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Existence and controllability result for fractional neutral stochastic integro-differential equations with infinite delay

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Abstract

This paper is concerned with the existence of mild solutions and the approximate controllability for a class of fractional neutral stochastic integro-differential equations with infinite delay in Hilbert spaces. Firstly, a sufficient condition for the existence is obtained under non-Lipschitz conditions by means of Sadovskii's fixed point theorem. Secondly, the approximate controllability of nonlinear fractional stochastic system is discussed, under the assumption that the corresponding linear system is approximately controllable. Finally, an example is given to illustrate the theory.

Key words and phrases: Existence result, approximate controllability, stochastic fractional differential equations, fixed point technique, infinite delay.
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1 Introduction

The subject of fractional calculus and its applications has gained a lot of importance during the past three decades, mainly because it has become a powerful tool in modeling several complex phenomena in numerous seemingly diverse and widespread fields of science and engineering [7, 10, 15]. Recently, there has been a significant development in the existence and uniqueness of solutions of initial and boundary value problem for fractional differential equations [24]. Neutral differential equations arise in many areas of applied mathematics and for this reason these equations have received much attention in the last decades. But the literature related to neutral fractional differential equations is very limited and we refer the reader to [23].

On the other hand, the study of controllability plays an important role in the control theory and engineering [2, 11]. In recent years, various controllability problems for different kinds of dynamical systems have been studied in many publications [1, 3, 5, 6]. From the mathematical point of view, the problems of exact and approximate controllability are to be distinguished. However, the concept of exact controllability is usually too strong and has limited applicability.

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Approximate controllability is a weaker concept than complete controllability and it is completely adequate in applications [4, 12]. In particular, the fixed point techniques are widely used in studying the controllability problems for nonlinear control systems. Klamka [8] studied the practical applicability of the fixed point theorem in solving various controllability problems for different types of dynamical control systems. Wang [22] derived a set of sufficient conditions for the approximate controllability of differential equations with multiple delays by implementing some natural conditions such as growth conditions for the nonlinear term and compactness of the semigroup. Sakthivel and Anandhi [18] investigated the problem of approximate controllability for a class of nonlinear impulsive differential equations with state-dependent delay by using semigroup theory and fixed point technique.

Sakthivel et al. [19] studied the approximate controllability of nonlinear deterministic and stochastic evolution systems with unbounded delay in abstract spaces. The same author et al. [20] studied the approximate controllability of deterministic semilinear fractional differential equations in Hilbert spaces. Kumar and Sukavanam [9] obtained a new set of sufficient conditions for the approximate controllability of a class of semilinear delay control systems of fractional order by using the contraction principle and the Schauder fixed point theorem. More recently, Sakthivel et al. [21] derived a new set of sufficient conditions for approximate controllability of fractional stochastic differential equations by using the Banach contraction principale.

In this paper, we are interested in the existence of mild solutions and the approximate controllability for a class of fractional neutral stochastic integro-differential equations with infinite delay of the form

$$\begin{cases} {}^cD_t^\alpha[x(t)+G(t,x_t)] = -Ax(t)+Bu(t)+f(t,x_t) \\ \qquad\qquad +\int_{-\infty}^t \sigma(t,s,x_s)dw(s), \quad t\in J:=[a,b], \\ x(t)=\phi(t), \qquad t\in(-\infty,0]. \end{cases} \tag{1}$$

Here, $x(.)$ takes value in a real separable Hilbert space $\mathcal{H}$ with inner product $(.,.)_{\mathcal{H}}$ and norm $\|.\|_{\mathcal{H}}$. The fractional derivative ${}^cD^\alpha$, $\alpha\in(0,1)$, is understood in the Caputo sense. $-A:\mathcal{D}(-A)\subset\mathcal{H}\to\mathcal{H}$ is the infinitesimal generator of a strongly continuous semigroup of a bounded linear operator $S(t), t\geq 0$, on $\mathcal{H}$, and the control function $u(.)$ is given in $\mathcal{L}^2_{\mathcal{F}}(J,\mathcal{U})$ of admissible control functions, $\mathcal{U}$ is a Hilbert space, B is a bounded linear operator from $\mathcal{U}$ into $\mathcal{H}$. Let $\mathcal{K}$ be another separable Hilbert space with inner product $(.,.)_{\mathcal{K}}$ and norm $\|.\|_{\mathcal{K}}$. w is a given $\mathcal{K}$-valued Wiener process with a finite trace nuclear covariance operator $Q\geq 0$ defined on a filtered complete probability space $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq 0},\mathbb{P})$. histories $x_t:\Omega\to\mathcal{C}_h$ defined by $x_t=\{x(t+\theta),\theta\in(-\infty,0]\}$ belong to the phase space $\mathcal{C}_h$ which will be defined in section 2. The initial data $\phi=\{\phi(t),t\in(-\infty,0]\}$ is an $\mathcal{F}_0$-measurable, $\mathcal{C}_h$-valued random variable independent of w with finite second moments, and $G:J\times\mathcal{C}_h\to\mathcal{H}$, $f:J\times\mathcal{H}\to\mathcal{H}$, $\sigma:J\times J\times\mathcal{H}\to\mathcal{L}^0_2(\mathcal{K},\mathcal{H})$ are appropriate mappings specified later, $\mathcal{L}^0_2(\mathcal{K},\mathcal{H})$ denotes the space of all Q-HilbertSchmidt operators from $\mathcal{K}$ into $\mathcal{H}$.

The paper is organized as follows. In section 2, we briefly present some basic notations and preliminaries. In section 3, we give the mild solution and existence result of the system (1) by Sadovskii's fixed point theorem. We also study the approximate controllability of the fractional stochastic system (1) under certain assumptions. An example is given to illustrate our result. To avoid some lengthy calculations arising from proofs of theorems, we give an appendix which consists of some basic estimates.

2 Preliminaries

In this section, we shall recall some basic definitions and lemmas from fractional calculus theory which will be used in the main results [7, 15].

Throughout this paper, $(\mathcal{H}, \|.\|_{\mathcal{H}})$ and $(\mathcal{K}, \|.\|_{\mathcal{K}})$ denote two real separable Hilbert spaces. We denote by $\mathcal{L}(\mathcal{K}, \mathcal{H})$ the set of all linear bounded operators from $\mathcal{K}$ into $\mathcal{H}$ equipped with the usual operator norm $\|.\|$. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a filtered complete probability space satisfying the usual condition, which means that the filtration is a right continuous increasing family and $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets. $w = (w_t)_{t\geq 0}$ be a Q-Wiener process defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ with the covariance operator Q such that $trQ < \infty$. We assume that there exists a complete orthonormal system $\{e_k\}_{k\geq 1}$ in $\mathcal{K}$, a bounded sequence of nonnegative real numbers λ_k such that $Qe_k = \lambda_k e_k$, $k = 1, 2, \dots$ and a sequence $\{\beta_k\}_{k\geq 1}$ of independent Brownian motions such that

$$(w(t), e)_{\mathcal{K}} = \sum_{k=1}^{\infty} \sqrt{\lambda_k}(e_k, e)_{\mathcal{K}}\beta_k(t), \quad e \in \mathcal{K}, t \in [0, b].$$

Let $\mathcal{L}_2^0 = \mathcal{L}_2(Q^{1/2}\mathcal{K}, \mathcal{H})$ be the space of all HilbertSchmidt operators from $Q^{1/2}\mathcal{K}$ into $\mathcal{H}$ with the inner product $\langle \psi, \pi\rangle_{\mathcal{L}_2^0} = tr[\psi Q \pi^\star]$.

Let $-A$ be the infinitesimal generator of an analytic semigroup $\{S(t)\}_{t\geq 0}$ of uniformly bounded linear operators on $\mathcal{H}$. For the semigroup $S(t)$, there is an $M \geq 1$ such that $\|S(t)\| \leq M$. Let $0 \in \rho(-A)$, the resolvent set of $-A$. Then, for $\beta \in (0, 1]$, it is possible to define the fractional power A^β as a closed linear operator on its domain $\mathcal{D}(A^\beta)$, being dense in $\mathcal{H}$, and we denote by $\mathcal{H}_\beta$ the Banach space $\mathcal{D}(A^\beta)$ endowed with the norm $\|x\|_\beta = \|A^\beta x\|$, which is equivalent to the graph norm of A^β.

Lemma 2.1 ([14]) *Suppose that the preceding conditions are satisfied.*

i. if $0 < \eta \leq \beta$ then the embedding $\mathcal{H}_\eta \subset \mathcal{H}_\beta$ is compact whenever the resolvent operator of A is compact.

ii. For every $\beta \in (0, 1]$, there exists a positive constant C_β such that $\|A^\beta S(t)\| \leq \frac{C_\beta}{t^\beta}$, $t > 0$.

Assume that $h : (-\infty, 0] \to (0, +\infty)$ with $l = \int_{-\infty}^0 h(t)dt < +\infty$ a continuous function. Recall that the abstract phase space $\mathcal{C}_h$ is defined by

$$\begin{aligned}\mathcal{C}_h \;=\; & \Big\{\varphi : (-\infty, 0] \to \mathcal{H}, \text{ for any } a > 0, (\mathbb{E}|\varphi(\theta)|^2)^{1/2} \text{ is bounded and measurable} \\ & \quad \text{function on } [-a, 0] \text{ with } \varphi(0) = 0 \text{ and } \int_{-\infty}^0 h(s) \sup_{s\leq\theta\leq 0} (\mathbb{E}|\varphi(\theta)|^2)^{1/2} ds < \infty\Big\}.\end{aligned}$$

If $\mathcal{C}_h$ is endowed with the norm

$$\|\varphi\|_{\mathcal{C}_h} = \int_{-\infty}^0 h(s) \sup_{s\leq\theta\leq 0} (\mathbb{E}|\varphi(\theta)|^2)^{1/2} ds, \quad \varphi \in \mathcal{C}_h,$$

then $(\mathcal{C}_h, \|.\|_{\mathcal{C}_h})$ is a Banach space.

Let us now recall some basic definitions and results of fractional calculus. For more details see [7].

Definition 2.2 *The fractional integral of order α with the lower limit 0 for a function f is defined as*

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s)}{(t-s)^{1-\alpha}} ds, \quad t > 0, \alpha > 0$$

provided the right-hand side is pointwise defined on $[0, \infty)$, where Γ is the gamma function.

Definition 2.3 *Riemann-Liouville derivative of order α with lower limit zero for a function $f : [0, \infty) \to \mathbb{R}$ can be written as*

$$^{L}D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t \frac{f(s)}{(t-s)^{\alpha+1-n}} ds, \quad t > 0, n-1 < \alpha < n. \tag{2}$$

Definition 2.4 *The Caputo derivative of order α for a function $f : [0, \infty) \to \mathbb{R}$ can be written as*

$$^{c}D^{\alpha} f(t) = {}^{L}D^{\alpha} \left(f(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} f^k(0) \right), \quad t > 0, n-1 < \alpha < n. \tag{3}$$

If $f(t) \in C^n[0, \infty)$, then

$$^{c}D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^n(s) ds = I^{n-\alpha} f^n(s), \quad t > 0, n-1 < \alpha < n$$

If f is an abstract function with values in $\mathcal{H}$, then the integrals appearing in the above definitions are taken in Bochner's sense.

At the end of this section, we recall the fixed point theorem of Sadovskii which is used to establish the existence of the mild solution to the fractional stochastic control system (1).

Lemma 2.5 ([16]) *Let Φ be a condensing operator on a Banach space $\mathcal{H}$, that is, Φ is continuous and takes bounded sets into bounded sets, and $\mu(\Phi(B)) \leq \mu(B)$ for every bounded set B of $\mathcal{H}$ with $\mu(B) > 0$. If $\Phi(N) \subset N$ for a convex, closed and bounded set N of $\mathcal{H}$, then Φ has a fixed point in $\mathcal{H}$ (where $\mu(.)$ denotes Kuratowski's measure of noncompactness).*

3 The main results

In this section, we consider the stochastic fractional control system (1). We first present the basic definition of the approximate controllability for the system.

Definition 3.1 ([17]) *Let $x_b(\phi; u)$ be the state value of (1) at the terminal time b corresponding to the control u and the initial value ϕ. Introduce the set*

$$\mathcal{R}(b, \phi) = \{x_b(\phi; u)(0) : u(.) \in \mathcal{L}^2(J, \mathcal{U})\},$$

which is called the reachable set of (1) at the terminal time b and its closure in $\mathcal{H}$ is denoted by $\overline{\mathcal{R}(b, \phi)}$. The system (1) is said to be approximately controllable on the interval J if $\overline{\mathcal{R}(b, \phi)} = \mathcal{H}$.

Secondly, we present the following definition of mild solutions for the system (1).

Definition 3.2 ([23]) *An $\mathcal{H}$-valued stochastic process $\{x(t), t \in (-\infty, b]\}$ is said to be a mild solution of the system* (1) *if*

i. *$x(t)$ is $\mathcal{F}_t$-adapted and measurable, $t \geq 0$;*

ii. *$x(t)$ is continuous on $[0, b]$ almost surely and for each $s \in [0, t)$, the function $(t-s)^{\alpha-1} A T_\alpha(t-s) G(s, x_s)$ is integrable such that the following stochastic integral equation is verified:*

$$\begin{aligned} x(t) &= S_\alpha(t)[\phi(0) + G(0,\phi)] - G(t, x_t) - \int_0^t (t-s)^{\alpha-1} A T_\alpha(t-s) G(s, x_s) ds \\ &+ \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) Bu(s) ds + \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) f(s, x_s) ds \\ &+ \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) \left[\int_{-\infty}^s \sigma(s, \tau, x_\tau) dw(\tau) \right] ds, \end{aligned} \tag{4}$$

where $S_\alpha(t)x = \int_0^\infty \xi_\alpha(\theta) S(t^\alpha \theta) x d\theta$, $T_\alpha(t)x = \alpha \int_0^\infty \theta \xi_\alpha(\theta) S(t^\alpha \theta) x d\theta$ and ξ_α is the probability density function defined on $(0, \infty)$.

iii. *$x(t) = \phi(t)$ on $(-\infty, 0]$ satisfying $\|\phi\|^2_{\mathcal{C}_h} < \infty$.*

Lemma 3.3 ([23]) *The operators $S_\alpha(t)$ and $T_\alpha(t)$ have the following properties:*

i. *For any fixed $t \geq 0$, $S_\alpha(t)$ and $T_\alpha(T)$ are linear and bounded operators such that for any $x \in \mathcal{H}$*

$$\|S_\alpha(t)x\|_{\mathcal{H}} \leq M\|x\|_{\mathcal{H}} \quad \text{and} \quad \|T_\alpha(t)x\|_{\mathcal{H}} \leq \frac{M\alpha}{\Gamma(1+\alpha)}\|x\|_{\mathcal{H}};$$

ii. *$S_\alpha(t)$ and $T_\alpha(t)$ are strongly continuous and compact;*

iii. *For any $x \in \mathcal{H}$, $\beta \in (0, 1)$ and $\eta \in (0, 1]$, we have*

$$A T_\alpha(t) x = A^{1-\beta} T_\alpha(t) A^\beta x \quad \text{and} \quad \|A^\eta T_\alpha(t)\| \leq \frac{\alpha C_\eta \Gamma(2-\eta)}{t^{\alpha\eta}\Gamma(1+\alpha(1-\eta))}, \quad t \in [0, b].$$

In order to explain our theorem on the existence of mild solutions, we need the following assumptions.
$(H1)$: The semigroup $S(t)$ is a compact operator for $t > 0$, and there exists a positive constant M such that $\|S(t)\| \leq M$.
$(H2)$: the function $G : J \times \mathcal{C}_h \to \mathcal{H}$ is continuous and there exist some constants $\beta \in (0, 1)$ and $M_G > 0$ such that G is $\mathcal{H}_\beta$-valued and

$$\begin{aligned} \mathbb{E}\|A^\beta G(t,x) - A^\beta G(t,y)\|^2_{\mathcal{H}} &\leq M_G \|x-y\|^2_{\mathcal{C}_h}, \quad t \in J, \quad x, y \in \mathcal{C}_h, \\ \mathbb{E}\|A^\beta G(t,x)\|^2_{\mathcal{H}} &\leq M_G\Big(1 + \|x\|^2_{\mathcal{C}_h}\Big). \end{aligned}$$

$(H3)$: For each $\varphi \in \mathcal{C}_h$,

$$K(t) = \lim_{a\to\infty} \int_{-a}^0 \sigma(t, s, \varphi) dw(s)$$

exists and is continuous. Further, there exists a positive constant M_k such that

$$\mathbb{E}\|K(t)\|^2_{\mathcal{H}} \leq M_k.$$

$(H4)$: $\sigma : J \times J \times \mathcal{C}_h \to \mathcal{L}(\mathcal{K},\mathcal{H})$ satisfies the following properties:

i. for each $(t,s) \in \mathcal{D} := J \times J$, $\sigma(t,s,.) : \mathcal{C}_h \to \mathcal{L}(\mathcal{K},\mathcal{H})$ is continuous and for each $x \in \mathcal{C}_h$, $\sigma(.,.,x) : \mathcal{D} \to \mathcal{L}(\mathcal{K},\mathcal{H})$ is strongly measurable;

ii. there is a positive integrable function $m \in L^1([0,b])$ and a continuous nondecreasing function $\Lambda_\sigma : [0,\infty) \to (0,\infty)$ such that for every $(t,s,x) \in J \times J \times \mathcal{C}_h$, we have

$$\int_0^t \mathbb{E}\|\sigma(t,s,x)\|^2_{\mathcal{L}_2^0} \leq m(t)\Lambda_\sigma(\|x\|^2_{\mathcal{C}_h}), \qquad \liminf_{r\to\infty} \frac{\Lambda_\sigma(r)}{r} ds = \vartheta < \infty.$$

$(H5)$: $f : J \times \mathcal{C}_h \to \mathcal{H}$ satisfies the following properties:

i. $f(t,.) : \mathcal{C}_h \to \mathcal{H}$ His continuous for each $t \in J$ and for each $x \in \mathcal{C}_h$, $f(.,x) : J \to \mathcal{H}$ is strongly measurable;

ii. there is a positive integrable function $n \in L^1([0,b])$ and a continuous nondecreasing function $\Lambda_f : [0,\infty) \to (0,\infty)$ such that for every $(t,x) \in J \times \mathcal{C}_h$, we have

$$\mathbb{E}\|f(t,x)\|^2_{\mathcal{H}} \leq n(t)\Lambda_f(\|x\|^2_{\mathcal{C}_h}), \qquad \liminf_{r\to\infty} \frac{\Lambda_f(r)}{r} ds = \gamma < \infty.$$

In order to study the approximate controllability for the fractional control system (1), we introduce the approximate controllability of its linear part

$$\begin{cases} {}^cD_t^\alpha x(t) &= Ax(t) + (Bu)(t), \quad t \in J, \\ x(0) &= \phi(0). \end{cases} \tag{5}$$

For this purpose, we need to introduce the relevant operator

$$\begin{aligned} \Theta_0^b &= \int_0^b (b-s)^{\alpha-1} S_\alpha(b-s)BB^\star S_\alpha^\star(b-s)ds, \\ R(\varepsilon,\Theta_0^b) &= (\varepsilon I + \Theta_0^b), \end{aligned}$$

where $B^\star$ denote the adjoint of B and $S_\alpha^\star(t)$ is the adjoint of $S_\alpha(t)$. It is straightforward that the operator Θ_0^b is a linear bounded operator.

We assume the following additional assumption:

$(H6)$: $\varepsilon R(\varepsilon,\Theta_0^b) \to 0$ as $\varepsilon \to 0^+$ in the strong operator topology.

Note that the assumption $(H6)$ is equivalent to the fact that the linear fractional control system (5) is approximately controllable on J (see [13]).

Lemma 3.4 ([17]) *Assume that $x \in \mathcal{C}_b$; then for all $t \in J$, $x_t \in \mathcal{C}_h$. Moreover*

$$l(\mathbb{E}\|x(t)\|^2)^{\frac{1}{2}} \leq \|x_t\|_{\mathcal{C}_h} \leq l \sup_{s\in[0,t]} (\mathbb{E}\|x(s)\|^2)^{\frac{1}{2}} + \|x_0\|_{\mathcal{C}_h},$$

where $l = \int_{-\infty}^0 h(s)ds < \infty.$

The following lemma is required to define the control function.

Lemma 3.5 ([11]) *For any* $\hat{x} \in \mathcal{L}^2(\mathcal{F}_b, \mathcal{H})$ *there exists* $\hat{\phi} \in \mathcal{L}^2_{\mathcal{F}}(\Omega; \mathcal{L}^2(0, b; \mathcal{L}^0_2))$ *such that* $\hat{x}_b = \mathbb{E}\hat{x}_b + \int_0^b \hat{\phi}(s)dw(s)$.

Now for any $\varepsilon > 0$ and $\hat{x} \in \mathcal{L}^2(\mathcal{F}_b, \mathcal{H})$, we define the control function

$$\begin{aligned} u^{\varepsilon}(t) &= B^{\star}T^{\star}_{\alpha}(b-t)(\varepsilon I + \Theta_0^b)^{-1} \\ &\quad \times \left\{ \mathbb{E}\hat{x}_b + \int_0^b \hat{\phi}(s)dw(s) - S_{\alpha}(b)[\phi(0) + G(0,\phi)] + G(b, x_b) \right\} \\ &\quad + B^{\star}T^{\star}_{\alpha}(b-t)\int_0^b (\varepsilon I + \Theta_s^b)^{-1}(b-s)^{\alpha-1}AT_{\alpha}(b-s)G(s,x_s)ds \\ &\quad - B^{\star}T^{\star}_{\alpha}(b-t)\int_0^b (\varepsilon I + \Theta_s^b)^{-1}(b-s)^{\alpha-1}T_{\alpha}(b-s)f(s,x_s)ds \\ &\quad - B^{\star}T^{\star}_{\alpha}(b-t)\int_0^b (\varepsilon I + \Theta_s^b)^{-1}(b-s)^{\alpha-1}T_{\alpha}(b-s)\left[\int_{-\infty}^s \sigma(s,\tau,x_{\tau})dw(\tau)\right]ds. \end{aligned}$$

Our first result is the following theorem on the existence of the mild solution to the fractional stochastic control system (1).

Theorem 3.6 *Assume that the assumptions* $(H1)$*-*$(H5)$ *hold. Then for each* $\varepsilon > 0$*, the system* (1) *has a mild solution on* $[0, b]$ *provided that*

$$\begin{aligned} &\left[4M_G\|A^{-\beta}\|^2 l^2 + 4M_G \frac{l^2 C^2_{1-\beta}\Gamma^2(1+\beta)b^{2\alpha\beta}}{\beta^2\Gamma^2(1+\alpha\beta)} + 4\gamma \frac{l^2M^2b^{2\alpha}}{\Gamma^2(1+\alpha)}\sup_{s\in J} n(s) \right. \\ &\left. \quad + 8\vartheta tr(Q)\frac{l^2M^2b^{2\alpha}}{\Gamma^2(1+\alpha)}\sup_{s\in J} m(s)\right] \times \left[6 + \frac{42}{\varepsilon^2}\left(\frac{\alpha M M_B}{\Gamma(\alpha+1)}\right)^4 \frac{b^{2\alpha}}{\alpha^2}\right] < 1. \end{aligned}$$

Proof. Let $\mathcal{C}((-\infty, b], \mathcal{H})$ be the space of all continuous $\mathcal{H}$-valued stochastic processes $\{\xi(t), t \in (-\infty, b]\}$ and $\mathcal{C}_b = \{x : x \in \mathcal{C}((-\infty, b], \mathcal{H}), x_0 = \phi \in \mathcal{C}_h\}$. Let $\|.\|_b$ be a seminorm defined by

$$\|x\|_b = \|x_0\|_{\mathcal{C}_h} + \sup_{0\leq s\leq b}(\mathbb{E}\|x(s)\|^2)^{\frac{1}{2}}, \quad x \in \mathcal{C}_b.$$

Using the control function, for any $\varepsilon > 0$, define the operator $\mathcal{P}^{\varepsilon} : \mathcal{C}_b \to \mathcal{C}_b$ by

$$(\mathcal{P}^{\varepsilon}x)(t) = \begin{cases} \phi(t), \qquad t \in (-\infty, 0]; \\ S_{\alpha}(t)[\phi(0) + G(0,\phi)] - G(t, x_t) - \int_0^t (t-s)^{\alpha-1}AT_{\alpha}(t-s)G(s,x_s)ds \\ \quad + \int_0^t (t-s)^{\alpha-1}T_{\alpha}(t-s)Bu^{\varepsilon}(s)ds + \int_0^t (t-s)^{\alpha-1}T_{\alpha}(t-s)f(s,x_s)ds \\ \quad + \int_0^t (t-s)^{\alpha-1}T_{\alpha}(t-s)\left[\int_{-\infty}^s \sigma(s,\tau,x_{\tau})dw(\tau)\right]ds, \quad t \in J. \end{cases}$$

Using Lemma 3.3, it follows that

$$\begin{aligned}
&\mathbb{E}\left\|\int_0^t (t-s)^{\alpha-1}AT_\alpha(t-s)G(s,x_s)ds\right\|_{\mathcal{H}}^2\\
\leq\ & \mathbb{E}\left(\int_0^t \|(t-s)^{\alpha-1}A^{1-\beta}T_\alpha(t-s)A^\beta G(s,x_s)\|_{\mathcal{H}}ds\right)^2\\
\leq\ & \frac{\alpha^2C_{1-\beta}^2\Gamma^2(1+\beta)}{\Gamma^2(1+\alpha\beta)}\mathbb{E}\left(\|(t-s)^{\alpha\beta-1}A^\beta G(s,x_s)\|_{\mathcal{H}}ds\right)^2.
\end{aligned}$$

Applying the Hölder inequality and assumption $(H2)$, we further derive that

$$\begin{aligned}
&\mathbb{E}\left\|\int_0^t (t-s)^{\alpha-1}AT_\alpha(t-s)G(s,x_s)ds\right\|_{\mathcal{H}}^2\\
\leq\ & \frac{\alpha^2C_{1-\beta}^2\Gamma^2(1+\beta)}{\Gamma^2(1+\alpha\beta)}\int_0^t(t-s)^{\alpha\beta-1}ds\int_0^t(t-s)^{\alpha\beta-1}\mathbb{E}\|A^\beta G(s,x_s)\|_{\mathcal{H}}^2ds\\
\leq\ & \frac{\alpha^2C_{1-\beta}^2\Gamma^2(1+\beta)b^{\alpha\beta}}{\Gamma^2(1+\alpha\beta)\alpha\beta}\int_0^t(t-s)^{\alpha\beta-1}\mathbb{E}\|A^\beta G(s,x_s)\|_{\mathcal{H}}^2ds\\
\leq\ & \frac{\alpha^2C_{1-\beta}^2\Gamma^2(1+\beta)M_Gb^{\alpha\beta}}{\Gamma^2(1+\alpha\beta)\alpha\beta}\int_0^t(t-s)^{\alpha\beta-1}(1+\|x_s\|_{\mathcal{C}_h}^2)ds,
\end{aligned}$$

which deduces that $(t-s)^{\alpha-1}AT_\alpha(t-s)G(s,x_s)$ is integrable on J by Bochner's theorem and Lemma 3.4.

We shall show that $\mathcal{P}^\varepsilon$ has a fixed point, by Sadovskii's theorem, which is then a mild solution for the system (1).
For $\phi\in\mathcal{C}_h$, define

$$\tilde{\phi}=\begin{cases}\phi(t), & t\in(-\infty,0];\\ S_\alpha(t)\phi(0), & t\in J.\end{cases}$$

Then $\tilde{\phi}\in\mathcal{C}_b$. Let $x(t)=\tilde{\phi}(t)+z(t)$, $t\in(-\infty,b]$. It is clear that x satisfies (1) if and only if $z_0=0$ and

$$\begin{aligned}
z(t) =\ & S_\alpha(t)G(0,\phi)-G(t,\tilde{\phi}_t+z_t)-\int_0^t(t-s)^{\alpha-1}AT_\alpha(t-s)G(s,\tilde{\phi}_s+z_s)ds\\
+\ & \int_0^t(t-s)^{\alpha-1}T_\alpha(t-s)Bu^\varepsilon(s)ds+\int_0^t(t-s)^{\alpha-1}T_\alpha(t-s)f(s,\tilde{\phi}_s+z_s)ds\\
+\ & \int_0^t(t-s)^{\alpha-1}T_\alpha(t-s)\left[\int_{-\infty}^s\sigma(s,\tau,\tilde{\phi}_\tau+z_\tau)\right]ds,
\end{aligned}$$

where

$$\begin{aligned}
u^\alpha(t) =\ & B^\star T_\alpha^\star(b-t)(\varepsilon I+\Theta_0^b)^{-1}\\
&\times\left\{\mathbb{E}\hat{x}_b+\int_0^b\hat{\phi}(s)dw(s)-S_\alpha(b)[\phi(0)+G(0,\phi)]+G(b,\tilde{\phi}_b+z_b)\right\}\\
&+B^\star T_\alpha^\star(b-t)\int_0^b(\varepsilon I+\Theta_s^b)^{-1}(b-s)^{\alpha-1}AT_\alpha(b-s)G(s,\tilde{\phi}_s+z_s)ds\\
&-B^\star T_\alpha^\star(b-t)\int_0^b(\varepsilon I+\Theta_s^b)^{-1}(b-s)^{\alpha-1}T_\alpha(b-s)f(s,\tilde{\phi}_s+z_s)ds\\
&-B^\star T_\alpha^\star(b-t)\int_0^b(\varepsilon I+\Theta_s^b)^{-1}(b-s)^{\alpha-1}T_\alpha(b-s)\left[\int_{-\infty}^s\sigma(s,\tau,\tilde{\phi}_\tau+z_\tau)dw(\tau)\right]ds.
\end{aligned}$$

Let $\mathcal{C}_b^0 = \{z \in \mathcal{C}_b, z_0 = 0 \in \mathcal{C}_h\}$. For any $z \in \mathcal{C}_b^0$, we have

$$\|z\|_b = \|z_0\|_{\mathcal{C}_h} + \sup_{0\le s\le b} (\mathbb{E}\|z(s)\|^2)^{\frac{1}{2}} = \sup_{0\le s\le b} (\mathbb{E}\|z(s)\|^2)^{\frac{1}{2}}.$$

Thus, $(\mathcal{C}_b^0, \|.\|_b)$ is a Banach space. For each positive number q, set

$$\mathcal{B}_q = \{y \in \mathcal{C}_b^0, \|y\|_b^2 \le q\}.$$

Then, for each q, $\mathcal{B}_q$ is clearly a bounded closed convex set in $\mathcal{C}_b^0$. From Lemma 3.4, for $z \in \mathcal{B}_q$, we see that

$$\begin{aligned}\|\tilde{\phi}_t + z_t\|_{\mathcal{C}_h}^2 &\le 2\Big(\|z_t\|_{\mathcal{C}_h}^2 + \|\tilde{\phi}_t\|_{\mathcal{C}_h}^2\Big)\\ &\le 4\Big(l^2 \sup_{0\le s\le t} \mathbb{E}\|z(s)\|^2 + \|z_0\|_{\mathcal{C}_h}^2 + l^2 \sup_{0\le s\le t} \mathbb{E}\|\tilde{\phi}(s)\|^2 + \|\tilde{\phi}_0\|_{\mathcal{C}_h}^2\Big) \qquad (6)\\ &\le 4l^2(q + M^2\mathbb{E}\|\phi(0)\|_{\mathcal{H}}^2) + 4\|\phi\|_{\mathcal{C}_h}^2.\end{aligned}$$

Consider the map Υ on $\mathcal{C}_b^0$ defined by

$$(\Upsilon z)(t) = \begin{cases} 0, & t \in (-\infty, 0];\\ S_\alpha(t)G(0,\phi) - G(t, \tilde{\phi}_t + z_t) - \int_0^t (t-s)^{\alpha-1} A T_\alpha(t-s) G(s, \tilde{\phi}_s + z_s) ds & \\ \quad + \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) B u^\varepsilon(s) ds + \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) f(s, \tilde{\phi}_s + z_s) ds & \\ \quad + \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) \left[\int_{-\infty}^s \sigma(s, \tau, \tilde{\phi}_\tau + z_\tau)\right] ds, & t \in J. \end{cases}$$

Observe that Υ is well defined on $\mathcal{B}_q$ for each $q > 0$.

Moreover, it is obvious that the operator $\mathcal{P}^\varepsilon$ has a fixed point if and only if Υ has a fixed point. Now, for $t \in J$, we decompose Υ as $\Upsilon = \Upsilon_1 + \Upsilon_2$, where the operator Υ_1 and Υ_2 are defined on $\mathcal{B}_q$ respectively, by

$$\begin{aligned}(\Upsilon_1 z)(t) &= S_\alpha(t)G(0,\phi) - G(t, \tilde{\phi}_t + z_t) - \int_0^t (t-s)^{\alpha-1} A T_\alpha(t-s) G(s, \tilde{\phi}_s + z_s) ds,\\ (\Upsilon_2 z)(t) &= \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) B u^\varepsilon(s) ds + \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) f(s, \tilde{\phi}_s + z_s) ds\\ &\quad + \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) \left[\int_{-\infty}^s \sigma(s, \tau, \tilde{\phi}_\tau + z_\tau)\right] ds.\end{aligned}$$

Thus, the theorem follows from the next theorem.

□

Theorem 3.7 *Assume that assumptions* $(H1)$-$(H5)$ *hold. Then,* Υ_1 *is a contractive mapping, while* Υ_2 *is compact.*

Proof. The theorem follows from lemmas in the appendix and Arzelá-Ascoli theorem.

□

Our second result is the following theorem on the approximate controllability of the fractional stochastic control system (1).

Theorem 3.8 *Assume that the assumptions of Theorem 3.6 hold and, in addition, the functions f, G and σ are uniformly bounded on their respective domains. Further, if $S(t)$ is compact, then the fractional control system* (1) *is approximately controllable on J.*

f Let x^ε be a fixed point of the operator $\mathcal{P}^\varepsilon$. Using the stochastic Fubini theorem, it is easy to see that

$$\begin{aligned} x^\varepsilon(b) &= \hat{x}_b - \varepsilon(\varepsilon I + \Theta_0^b)^{-1}\left[\mathbb{E}\hat{x}_b + \int_0^b \hat{\phi}(s)dw(s) - S_\alpha(t)[\phi(0) + G(0,\phi)] - G(b, x_b^\varepsilon)\right] \\ &- \varepsilon \int_0^b (\varepsilon I + \Theta_s^b)^{-1}(b-s)^{\alpha-1} A T_\alpha(b-s) G(s, x_s^\varepsilon) ds \\ &+ \varepsilon \int_0^b (\varepsilon I + \Theta_s^b)^{-1}(b-s)^{\alpha-1} T_\alpha(b-s) f(s, x_s^\varepsilon) ds \\ &+ \varepsilon \int_0^b (\varepsilon I + \Theta_s^b)^{-1}(b-s)^{\alpha-1} T_\alpha(b-s)\left[\int_{-\infty}^s \sigma(s,\tau,x_\tau^\varepsilon)dw(\tau)\right] ds. \end{aligned}$$

It follows from the properties of G, f and σ that $\|f(s,x_s^\varepsilon)\|^2 + \|\sigma(s,\tau,x_\tau^\varepsilon)\|^2 \le \kappa_1$, $\|A^\beta G(s,x_s^\varepsilon)\|^2 \le \kappa_2$ Then there is a subsequence still denoted by $\{A^\beta G(s,x_s^\varepsilon), f(s,x_s^\varepsilon), \sigma(s,\tau,x_\tau^\varepsilon)\}$ which converges to weakly to, say, $\{G(s), f(s), \sigma(s,\tau)\}$.
From the above equation, we have

$$\begin{aligned} \mathbb{E}\|x^\varepsilon(b) - \hat{x}_b\|^2 &\le 9\|\varepsilon(\varepsilon I + \Theta_0^b)^{-1}[\mathbb{E}\hat{x}_b - S_\alpha(b)(\phi(0) + G(0,\phi))]\|^2 \\ &+ 9\mathbb{E}\left(\int_0^b \|\varepsilon(\varepsilon I + \Theta_0^b)^{-1}\hat{\phi}(s)\|_{\mathcal{L}_2^0}^2 ds\right) + 9\mathbb{E}\|\varepsilon(\varepsilon I + \Theta_0^b)^{-1} G(b, x_b^\varepsilon)\|^2 \\ &+ 9\mathbb{E}\left(\int_0^b (b-s)^{\alpha-1}\|\varepsilon(\varepsilon I + \Theta_s^b)^{-1} A T_\alpha(b-s)[G(s,x_s^\varepsilon) - G(s)]\| ds\right)^2 \\ &+ 9\mathbb{E}\left(\int_0^b (b-s)^{\alpha-1}\|\varepsilon(\varepsilon I + \Theta_s^b)^{-1} A T_\alpha(b-s) G(s)\| ds\right)^2 \\ &+ 9\mathbb{E}\left(\int_0^b (b-s)^{\alpha-1}\|\varepsilon(\varepsilon I + \Theta_s^b)^{-1} T_\alpha(b-s)[f(s,x_s^\varepsilon) - f(s)]\| ds\right)^2 \\ &+ 9\mathbb{E}\left(\int_0^b (b-s)^{\alpha-1}\|\varepsilon(\varepsilon I + \Theta_s^b)^{-1} T_\alpha(b-s) f(s)\| ds\right)^2 \\ &+ 9\mathbb{E}\left(\int_0^b (b-s)^{\alpha-1}\|\varepsilon(\varepsilon I + \Theta_s^b)^{-1} T_\alpha(b-s)\left[\int_{-\infty}^s [\sigma(s,\tau,x_\tau^\varepsilon) - \sigma(s,\tau)]\right]\| ds\right)^2 \\ &+ 9\mathbb{E}\left(\int_0^b (b-s)^{\alpha-1}\|\varepsilon(\varepsilon I + \Theta_s^b)^{-1} T_\alpha(b-s)\left[\int_{-\infty}^s \sigma(s,\tau)\right]\| ds\right)^2. \end{aligned}$$

On the other hand, by assumption $(H6)$ for all $0 \le s \le b$, the operator $\varepsilon(\varepsilon I + \Theta_s^b)^{-1} \to 0$ strongly as $\varepsilon \to 0^+$ and moreover $\|\varepsilon(\varepsilon I + \Theta_s^b)^{-1}\| \le 1$. Thus, by the Lebesgue dominated convergence theorem and the compactness of $S_\alpha(t)$ implies that $\mathbb{E}\|x^\varepsilon(b) - \hat{x}_b\|^2 \to 0$. This gives the approximate controllability of (1).

□

At last, an example is provided to illustrate our results.

Example 3.9

Consider the following fractional neutral stochastic partial differential equation with infinite delays of the form:

$$\begin{cases} {}^cD_t^{\alpha}[z(t,x)-\hat{G}(t,z(t-h,x))] = \frac{\partial^2}{\partial x^2}z(t,x)+\mu(t,x)+\hat{f}(t,z(t-h,x)) \\ \quad +\int_{-\infty}^{t}\hat{\sigma}(t,s,z(s-h,x))dw(s), \quad 0\le x\le \pi, \quad h>0, \quad t\in J:=[0,b], \\ z(t,0)=z(t,\pi)=0, \qquad t\in J, \\ z(t,x)=\phi(t,x), \qquad t\in(-\infty,0], \end{cases} \tag{7}$$

where ${}^cD_t^{\alpha}$ is a Caputo fractional partial derivative of order $0<\alpha<1$; $\phi(t,x)$ is continuous; $w(t)$ denotes a standard cylindrical Wiener process defined on a stochastic basis $(\Omega,\{\mathcal{F}_t\},\mathcal{F},\mathbb{P})$. To rewrite this system into the abstract form (1), let $\mathcal{H}=\mathcal{L}^2([0,\pi])$ with the norm $\|.\|$.
Define $A:\mathcal{H}\to\mathcal{H}$ by $AX=x''$ with the domain

$$\mathcal{D}(A)=\{x\in\mathcal{H}; x,x' \text{ are absolutely continuous, } x''\in\mathcal{H} \text{ and } x(0)=x(\pi)=0\}.$$

Then A generates a symmetric $\mathcal{C}_0$-semigroup e^{-tA} in $\mathcal{H}$ and there exists a complete orthonormal set $\{\omega_n, n=1,2,\dots\}$ of eigenvectors of A with $\omega_n(s)=\sqrt{\frac{2}{\pi}}\sin(ns)$, $n=1,2,\dots$.
Then the operator $A^{-\frac{1}{2}}$ is given by $A^{-\frac{1}{2}}\xi=\sum_{n=1}^{\infty}n(\xi,\omega_n)\omega_n$ on the space $\mathcal{D}(A^{-\frac{1}{2}})=\{\xi(.)\in\mathcal{H},\sum_{n=1}^{\infty}n(\xi,\omega_n)\omega_n\in\mathcal{H}\}$.

Now, we present a special phase space $\mathcal{C}_h$. Let $h(s)=e^{2s}$, $s<0$; then $l=\int_{-\infty}^{0}h(s)ds=\frac{1}{2}$. Let

$$\|\varphi\|_{\mathcal{C}_h}=\int_{-\infty}^{0}h(s)\sup_{s\le\theta\le 0}(\mathbb{E}\|\varphi(\theta)\|^2)^{\frac{1}{2}}ds.$$

Then $(\mathcal{C}_h,\|.\|_{\mathcal{C}_h})$ is a Banach space.
Define an infinite dimensional space $\mathcal{U}$ by $\mathcal{U}=\{u|u=\sum_{n=2}^{\infty}u_n\omega_n, \text{ with } \sum_{n=2}^{\infty}\mathcal{U}_n^2<\infty\}$. The norm in $\mathcal{U}$ is defined by $\|u\|_{\mathcal{U}}=(\sum_{n=2}^{\infty}\mathcal{U}_n^2)^{1/2}$. Now define a continuous linear mapping B from $\mathcal{U}$ into $\mathcal{H}$ as $Bu=2u_2\omega_1+\sum_{n=2}^{\infty}u_n\omega_n$ for $u=\sum_{n=2}^{\infty}u_n\omega_n\in\mathcal{U}$.

Define the bounded linear operator $B:\mathcal{U}\to\mathcal{H}$ by $Bu(t)(x)=\mu(t,x)$, $0\le x\le 1$. For $(t,\varphi)\in J\times\mathcal{C}_h$, where $\varphi(\theta)(.)=\phi(\theta,.)$, $(\theta,.)\in(-\infty,0]\times[0,\pi]$, let $z(t)(.)=z(t,.)$ and define $f(t,z)(.)=\hat{f}(t,z(.))$, $G(t,z)(.)=\hat{G}(t,z(.))$ and $\sigma(t,s,z)(.)=\hat{\sigma}(t,s,z(.))$. Therefore, with the above choices, the system (7) can be written to the abstract form (1) and all the conditions of Theorem 3.6 are satisfied. Thus, there exists a mild solution for the system (7). On the other hand, the linear system corresponding to (7) is approximately controllable (but not exactly controllable). Hence, all the conditions of Theorem 3.8 are satisfied. Thus by Theorem 3.8, fractional stochastic control system (7) is approximately controllable on $[0,b]$.

Appendix. Some basic estimates
In this appendix, our main object is to prove theorem 3.7.

Lemma 3.10 *Under assumptions $(H1)$-$(H5)$, for each $\varepsilon>0$, there exists a positive number q such that $\Upsilon(\mathcal{B}_q)\subset\mathcal{B}_q$.*

f If it is not true, then for each positive number q, there exists a function $z^q(.)\in\mathcal{B}_q$, but $\Upsilon(z^q)\notin\mathcal{B}_q$, that is, $\mathbb{E}\|(\Upsilon z^q)(t)\|_{\mathcal{H}}^2>q$ for some $t=t(q)\in J$. For such $\varepsilon>0$, an elementary inequality can show that

$$\begin{aligned}
q &\leq \mathbb{E}\|\Upsilon(z^q)(t)\|_{\mathcal{H}}^2 \\
&\leq 6\mathbb{E}\|S_\alpha(t)G(0,\phi)\|_{\mathcal{H}}^2 + 6\mathbb{E}\|G(t,\tilde{\phi}_t + z_t^q)\|_{\mathcal{H}}^2 + 6\mathbb{E}\left\|\int_0^t (t-s)^{\alpha-1}AT_\alpha(t-s)G(s,\tilde{\phi}_s + z_s^q)ds\right\|_{\mathcal{H}}^2 \\
&+ 6\mathbb{E}\left\|\int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)f(s,\tilde{\phi}_s + z_s^q)ds\right\|_{\mathcal{H}}^2 + 6\mathbb{E}\left\|\int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)Bu^\varepsilon(s)ds\right\|_{\mathcal{H}}^2 \\
&+ 6\mathbb{E}\left\|\int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)\left[\int_{-\infty}^s \sigma(s,\tau,z_\tau^q + \tilde{\phi}_\tau)\right]ds\right\|_{\mathcal{H}}^2 \\
&= 6\sum_{i=1}^6 I_i.
\end{aligned} \tag{8}$$

In what follows, $K(\alpha,\beta)$ is the number defined by

$$K(\alpha,\beta) := \frac{\alpha^2 C_{1-\beta}^2 \Gamma^2(1+\beta)}{\Gamma^2(1+\alpha\beta)}. \tag{9}$$

Let us now estimate each term above I_i, $i = 1,\ldots,6$. By Lemma 3.4 and assumptions $(H1)$-$(H2)$, we have

$$I_1 \leq M^2\|A^{-\beta}\|^2\mathbb{E}\|A^\beta G(0,\phi)\|_{\mathcal{H}}^2 \leq M^2\|A^{-\beta}\|^2 M_G\Big(1 + \|\phi\|_{\mathcal{C}_h}^2\Big). \tag{10}$$

$$\begin{aligned}
I_2 &\leq \|A^{-\beta}\|^2\mathbb{E}\|A^\beta G(t,\tilde{\phi}_t + z_t^q)\|_{\mathcal{H}}^2 \leq M_G\|A^{-\beta}\|^2\Big(1 + \|\tilde{\phi}_t + z_t^q\|_{\mathcal{C}_h}^2\Big) \\
&\leq M_G\|A^{-\beta}\|^2\Big(1 + 4l^2(q + M^2\mathbb{E}\|\phi(0)\|_{\mathcal{H}}^2) + 4\|\phi\|_{\mathcal{C}_h}^2\Big).
\end{aligned} \tag{11}$$

By Lemma 3.3 and the Hölder inequality, we can deduce that

$$\begin{aligned}
I_3 &\leq \mathbb{E}\bigg(\int_0^t \|(t-s)^{\alpha-1}A^{1-\beta}T_\alpha(t-s)A^\beta G(s,\tilde{\phi}_s + z_s^q)\|_{\mathcal{H}}ds\bigg)^2 \\
&\leq K(\alpha,\beta)\int_0^t (t-s)^{\alpha\beta-1}ds\int_0^t (t-s)^{\alpha\beta-1}\mathbb{E}\|A^\beta G(s,\tilde{\phi}_s + z_s^q)\|_{\mathcal{H}}ds \\
&\leq \frac{K(\alpha,\beta)b^{\alpha\beta}}{\alpha\beta}\int_0^t (t-s)^{\alpha\beta-1}\mathbb{E}\|A^\beta G(s,\tilde{\phi}_s + z_s^q)\|_{\mathcal{H}}ds;
\end{aligned}$$

using the assumption $(H2)$ and (6), we derive that

$$\begin{aligned}
I_3 &\leq \frac{K(\alpha,\beta)b^{\alpha\beta}}{\alpha\beta}\int_0^t (t-s)^{\alpha\beta-1}M_G\Big(1 + \|\tilde{\phi}_s + z_s^q\|_{\mathcal{C}_h}^2\Big)ds \\
&\leq \frac{M_G K(\alpha,\beta)b^{2\alpha\beta}}{(\alpha\beta)^2}(1 + 4l^2(q + M^2\mathbb{E}\|\phi(0)\|_{\mathcal{H}}^2) + 4\|\phi\|_{\mathcal{C}_h}^2).
\end{aligned} \tag{12}$$

By (6) and the assumption $(H5)$, we have

$$\begin{aligned}
I_4 &\leq \mathbb{E}\bigg(\int_0^t \|(t-s)^{\alpha-1}T_\alpha(t-s)f(s,\tilde{\phi}_s+z_s^q)\|_{\mathcal{H}}ds\bigg)^2 \\
&\leq \bigg(\frac{M\alpha}{\Gamma(1+\alpha)}\bigg)^2\int_0^t(t-s)^{\alpha-1}ds\int_0^t(t-s)^{\alpha-1}\mathbb{E}\|f(s,\tilde{\phi}_s+z_s^q)\|^2_{\mathcal{H}}ds \\
&\leq \bigg(\frac{M\alpha}{\Gamma(1+\alpha)}\bigg)^2\frac{b^\alpha}{\alpha}\int_0^t(t-s)^{\alpha-1}\mathbb{E}\|f(s,\tilde{\phi}_s+z_s^q)\|^2_{\mathcal{H}}ds \qquad (13)\\
&\leq \bigg(\frac{M\alpha}{\Gamma(1+\alpha)}\bigg)^2\frac{b^\alpha}{\alpha}\int_0^t(t-s)^{\alpha-1}n(s)\Lambda_f\Big(\|\tilde{\phi}_s+z_s^q\|^2_{\mathcal{C}_h}\Big)ds \\
&\leq \bigg(\frac{M\alpha}{\Gamma(1+\alpha)}\bigg)^2\frac{b^{2\alpha}}{\alpha^2}\Lambda_f\Big(4l^2(q+M^2\mathbb{E}\|\phi(0)\|^2_{\mathcal{H}})+4\|\phi\|^2_{\mathcal{C}_h}\Big)\sup_{s\in J}n(s).
\end{aligned}$$

We have

$$\begin{aligned}
I_5 &\leq \mathbb{E}\bigg(\int_0^t \|(t-s)^{\alpha-1}T_\alpha(t-s)Bu^\varepsilon(s)\|ds\bigg)^2 \\
&\leq \bigg(\frac{M\alpha}{\Gamma(1+\alpha)}\bigg)^2\int_0^t(t-s)^{\alpha-1}ds\int_0^t(t-s)^{\alpha-1}\mathbb{E}\|Bu^\varepsilon(s)\|^2ds \\
&\leq \bigg(\frac{M\alpha}{\Gamma(1+\alpha)}\bigg)^2 M_B^2\frac{b^\alpha}{\alpha}\int_0^t(t-s)^{\alpha-1}\mathbb{E}\|u^\varepsilon(s)\|^2ds,
\end{aligned}$$

where $M_B = \|B\|$. Further, by using the assumptions $(H3)$-$(H5)$, Hölder inequality and Lemma 3.3, we get

$$\begin{aligned}
\mathbb{E}\|u^\varepsilon(s)\|^2 &\leq \frac{1}{\varepsilon^2}M_B^2\bigg(\frac{M\alpha}{\Gamma(1+\alpha)}\bigg)^2\bigg[7\|\mathbb{E}\hat{x}_b+\int_0^b\hat{\phi}(s)dw(s)\|^2 \\
&+ 7\mathbb{E}\|S_\alpha(b)\phi(0)\|^2+7\mathbb{E}\|G(0\phi)\|^2+7\mathbb{E}\|G(b,\tilde{\phi}_b+z_b)\|^2 \\
&+ 7\mathbb{E}\bigg\|\int_0^b(b-s)^{\alpha-1}AT_\alpha(b-s)G(s,\tilde{\phi}_s+z_s)ds\bigg\|^2 \\
&+ 7\mathbb{E}\bigg\|\int_0^b(b-s)^{\alpha-1}T_\alpha(b-s)f(s,\tilde{\phi}_s+z_s)ds\bigg\|^2 \\
&+ 7\mathbb{E}\bigg\|\int_0^b(b-s)^{\alpha-1}T_\alpha(b-s)\bigg[\int_{-\infty}^s\sigma(s,\tau,\tilde{\phi}_\tau+z_\tau)dw(\tau)\bigg]ds\bigg\|^2\bigg] \\
&\leq \frac{7}{\varepsilon^2}\bigg(\frac{MM_B\alpha}{\Gamma(1+\alpha)}\bigg)^2\bigg[2\|\mathbb{E}\hat{x}_b\|^2+2\int_0^b\mathbb{E}\|\hat{\phi}(s)\|^2ds+M^2\|\phi\|^2_{\mathcal{C}_h} \\
&+ M^2M_G\|A^{-\beta}\|^2(1+\|\phi\|^2_{\mathcal{C}_h})+M_G\|A^{-\beta}\|^2(1+4(l^2q+l^2M^2\mathbb{E}\|\phi(0)\|^2+\|\phi\|^2_{\mathcal{C}_h})) \\
&+ \frac{M_GK(\alpha,\beta)b^{2\alpha\beta}}{(\alpha\beta)^2}(1+4(l^2q+l^2M^2\mathbb{E}\|\phi(0)\|^2+\|\phi\|^2_{\mathcal{C}_h})) \\
&+ \bigg(\frac{Mb^\alpha}{\Gamma(1+\alpha)}\bigg)^2\Lambda_f(4(l^2q+l^2M^2\mathbb{E}\|\phi(0)\|^2+\|\phi\|^2_{\mathcal{C}_h}))\sup_{s\in J}n(s)+2\bigg(\frac{Mb^\alpha}{\Gamma(1+\alpha)}\bigg)^2M_k \\
&+ 2\bigg(\frac{Mb^\alpha}{\Gamma(1+\alpha)}\bigg)^2tr(Q)\Lambda_\sigma(4(l^2q+l^2M^2\mathbb{E}\|\phi(0)\|^2+\|\phi\|^2_{\mathcal{C}_h}))\sup_{s\in J}m(s)\bigg].
\end{aligned}$$

Now, we have

$$I_5 \leq \left(\frac{M_B M\alpha}{\Gamma(1+\alpha)}\right)^2 \frac{b^{2\alpha}}{\alpha^2}\frac{7}{\varepsilon^2}\left(\frac{MM_B\alpha}{\Gamma(1+\alpha)}\right)^2 M_C. \tag{14}$$

A similar argument involves Burkholder-Davis-Gundy's inequality and assumptions $(H3)$-$(H4)$, we obtain

$$\begin{aligned}
I_6 &\leq \mathbb{E}\left(\int_0^t \left\|(t-s)^{\alpha-1}T_\alpha(t-s)\left[\int_{-\infty}^s \sigma(s,\tau,\tilde{\phi}_\tau+z_\tau^q)dw(\tau)\right]\right\|_{\mathcal{H}} ds\right)^2 \\
&\leq \left(\frac{M\alpha}{\Gamma(1+\alpha)}\right)^2 \frac{b^\alpha}{\alpha}\int_0^t (t-s)^{\alpha-1}\mathbb{E}\left\|\int_{-\infty}^s \sigma(s,\tau,\tilde{\phi}_\tau+z_\tau^q)dw(\tau)\right\|_{\mathcal{H}}^2 ds \\
&\leq \left(\frac{M\alpha}{\Gamma(1+\alpha)}\right)^2 \frac{b^\alpha}{\alpha}\int_0^t (t-s)^{\alpha-1}\left(2M_k + 2tr(Q)\int_0^t \mathbb{E}\|\sigma(s,\tau,\tilde{\phi}_\tau+z_\tau^q)\|_{\mathcal{L}_2^0}^2 d\tau\right)ds \\
&\leq \left(\frac{M\alpha}{\Gamma(1+\alpha)}\right)^2 \frac{b^\alpha}{\alpha}\int_0^t (t-s)^{\alpha-1}(2M_k + 2tr(Q)m(s)\Lambda_\sigma(\|\tilde{\phi}_s+z_s^q\|_{\mathcal{C}_h}^2))ds \\
&\leq \left(\frac{M\alpha}{\Gamma(1+\alpha)}\right)^2 \frac{2M_k b^{2\alpha}}{\alpha^2} + \left(\frac{M\alpha}{\Gamma(1+\alpha)}\right)^2 \frac{2tr(Q)b^{2\alpha}}{\alpha^2}\Lambda_\sigma(4(l^2q + l^2M^2\mathbb{E}\|\phi(0)\|^2 + \|\phi\|_{\mathcal{C}_h}^2))\sup_{s\in J} m(s).
\end{aligned} \tag{15}$$

Combining these estimates (10)-(15) yields

$$\begin{aligned}
q &\leq \mathbb{E}\|\Upsilon(z^q)(t)\|_{\mathcal{H}}^2 \\
&\leq L_0 + 24M_G\|A^{-\beta}\|^2 l^2 q + \frac{24l^2M_GC_{1-\beta}^2\Gamma^2(1+\beta)b^{2\alpha\beta}}{\beta^2\Gamma^2(1+\alpha\beta)}q + 42\left(\frac{M_BM\alpha}{\Gamma(1+\alpha)}\right)^2\frac{b^{2\alpha}}{\alpha^2\varepsilon^2}\left(\frac{MM_B\alpha}{\Gamma(1+\alpha)}\right)^2 M_C \\
&+ 6\left(\frac{Mb^\alpha}{\Gamma(1+\alpha)}\right)^2\Lambda_f\Big(4l^2(q + M^2\mathbb{E}\|\phi(0)\|_{\mathcal{H}}^2) + 4\|\phi\|_{\mathcal{C}_h}^2\Big)\sup_{s\in J} n(s) \\
&+ 12\left(\frac{Mb^\alpha}{\Gamma(1+\alpha)}\right)^2 tr(Q)\Lambda_\sigma(4(l^2q + l^2M^2\mathbb{E}\|\phi(0)\|^2 + \|\phi\|_{\mathcal{C}_h}^2))\sup_{s\in J} m(s).
\end{aligned} \tag{16}$$

where

$$\begin{aligned}
L_0 &= 6M^2\|A^{-\beta}\|^2 M_G\Big(1+\|\phi\|_{\mathcal{C}_h}^2\Big) + 6M_G\|A^{-\beta}\|^2\Big(1 + 4l^2M^2\mathbb{E}\|\phi(0)\|_{\mathcal{H}}^2) + 4\|\phi\|_{\mathcal{C}_h}^2\Big) \\
&+ \frac{6M_GC_{1-\beta}^2\Gamma^2(1+\beta)}{\Gamma^2(1+\alpha\beta)}\frac{b^{2\alpha\beta}}{\beta^2}(1 + 4l^2M^2\mathbb{E}\|\phi(0)\|_{\mathcal{H}}^2) + 4\|\phi\|_{\mathcal{C}_h}^2) + \frac{12M_kM^2b^{2\alpha}}{\Gamma^2(1+\alpha)}.
\end{aligned}$$

Dividing both sides of (16) by q and taking $q \to \infty$, we obtain that

$$\begin{aligned}
&\Bigg[4M_G\|A^{-\beta}\|^2 l^2 + 4M_G\frac{l^2C_{1-\beta}^2\Gamma^2(1+\beta)b^{2\alpha\beta}}{\beta^2\Gamma^2(1+\alpha\beta)} + 4\gamma\frac{l^2M^2b^{2\alpha}}{\Gamma^2(1+\alpha)}\sup_{s\in J} n(s) \\
&+8\vartheta tr(Q)\frac{l^2M^2b^{2\alpha}}{\Gamma^2(1+\alpha)}\sup_{s\in J} m(s)\Bigg] \times \left[6 + \frac{42}{\varepsilon^2}\left(\frac{\alpha MM_B}{\Gamma(\alpha+1)}\right)^4\frac{b^{2\alpha}}{\alpha^2}\right] \geq 1,
\end{aligned}$$

which is a contradiction to our assumption. Thus for $\varepsilon > 0$, for some positive number q, $\Upsilon(\mathcal{B}_q) \subset \mathcal{B}_q$.

□

Lemma 3.11 *Let assumptions* $(H1)$*-*$(H5)$ *hold. Then* Υ_1 *is contractive.*

f Let $u, v \in \mathcal{B}_q$. Then

$$\begin{aligned}
& \mathbb{E}\|(\Upsilon_1 u)(t) - (\Upsilon_1 v)(t)\|^2_{\mathcal{H}} \\
\leq\ & 2\mathbb{E}\|G(t, \tilde{\phi}_t + u_t) - G(t, \tilde{\phi}_t + v_t)\|^2_{\mathcal{H}} + 2\mathbb{E}\left\| \int_0^t (t-s)^{\alpha-1} A T_\alpha(t-s) G(s, \tilde{\phi}_s + z_s) ds \right\|^2_{\mathcal{H}} \\
\leq\ & 2\|A^{-\beta}\|^2 M_G \|u_t - v_t\|^2_{\mathcal{C}_h} + 2K(\alpha,\beta)\mathbb{E}\left[\int_0^t (t-s)^{\alpha\beta-1} \|A^\beta(G(s, \tilde{\phi}_s + u_s) - G(s, \tilde{\phi}_s + v_s))\|_{\mathcal{H}} ds \right]^2 \\
\leq\ & 2\|A^{-\beta}\|^2 M_G \|u_t - v_t\|^2_{\mathcal{C}_h} + \frac{2K(\alpha,\beta) b^{\alpha\beta}}{\alpha\beta} \int_0^t (t-s)^{\alpha\beta-1} \mathbb{E}\|A^\beta(G(s, \tilde{\phi}_s + u_s) - G(s, \tilde{\phi}_s + v_s))\|^2_{\mathcal{H}} ds \\
\leq\ & 2\|A^{-\beta}\|^2 M_G \|u_t - v_t\|^2_{\mathcal{C}_h} + \frac{2M_G K(\alpha,\beta) b^{\alpha\beta}}{\alpha\beta} \int_0^t (t-s)^{\alpha\beta-1} \|u_s - v_s\|^2_{\mathcal{C}_h} ds.
\end{aligned}$$

Hence,

$$\mathbb{E}\|(\Upsilon_1 u)(t) - (\Upsilon_1 v)(t)\|^2_{\mathcal{H}} \leq 4M_G l^2 \left(\|A^{-\beta}\|^2 + K(\alpha,\beta) \frac{b^{2\alpha\beta}}{(\alpha\beta)^2} \right) \sup_{0\leq s\leq t} \mathbb{E}\|u(s) - v(s)\|^2_{\mathcal{H}},$$

where we have used the fact that $u_0 = v_0 = 0$; $K(\alpha,\beta)$ is defined in (9).
Thus,

$$\mathbb{E}\|\Upsilon_1 u - \Upsilon_1 v\|^2_b \leq 4M_G l^2 \left(\|A^{-\beta}\|^2 + K(\alpha,\beta) \frac{b^{2\alpha\beta}}{(\alpha\beta)^2} \right) \sup_{0\leq s\leq t} \mathbb{E}\|u - v\|^2_b,$$

so, Υ_1 is a contraction by our assumption in Theorem 3.6.

□

Let $q > 0$ an $\Upsilon_2(\mathcal{B}_q) \subset \mathcal{B}_q$.

Lemma 3.12 *Let assumptions* $(H1)$*-*$(H5)$ *hold. Then* Υ_2 *maps bounded sets to bounded sets in* $\mathcal{B}_q$.

Proof. For each $t \in J$, $z \in \mathcal{B}_q$ and $\varepsilon > 0$, from (6), we have

$$\|z_t + \tilde{\phi}_t\|^2_{\mathcal{C}_h} \leq 4l^2(q + M^2\mathbb{E}\|\phi(0)\|^2_{\mathcal{H}}) + 4\|\phi\|^2_{\mathcal{C}_h} := q'.$$

By the similar argument as Lemma 3.10, we obtain

$$\begin{aligned}
\mathbb{E}\|\Upsilon_2 z(t)\|^2_{\mathcal{H}} &\leq 3\mathbb{E}\left\|\int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)f(s,\tilde{\phi}_s+z_s)ds\right\|^2_{\mathcal{H}} + 3\mathbb{E}\left\|\int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)Bu^\varepsilon(s)ds\right\|^2_{\mathcal{H}} \\
&+ 3\mathbb{E}\left\|\int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)\left[\int_{-\infty}^s \sigma(s,\tau,\tilde{\phi}_{s\tau}+z_{s\tau})dw(\tau)\right]ds\right\|^2_{\mathcal{H}} \\
&\leq 3\left(\frac{M\alpha}{\Gamma(1+\alpha)}\right)^2 \frac{b^\alpha}{\alpha}\int_0^t (t-s)^{\alpha-1}n(s)\Lambda_f\left(\|\tilde{\phi}_s+z^q_s\|^2_{\mathcal{C}_h}\right)ds \\
&+ 3\left(\frac{M_B M\alpha}{\Gamma(1+\alpha)}\right)^4 \frac{b^{2\alpha}}{\alpha^2}\frac{7}{\varepsilon^2}M_C \\
&+ 3\left(\frac{M\alpha}{\Gamma(1+\alpha)}\right)^2 \frac{b^\alpha}{\alpha}\int_0^t (t-s)^{\alpha-1}(2M_k+2tr(Q)m(s)\Lambda_\sigma(\|\tilde{\phi}_s+z^q_s\|^2_{\mathcal{C}_h}))ds \\
&\leq 3\left(\frac{M\alpha}{\Gamma(1+\alpha)}\right)^2 \frac{b^{2\alpha}}{\alpha^2}\Lambda_f(q')\sup_{s\in J} n(s) + 3\left(\frac{M_B M\alpha}{\Gamma(1+\alpha)}\right)^4 \frac{b^{2\alpha}}{\alpha^2}\frac{7}{\varepsilon^2}M_C \\
&+ 4\left(\frac{M\alpha}{\Gamma(1+\alpha)}\right)^2 \frac{M_k b^\alpha}{\alpha^2} + 4\left(\frac{M\alpha}{\Gamma(1+\alpha)}\right)^2 \frac{tr(Q)b^\alpha}{\alpha^2}\Lambda_\sigma(q')\sup_{s\in J} m(s) \\
&:= \Delta,
\end{aligned}$$

which implies that for each $z \in \mathcal{B}_q$, $\|\Upsilon_2 z\|^2_b \leq \Delta$.

□

Lemma 3.13 *Let assumptions (H1)-(H5) hold. Then the set $\{\Upsilon_2 z, z\in\mathcal{B}_q\}$ is an equicontinuous family of functions on J.*

Proof. Let $0<\epsilon<t<b$ and $\delta>0$ such that $\|T_\alpha(s_1)-T_\alpha(s_2)\|<\epsilon$, for every $s_1,s_2\in J$ with $|s_1-s_2|<\delta$. For $z\in\mathcal{B}_q$, $0<|h|<\delta$, $t+h\in J$, we have

$$\begin{aligned}
&\mathbb{E}\|\Upsilon_2 z(t+h)-\Upsilon_2 z(t)\|^2_{\mathcal{H}} \\
\leq\ & 9\mathbb{E}\left\|\int_0^t [(t+h-s)^{\alpha-1}-(t-s)^{\alpha-1}]T_\alpha(t+h-s)Bu^\varepsilon(s)ds\right\|^2_{\mathcal{H}} \\
+\ & 9\mathbb{E}\left\|\int_t^{t+h} (t+h-s)^{\alpha-1}T_\alpha(t+h-s)Bu^\varepsilon(s)ds\right\|^2_{\mathcal{H}} \\
+\ & 9\mathbb{E}\left\|\int_0^t (t-s)^{\alpha-1}[T_\alpha(t+h-s)-T_\alpha(t-s)]Bu^\varepsilon(s)ds\right\|^2_{\mathcal{H}} \\
+\ & 9\mathbb{E}\left\|\int_0^t [(t+h-s)^{\alpha-1}-(t-s)^{\alpha-1}]T_\alpha(t+h-s)f(s,\tilde{\phi}_s+z_s)ds\right\|^2_{\mathcal{H}} \\
+\ & 9\mathbb{E}\left\|\int_t^{t+h} (t+h-s)^{\alpha-1}T_\alpha(t+h-s)f(s,\tilde{\phi}_s+z_s)ds\right\|^2_{\mathcal{H}} \\
+\ & 9\mathbb{E}\left\|\int_0^t (t-s)^{\alpha-1}[T_\alpha(t+h-s)-T_\alpha(t-s)]f(s,\tilde{\phi}_s+z_s)ds\right\|^2_{\mathcal{H}}
\end{aligned}$$

$$
\begin{aligned}
&+ \quad 9\mathbb{E}\left\| \int_0^t [(t+h-s)^{\alpha-1} - (t-s)^{\alpha-1}] T_\alpha(t+h-s) \left[\int_{-\infty}^s \sigma(s,\tau,\tilde{\phi}_\tau + z_\tau) dw(\tau) \right] ds \right\|_{\mathcal{H}}^2 \\
&+ \quad 9\mathbb{E}\left\| \int_t^{t+h} (t+h-s)^{\alpha-1} T_\alpha(t+h-s) \left[\int_{-\infty}^s \sigma(s,\tau,\tilde{\phi}_\tau + z_\tau) dw(\tau) \right] ds \right\|_{\mathcal{H}}^2 \\
&+ \quad 9\mathbb{E}\left\| \int_0^t (t-s)^{\alpha-1} [T_\alpha(t+h-s) - T_\alpha(t-s)] \left[\int_{-\infty}^s \sigma(s,\tau,\tilde{\phi}_\tau + z_\tau) dw(\tau) \right] ds \right\|_{\mathcal{H}}^2 .
\end{aligned}
$$

Applying Lemma 3.3, assumptions $(H4)$-$(H5)$ and the Hölder inequality, we obtain

$$
\begin{aligned}
&\mathbb{E}\|\Upsilon_2 z(t+h) - \Upsilon_2 z(t)\|_{\mathcal{H}}^2 \\
\leq \quad & 9\left(\frac{M_B M\alpha}{\Gamma(1+\alpha)}\right)^2 \int_0^t [(t+h-s)^{\alpha-1} - (t-s)^{\alpha-1}] ds \int_0^t [(t+h-s)^{\alpha-1} - (t-s)^{\alpha-1}] \mathbb{E}\|u^\varepsilon(s)\|^2 ds \\
+ \quad & 9\left(\frac{M_B M\alpha}{\Gamma(1+\alpha)}\right)^2 \frac{h^\alpha}{\alpha} \int_t^{t+h} (t+h-s)^{\alpha-1} \mathbb{E}\|u^\varepsilon(s)\|^2 ds + 9\epsilon^2 M_B^2 \frac{b^\alpha}{\alpha} \int_0^t (t-s)^{\alpha-1} \mathbb{E}\|u^\varepsilon(s)\|^2 ds \\
+ \quad & 9\left(\frac{M\alpha}{\Gamma(1+\alpha)}\right)^2 \int_0^t [(t+h-s)^{\alpha-1} - (t-s)^{\alpha-1}] ds \\
& \times \int_0^t [(t+h-s)^{\alpha-1} - (t-s)^{\alpha-1}] n(s) \Lambda_f(q') ds \\
+ \quad & 9\left(\frac{M\alpha}{\Gamma(1+\alpha)}\right)^2 \frac{h^\alpha}{\alpha} \int_t^{t+h} (t+h-s)^{\alpha-1} n(s) \Lambda_f(q') ds \\
+ \quad & 9\epsilon^2 \frac{b^\alpha}{\alpha} \int_0^t (t-s)^{\alpha-1} n(s) \Lambda_f(q') ds \\
+ \quad & 9\left(\frac{M\alpha}{\Gamma(1+\alpha)}\right)^2 \int_0^t [(t+h-s)^{\alpha-1} - (t-s)^{\alpha-1}] ds \\
& \times \int_0^t [(t+h-s)^{\alpha-1} - (t-s)^{\alpha-1}] [2M_k + 2tr(Q) m(s) \Lambda_\sigma(q')] ds \\
+ \quad & 9\left(\frac{M\alpha}{\Gamma(1+\alpha)}\right)^2 \frac{h^\alpha}{\alpha} \int_t^{t+h} (t+h-s)^{\alpha-1} [2M_k + 2tr(Q) m(s) \Lambda_\sigma(q')] ds \\
+ \quad & 9\epsilon^2 \frac{b^\alpha}{\alpha} \int_0^t (t-s)^{\alpha-1} [2M_k + 2tr(Q) m(s) \Lambda_\sigma(q')] ds.
\end{aligned}
$$

Therefore, for ϵ sufficiently small, the right-hand side of the above inequality tends to zero as $h \to 0$. On the other hand, the compactness of $T_\alpha(t), t > 0$ implies the continuity in the uniform operator topology. Thus, the set $\{\Upsilon_2 z, z \in \mathcal{B}_q\}$ is equicontinuous.

□

Lemma 3.14 *Let assumptions* $(H1)$-$(H5)$ *hold. then* Υ_2 *maps* $\mathcal{B}_q$ *into a precompact set in* $\mathcal{B}_q$.

Proof. Let $0 < t \leq b$ be fixed and ϵ be a real number satisfying $0 < \epsilon < t$. For $\delta > 0$, define an operator $\Upsilon_2^{\epsilon,\delta}$ on $\mathcal{B}_q$ by

$$\begin{aligned}
(\Upsilon_2^{\epsilon,\delta})(t) &= \alpha \int_0^{t-\epsilon} \int_\delta^\infty \theta(t-s)^{\alpha-1}\xi_\alpha(\theta)S((t-s)^\alpha\theta)Bu^\varepsilon(s)d\theta ds \\
&+ \alpha \int_0^{t-\epsilon} \int_\delta^\infty \theta(t-s)^{\alpha-1}\xi_\alpha(\theta)S((t-s)^\alpha\theta)f(s,\tilde{\phi}_s+z_s)d\theta ds \\
&+ \alpha \int_0^{t-\epsilon} \int_\delta^\infty \theta(t-s)^{\alpha-1}\xi_\alpha(\theta)S((t-s)^\alpha\theta)\left[\int_{-\infty}^s \sigma(s,\tau,\tilde{\phi}_\tau+z_\tau)dw(\tau)\right]d\theta ds \\
&= S(\epsilon^\alpha\delta)\alpha \int_0^{t-\epsilon} \int_\delta^\infty \theta(t-s)^{\alpha-1}\xi_\alpha(\theta)S((t-s)^\alpha\theta-\epsilon^\alpha\delta)Bu^\varepsilon(s)d\theta ds \\
&+ S(\epsilon^\alpha\delta)\alpha \int_0^{t-\epsilon} \int_\delta^\infty \theta(t-s)^{\alpha-1}\xi_\alpha(\theta)S((t-s)^\alpha\theta-\epsilon^\alpha\delta)f(s,\tilde{\phi}_s+z_s)d\theta ds \\
&+ S(\epsilon^\alpha\delta)\alpha \int_0^{t-\epsilon} \int_\delta^\infty \theta(t-s)^{\alpha-1}\xi_\alpha(\theta)S((t-s)^\alpha\theta-\epsilon^\alpha\delta)\left[\int_{-\infty}^s \sigma(s,\tau,\tilde{\phi}_\tau+z_\tau)dw(\tau)\right]d\theta ds.
\end{aligned}$$

Since $S(t)$, $t > 0$, is a compact operator, the set $\{(\Upsilon_2^{\epsilon,\delta}z)(t), z \in \mathcal{B}_q\}$ is precompact in $\mathcal{H}$ for every $\epsilon \in (0,t)$, $\delta > 0$. Moreover, for each $z \in \mathcal{B}_q$, we have

$$\begin{aligned}
&\mathbb{E}\|(\Upsilon_2 z)(t) - (\Upsilon_2^{\epsilon,\delta}z)(t)\|_{\mathcal{H}}^2 \\
\leq\ & 6\alpha^2\mathbb{E}\left\|\int_0^t\int_0^\delta \theta(t-s)^{\alpha-1}\xi_\alpha(\theta)S((t-s)^\alpha\theta)Bu^\varepsilon(s)d\theta ds\right\|_{\mathcal{H}}^2 \\
+\ & 6\alpha^2\mathbb{E}\left\|\int_{t-\epsilon}^t\int_\delta^\infty \theta(t-s)^{\alpha-1}\xi_\alpha(\theta)S((t-s)^\alpha\theta)Bu^\varepsilon(s)d\theta ds\right\|_{\mathcal{H}}^2 \\
+\ & 6\alpha^2\mathbb{E}\left\|\int_0^t\int_0^\delta \theta(t-s)^{\alpha-1}\xi_\alpha(\theta)S((t-s)^\alpha\theta)f(s,\tilde{\phi}_s+z_s)d\theta ds\right\|_{\mathcal{H}}^2 \\
+\ & 6\alpha^2\mathbb{E}\left\|\int_{t-\epsilon}^t\int_\delta^\infty \theta(t-s)^{\alpha-1}\xi_\alpha(\theta)S((t-s)^\alpha\theta)f(s,\tilde{\phi}_s+z_s)d\theta ds\right\|_{\mathcal{H}}^2 \\
+\ & 6\alpha^2\mathbb{E}\left\|\int_0^t\int_0^\delta \theta(t-s)^{\alpha-1}\xi_\alpha(\theta)S((t-s)^\alpha\theta)\left[\int_{-\infty}^s \sigma(s,\tau,\tilde{\phi}_\tau+z_\tau)dw(\tau)\right]d\theta ds\right\|_{\mathcal{H}}^2 \\
+\ & 6\alpha^2\mathbb{E}\left\|\int_{t-\epsilon}^t\int_\delta^\infty \theta(t-s)^{\alpha-1}\xi_\alpha(\theta)S((t-s)^\alpha\theta)\left[\int_{-\infty}^s \sigma(s,\tau,\tilde{\phi}_\tau+z_\tau)dw(\tau)\right]d\theta ds\right\|_{\mathcal{H}}^2.
\end{aligned}$$

A similar argument as before can show that

$$\begin{aligned}
&\mathbb{E}\|(\Upsilon_2 z)(t) - (\Upsilon_2^{\epsilon,\delta}z)(t)\|_{\mathcal{H}}^2 \\
\leq\ & 6\alpha M^2 b^\alpha \int_0^t (t-s)^{\alpha-1}\left[M_B^2\frac{7}{\alpha^2}\left(\frac{\alpha M_B M}{\Gamma(1+\alpha)}\right)^2 M_C + n(s)\Lambda_f(q') + [2M_k + 2tr(Q)m(s)\Lambda_\sigma(q')\right]ds\times \\
&\left(\int_0^\delta \theta\xi_\alpha(\theta)d\theta\right)^2 + \frac{6\alpha M^2\epsilon^\alpha}{\Gamma(1+\alpha)}\int_{t-\epsilon}^t (t-s)^{\alpha-1}\left[M_B^2\frac{7}{\alpha^2}\left(\frac{\alpha M_B M}{\Gamma(1+\alpha)}\right)^2 M_C + n(s)\Lambda_f(q')\right. \\
&\left.+2M_k + 2tr(Q)m(s)\Lambda_\sigma(q')\right]ds \to 0 \quad \text{as } \epsilon,\delta \to 0^+.
\end{aligned}$$

Therefore, there are relatively compact sets arbitrary close to the set $\{\Upsilon_2(t), z \in \mathcal{B}_q\}$; hence the set $\{(\Upsilon_2 z)(t), z \in \mathcal{B}_q\}$ is also precompact in $\mathcal{B}_q$.

□

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