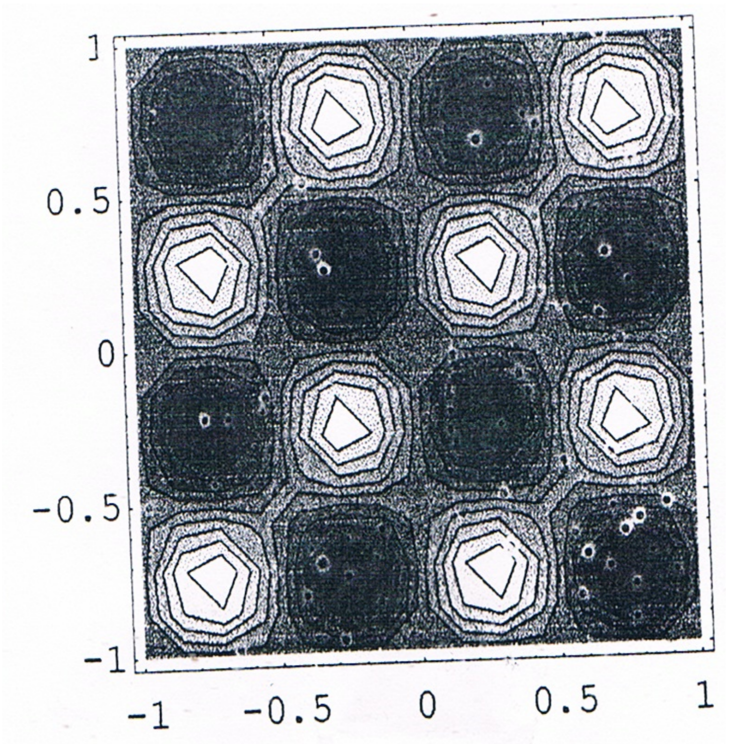

FIGURE 9. Approach solution for N=11

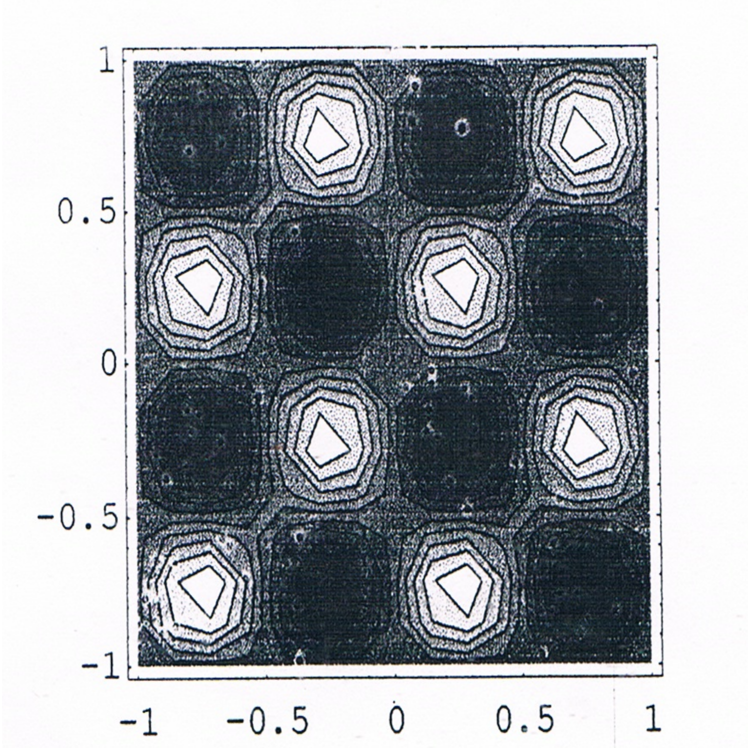

FIGURE 10. Approach solution for N=13

4. Conclusions

Numerical tests prove the best quality of these basis functions. The choice is very efficient in practice. Further, associated matrices to this choice are positive definite, symmetric and sufficiently deep. Numerical tests prove our good choice and the convergence.

References

[1] C. Bernardi and Y. Maday, Approximation spectrale de problèmes aux limites elliptiques, Springer-Verlag, Berlin, 1992.

[2] C. Canuto, M.Y. Hussaini, A. Quarteroni and T.A. Zang, Spectral Methods in Fluid Dynamic, Springer-Verlag, Berlin, Heidelberg, 1987.

[3] C. Canuto and A. Quarteroni, Approximation results for orthogonal polynomials in Sobolev spaces, Math. Comput. Vol. 38 (1982), pp. 67-86.

[4] R. Dautray and J.L. Lions, Analyse mathématique et calcul numérique pour les sciences et les techniques, Vol. 1 et 3, Masson, Paris, 1987.

[5] D. Gottlieb and S.A. Orszag, Numerical Analysis of Spectral Methods, Theory and Application, SIAM Publications, Philadelphia, 1977.

[6] Y. Maday and A. Quarteroni, Legendre and Chebyshev Spectral approximations of Burgers Equations, Numer. Math. Vol. 37 (1981), pp. 321-332.

[7] P.A. Raviart and J.M. Thomas, Introduction à l'analyse numérique des équations aux dérivées partielles, Masson, Paris, 1983.

[8] J. Shen, Efficient spectral-Galerkin method, I. Direct solvers of second and fourth-order using Legendre polynomials, SIAM. Jour. Sci. Comput. 5 (1994), pp. 1489-1505.

[9] G. Szegö, Orthogonal Polynomials, Colloquium, Publications AMS, Providence, 1978.