AMO - Advanced Modeling and Optimization, Volume 15, Number 2, 2013

Fuzzy stochastic EOQ inventory model for items with imperfect quality and shortages are backlogged

Ravi Shankar Kumar[a] and A. Goswami[b]

Department of Mathematics, Indian Institute of Technology, Kharagpur - 721302, India

Abstract

This article deals with an economic order quantity (EOQ) inventory model for items with imperfect quality in fuzzy stochastic environment, wherein shortages are allowed and completely backlogged. Fuzzy stochastic environment means linguistic 'impreciseness' and statistical 'uncertainty' both appear simultaneously. Due to uncertain demand trend, imperfect production process, natural disaster etc., the demand rate or imperfect quality items in the lot size can't predict precisely or to fit the exact probability density function. In this context, we assume that demand rate as a fuzzy number and fraction of defective items as a fuzzy random variable. We formulate the model and derive the total profit which is a function of fuzzy random variable. The fuzzy random renewal reward theorem is used to find the fuzzy expected total profit per unit time. The fuzzy expected total profit function is defuzzified by using the signed distance method. The closed form solution of the model is derived and subsequently the concavity of the total profit function is proved. The solution procedure is illustrated with the help of numerical examples. Sensitivity of decision variables for change in different parameters is examined and discussed.

Keywords: EOQ inventory model; Imperfect quality; Fuzzy random variable; Fuzzy renewal reward theorem; Signed distance method.

1 Introduction

Incorporation of real life situations in the inventory modeling acquired a keen area of research in the recent trends. It provides the competitive advantage as well as keep consistency in maintenance of authenticity of the organization. Though the EOQ (economic order quantity) and the EPQ (economic production quantity) models have been successfully applied in industry for a long period of time, but still some of the assumptions of these models are not realistic. One of the unrealistic assumption in an inventory model is that all items produced/received are of good quality. But, in a production system due to defective production process, natural disasters, damage or breakage in transit, or for many other reasons, the lot sizes produced/received may contain some defective items. Models related to imperfect quality production process, initially considered by [Rosenblatt and Lee, 1986] and [Porteus, 1986]. Later, [Salameh and Jaber,

[a]E-mail: ravikhushi412@yahoo.co.in

[b]All Correspondence to: E-mail: goswami@maths.iitkgp.ernet.in

AMO - Advanced Modeling and Optimization. ISSN: 1841-4311

2000] discussed an inventory model, where all items were screened before meeting up the demand and imperfect items sold as a single batch at a discounted price prior to receiving next shipment. Several authors [Chung et al., 2009; Eroglu and Ozdemir, 2007; Maddah and Jaber, 2008; Maddah et al., 2010; Papachristos and Konstantaras, 2006; Wee et al., 2007] have modified or extended the [Salameh and Jaber, 2000] model in various directions. [Papachristos and Konstantaras, 2006] emphasized on the assumptions of Salameh and Jaber's model, especially to avoid the shortages during screening period. [Eroglu and Ozdemir, 2007; Wee et al., 2007] extended that model in the case of backorder. [Eroglu and Ozdemir, 2007] considered that the defective items are two types, one is imperfect quality and the other is scrap items. [Chung et al., 2009] assumed that inventories were carried in two warehouse namely, owns warehouse and ranted warehouse. [Maddah and Jaber, 2008], [Maddah et al., 2010] and [Chang and Ho, 2010] enhanced the [Salameh and Jaber, 2000] and [Wee et al., 2007] models by implementing the renewal reward theorem to obtain the expected annual profit.

From the article of [Salameh and Jaber, 2000], we notice that some of the previous researchers have assumed that the imperfect quality items and demand rate are deterministic variables or random variables with a known probability distribution function. But real fact is quite different. It may not always be possible to estimate the probability distribution function or predict precise values of these variables. With the development of fuzzy set theory, uncertainty theory, rough theory, etc., the technique and approaches of these theories are being widely accepted and implemented in each and every area of science, engineering and mathematics. Here, we focus on the implementation of fuzzy set theory in inventory management. [Chang et al., 1998] extended classical EOQ model with backorder by assuming backorder quantity as fuzzy number. [Yao et al., 2000] extended the classical EOQ model in fuzzy environment with the assumption that demand rate is fuzzy number. [Chang et al., 2006] developed a fuzzy inventory model for deteriorating items and shortage, where cost coefficients were triangular fuzzy numbers. [Chang, 2004] extended the [Salameh and Jaber, 2000] model in fuzzy environment by assuming that the annual demand and fraction of defective items are fuzzy numbers. [Kazemi et al., 2010] developed a fuzzy EOQ model with backorder, where demand rate and cost coefficients are fuzzy numbers. [Björk, 2009] extended the classical EOQ model with backorder in fuzzy environment by assuming that demand rate and lead time are fuzzy numbers. [Liu and Zheng, 2012] extended the [Eroglu and Ozdemir, 2007] model in fuzzy environment by considering the fraction of defective items as fuzzy variable. [Kumar et al., 2012] address fuzzy EOQ models with ramp type demand rate and Weibull deterioration rate, wherein cost parameters and backlogging rate are taken as fuzzy numbers.

Fuzzy set theory concern with the linguistic 'impreciseness' or 'vagueness'. But, in real life situations many systems or events concern not only with this linguistic 'impreciseness' or 'vagueness', but also with statistical 'uncertainty' together. In 1978, [Kwakernaak, 1978] introduced the concept of fuzzy random variable and its fuzzy expectation. Later, [Puri and Ralescu, 1986] developed and discussed it in another way. [Gil et al., 2006] and [Shapiro, 2009] closely studied the both type of fuzzy random variables. Some authors like [Chang et al., 2006; Dey and Chakraborty, 2009; Dutta et al., 2007; Lin, 2008] etc. extended the classical continuous review or periodic review inventory model in fuzzy random environment. [Hu et al., 2010] developed a single-period supply chain model with defective products and fuzzy random demand. Recently, [Das et al., 2011] developed a fuzzy-stochastic inventory model with imperfect quality items, where fraction of imperfect quality items and machine failure time are random variables while cost coefficients are fuzzy numbers.

In this study, we extend the model of [Chang and Ho, 2010] in fuzzy random environment. We assume that the annual demand is a fuzzy number and fraction of imperfect quality items is a fuzzy random variable. Consequently, the total profit per cycle and scheduling period become as functions of fuzzy random variable. Between the two consecutive replenishment namely $(n-1)$th and nth, the inter-arrival time (scheduling period) is a fuzzy random variable. The process continuously repeats itself for infinite time horizon. So, this process can be referred as a fuzzy renewal process [Wang and Watada, 2009; Zhao et al., 2007]. Hence, we apply fuzzy random renewal reward theorem [Hwang, 2000; Hwang and Yang, 2011] to estimate the fuzzy expected annual profit function. The fuzzy expected total profit and fuzzy expected scheduling period are unique fuzzy numbers. Further, for each case, we employ Yao and Wu [2000] ranking criterion for fuzzy numbers to find the crisp estimate of total average annual profit along with the corresponding optimal lot size and backorder quantity. We show analytically that the defuzzified cost function is concave. We consider some numerical examples in the support of developed model. Lastly, we have studied the effect of decision variables and cost function for changes in different parameter values.

2 Preliminaries

Before presenting the proposed fuzzy inventory model with backorders, we need to introduce some definitions and basics about fuzzy set [Kaufmann and Gupta, 1991; Yao and Wu, 2000; Zimmermann, 2001], fuzzy random variable [Gil et al., 2006; Kwakernaak, 1978; Shapiro, 2009] and fuzzy random renewal process [Hwang, 2000; Hwang and Yang, 2011; Popova and Wu, 1999; Wang and Watada, 2009].

2.1 Fuzzy set

Definition 2.1 A fuzzy set $\tilde{A}$ on the given universe X is a set of ordered pairs

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) : x \in X\},$$

where $\mu_{\tilde{A}} : X \longrightarrow [0,1]$ is called membership function or grade of membership of x in $\tilde{A}$.

Remark 2.1 *If the range of $\mu_{\tilde{A}}$ admits only two values 0 and 1, then $\mu_{\tilde{A}}$ degenerates to a usual set characteristic function.*

Definition 2.2 If $\tilde{A}$ is a fuzzy set in X, then the crisp set $A_\alpha = \{x \in X : \mu_{\tilde{A}}(x) \geq \alpha\}$ is called α-cut (or α-level) set.

Definition 2.3 A fuzzy set $\tilde{A}$ is convex if $\mu_{\tilde{A}}(\lambda x_1 + (1-\lambda)x_2) \geq \min\big(\mu_{\tilde{A}}(x_1), \mu_{\tilde{A}}(x_2)\big)$.

Definition 2.4 A fuzzy set $\tilde{A}$ of X is called normal if there exists a $x \in X$ such that $\mu_{\tilde{A}}(x) = 1$.

Definition 2.5 A fuzzy number $\tilde{M}$, is a convex normalized fuzzy set on real line $\mathbb{R}$ and its membership function $\mu_{\tilde{M}}$ is piecewise continuous.

Definition 2.6 A fuzzy set $\tilde{A} = (a, b, c)$, where $a < b < c$ and defined on $\mathbb{R}$, is called triangular fuzzy number, if membership function of $\tilde{A}$ is given by

$$\mu_{\tilde{A}}(x) = \begin{cases} \frac{x-a}{b-a}, & a \leq x \leq b; \\ \frac{c-x}{c-b}, & b \leq x \leq c; \\ 0, & \text{otherwise.} \end{cases}$$

Interval arithmetic [Kaufmann and Gupta, 1991]
Let us suppose that $I_1 = [a, b]$ and $I_2 = [c, d]$, where $a < b$ and $c < d$, be two intervals defined by ordered pairs of real numbers with lower and upper bounds. Then the following relation hold.

$$\begin{aligned} [a, b] + [c, d] &= [a + c, b + d], \\ [a, b] - [c, d] &= [a - d, b - c], \\ [a, b].[c, d] &= [\min(ac, ad, bc, bd), \ \ \max(ac, ad, bc, bd)], \\ [a, b] \div [c, d] &= [a, b].\left[\frac{1}{d}, \frac{1}{c}\right] \quad \text{provided that } 0 \notin [c, d] \\ \text{and } k[a, b] &= \begin{cases} [ka, kb], & \text{for } k > 0; \\ [kb, ka], & \text{for } k < 0. \end{cases} \end{aligned}$$

Decomposition principle [Kaufmann and Gupta, 1991; Yao and Wu, 2000]
Suppose $\mathcal{F}_{\mathcal{R}}$ be family of fuzzy numbers on real line $\mathbb{R}$, whose elements satisfy the properties of definition 2.5. For each $\tilde{M} \in \mathcal{F}_{\mathcal{R}}$ and $0 \leq \alpha \leq 1$, the α-cut of $\tilde{M}$ is $M_\alpha = [M_\alpha^-, M_\alpha^+]$, a closed interval. The decomposition principle allow us to express the $\tilde{M}$ as

$$\tilde{M} = \bigcup_{\alpha \in [0,1]} [M_\alpha^-, M_\alpha^+]$$

with membership function

$$\mu_{\tilde{M}}(x) = \bigvee_{\alpha \in [0,1]} \alpha \wedge C_{M_\alpha}(x) = \bigvee_{\alpha \in [0,1]} \mu_{M_\alpha}(x),$$

where M_α^- and M_α^+ are the left and right end points of the closed interval $[M_\alpha^-, M_\alpha^+]$, and

$$C_{M_\alpha}(x) = \begin{cases} 1, & x \in M_\alpha; \\ 0, & \text{othewise.} \end{cases}$$

Definition 2.7 [Yao and Wu, 2000]
Let $\tilde{M} \in \mathcal{F}_{\mathcal{R}}$, then the signed distance of $\tilde{M}$ can be defined as

$$d(\tilde{M}, 0) = \frac{1}{2} \int_0^1 [M_\alpha^- + M_\alpha^+] d\alpha.$$

2.2 Fuzzy random renewal reward process

Definition 2.8 [Kwakernaak, 1978; Shapiro, 2009]
Let $(\Omega, \mathcal{B}, P)$ be a probability space, where Ω is sample space, $\mathcal{B}$ is σ-algebra of subsets of Ω, and P is the probability measure. Then the fuzzy random variable $\tilde{X}$, is measurable function from Ω to family of fuzzy numbers $\mathcal{F}_{\mathcal{R}}$. For each $\alpha \in (0,1]$ and $\omega \in \Omega$, $\tilde{X}(\omega) \in \mathcal{F}_{\mathcal{R}}$ satisfies the following properties:
$X_\alpha^-(\omega) = \inf X_\alpha(\omega)$ and $X_\alpha^+(\omega) = \sup X_\alpha(\omega)$ are real valued random variables on $(\Omega, \mathcal{B}, P)$; and their expectations EX_α^- and EX_α^+ exist. If the expectations EX_α^- and EX_α^+ exist, then $EX_\alpha^- = E[X_\alpha^-] = \int_\Omega X_\alpha^- f(x)dx$ and $EX_\alpha^+ = E[X_\alpha^+] = \int_\Omega X_\alpha^+ f(x)dx$, where $f(x)$ is probability density function.

Let $\tilde{X}_n$ be a fuzzy random variable denote the interarrival time between the $(n-1)$th and nth events, $n = 1, 2, \ldots$, respectively. Define $\tilde{S}_0 = 0$ and

$$\tilde{S}_n = \sum_{i=1}^{n} \tilde{X}_i, \ \forall \ n \geq 1, \tag{2.1}$$

then the process $\{\tilde{S}_n, \ n \geq 1\}$ is called a fuzzy stochastic process [Zhao et al., 2007]. For each $\omega \in \Omega$ and $n \in N$(set of positive integers), $\tilde{S}_n(\omega)$ is a fuzzy number. For $\alpha \in (0,1]$, the α-cut of $\tilde{S}_n(\omega)$ is denoted by $S_{n,\alpha}(\omega)$ and is defined as

$$S_{n,\alpha}(\omega) = [S_{n,\alpha}^-(\omega), \ S_{n,\alpha}^+(\omega)] = \left[\inf\{t : \mu_{\tilde{S}_n(\omega)}(t) \geq \alpha\}, \ \sup\{t : \mu_{\tilde{S}_n(\omega)}(t) \geq \alpha\}\right]. \tag{2.2}$$

Let $\tilde{N}(t)$ denote the total number of the events that have occurred by time t, i.e.,

$$\tilde{N}(t) = \max\{n : 0 \leq \tilde{S}_n \leq t\}.$$

For any $\omega \in \Omega$ and $\alpha \in [0,1]$, we can define

$$N_\alpha^-(t)(\omega) = \max\{n : 0 \leq S_{n,\alpha}^-(\omega) \leq t\} \tag{2.3}$$

$$\text{and } N_\alpha^+(t)(\omega) = \max\{n : 0 \leq S_{n,\alpha}^+(\omega) \leq t\}. \tag{2.4}$$

For a $\omega \in \Omega$, $\tilde{N}(t)(\omega)$ is a non-negative integer-valued fuzzy number. Hence $\tilde{N}(t)$ is a fuzzy random variable.

Definition 2.9 Fuzzy renewal process [Hwang, 2000]
Let $\tilde{X}_n$ denote the fuzzy random time interval between the $(n-1)$th and nth events. If $\{\tilde{X}_1, \tilde{X}_2, \ldots, \tilde{X}_n, \ldots\}$ is a sequence of independent and identically distributed fuzzy random variables, the fuzzy counting process $\{\tilde{N}(t), t \geq 0\}$ is called fuzzy random renewal process. $\tilde{N}(t)$ can be defined with the help of decomposition principle [[Yao and Wu, 2000]]as

$$\tilde{N}(t) = \bigcup_{\alpha \in (0,1]} [N_\alpha^-(t), N_\alpha^+(t)], \tag{2.5}$$

where $[N_\alpha^-(t), N_\alpha^+(t)]$ is a random interval, for a $\omega \in \Omega$ the end points $N_\alpha^-(t)$ and $N_\alpha^-(t)$ are defined as in equations (2.3) and (2.4).

Theorem 2.1 Elementary fuzzy renewal theorem [Hwang, 2000]

$$\frac{E\tilde{N}(t)}{t} \to \frac{1}{E\tilde{X}_1} \text{ as } t \to \infty. \tag{2.6}$$

Let us consider the sequence of pair of independent and identically distributed fuzzy random variables, $(\tilde{X}_1, \tilde{Y}_1), (\tilde{X}_2, \tilde{Y}_2), \ldots$ on the probability space $(\Omega, \mathcal{B}, P)$, where $\tilde{X}_n$ is the interarrival time between the $(n-1)$th and nth event and $\tilde{Y}_n$ is the reward associated with the nth interarrival time $\tilde{X}_n,\ n = 1, 2, \ldots$, respectively.

Let $\tilde{C}(t)$ denote the total reward earned by the time t, i.e.,

$$\tilde{C}(t) = \sum_{i=1}^{\tilde{N}(t)} \tilde{Y}_i, \tag{2.7}$$

where $\tilde{N}(t)$ is fuzzy random renewal variable.

Theorem 2.2 Fuzzy renewal reward theorem [Hwang and Yang, 2011]
If $EY^{-}_{1,\alpha} < \infty$, $EY^{+}_{1,\alpha} < \infty$, $EX^{-}_{1,\alpha} < \infty$ *and* $EX^{+}_{1,\alpha} < \infty$ *for* $0 \le \alpha \le 1$, *then*

$$\lim_{t\to\infty} \frac{E\tilde{C}(t)}{t} = \frac{E\tilde{Y}_1}{E\tilde{X}_1}. \tag{2.8}$$

3 Notations and assumptions

The following notations and assumptions are used to develop the model.

3.1 Notations

- λ annual demand rate
- x per year screening rate
- p random fraction of defective items in the lot size y, with density function $f(p)$
- c per unit purchasing cost
- s per unit selling price
- v salvage value of per unit defective item
- h holding cost per unit per year
- b back order cost per unit per year
- d per unit screening cost
- K ordering cost per order
- y order quantity
- w back order quantity
- t time interval, during which all items are screened
- T the time interval between two replenishment
- $\tilde{p} = (p - \Delta_1, p, p + \Delta_2)$ is a fuzzy random variable, denoting the fraction of the defective items in the lot size
- $\tilde{\lambda} = (\lambda - \Delta_3, \lambda, \lambda + \Delta_4)$ is the fuzzy annual demand.

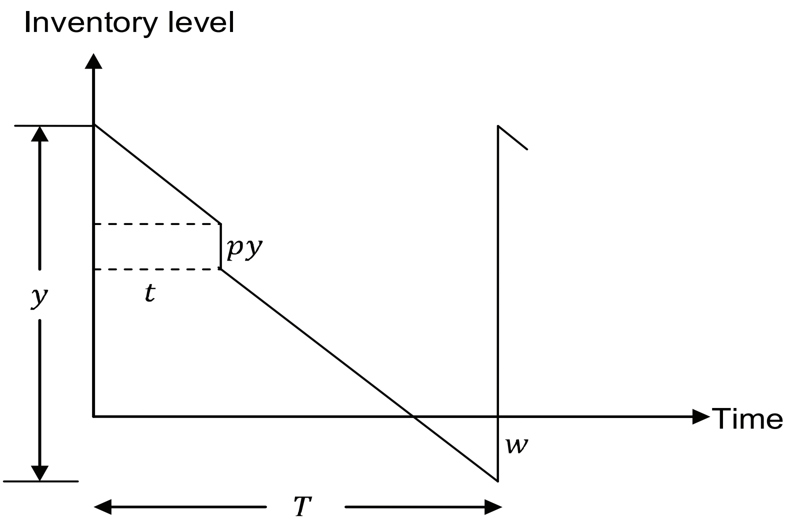


Fig. 1: Behavior of the inventory level with complete backordering

3.2 Assumptions

1. The items are replenished instantaneously.

2. The lead time is zero.

3. A fraction p of each lot size y contains defective items, and p is uniformly distributed in the interval $[a_l, a_u]$, $0 \leq a_l < a_u < 1$. The expected defective items $E[p]y$, are sold in a single batch at a discounted price prior to receiving next shipment.

4. The screening process and demand proceeds simultaneously, but screening rate is greater than demand rate (i.e., $x > \lambda$).

5. For avoiding the shortage during the screening period t, the minimum fraction of good-quality item $(1-p-\Delta_2)y$ must be greater than or equal to the maximum demand $(\lambda+\Delta_4)t$ [see Salameh and Jaber, 2000], that is,

$$p + \Delta_2 \leq 1 - \frac{\lambda + \Delta_4}{x}. \tag{3.1}$$

6. Shortage are allowed and completely backlogged. The backorder quantity is delivered without any defects.

7. A single product is considered.

4 Mathematical formulation of the model

The inventory level is illustrated in Fig. 1. At the beginning of the scheduling period a batch of product of size y is replenished by the vendor. It is assumed that even though the 100%

screening process has not been conducted when they received the batch of the product, the backorder quantity delivered without any defects [see Chang and Ho, 2010].

The time interval T, between two consecutive replenishment is

$$T = \frac{(1-p)y}{\lambda}. \tag{4.1}$$

Total revenue per cycle is

$$TR = s(1-p)y + vpy. \tag{4.2}$$

The inventory total cost per cycle is sum of ordering, purchasing, screening, holding and back-ordering costs.

$$TC = K + cy + dy + h\left[\frac{(y-py-w)^2}{2\lambda} + \frac{py^2}{x}\right] + \frac{bw^2}{2\lambda}. \tag{4.3}$$

The net profit per cycle is equal to the difference of total revenue and total cost, i.e.,

$$\begin{aligned} TP &= TR - TC \\ &= s(1-p)y + vpy - K - cy - dy - h\Big[\frac{(y-py-w)^2}{2\lambda} + \frac{py^2}{x}\Big] - \frac{bw^2}{2\lambda}. \end{aligned} \tag{4.4}$$

For the purpose of easier fuzzy arithmetic operation, we re-write the total profit per cycle from equation (4.4) as

$$\begin{aligned} TP(w,y) &= (v-c-d)y - K - \frac{hy^2}{x} + \Big(s - v + \frac{hy}{x}\Big)yq + hwy\frac{q}{\lambda} \\ &\quad -\frac{hy^2}{2}\frac{q^2}{\lambda} - \frac{(b+h)w^2}{2}\frac{1}{\lambda}, \end{aligned} \tag{4.5}$$

where $q = 1-p$.

As discussed earlier, due to various reason, the lot size received by the retailer is not cent percent perfect. A fraction p of the lot size is defective. Sometime the value of p is determined by the experts' experiences, such as "the fraction of defective items is about p" with probability $\mathcal{P}$. However, the linguistic information of experts may varies randomly. So, keep it in mind, we assume that the fraction of defective items is a fuzzy random variable, $\tilde{p} = (p-\Delta_1, p, p+\Delta_2)$, where $0 \le \Delta_1 \le E[p]$, $0 \le \Delta_2 \le 1 - E[p]$, and p is uniformly distributed. Also, in real life business transaction, it is not always possible to estimate the exact annual demand. The annual demand may have some fluctuation, especially in a perfect competitive market. So, instead of a crisp annual demand λ, the experts may suggest in linguistic sense as, the annual demand is about $\tilde{\lambda} = (\lambda-\Delta_3, \lambda, \lambda+\Delta_4)$, where $0 \le \Delta_3 < \lambda$ and $\Delta_4 \ge 0$.

The fuzzy random total profit per cycle is

$$\begin{aligned} \widetilde{TP}(w,y) &= (v-c-d)y - K - \frac{hy^2}{x} + \Big(s - v + \frac{hy}{x}\Big)y\tilde{q} + hwy\frac{\tilde{q}}{\tilde{\lambda}} \\ &\quad -\frac{hy^2}{2}\frac{\tilde{q}^2}{\tilde{\lambda}} - \frac{(b+h)w^2}{2}\frac{1}{\tilde{\lambda}} \end{aligned} \tag{4.6}$$

and fuzzy random scheduling period is

$$\tilde{T} \;=\; y\frac{\tilde{q}}{\tilde{\lambda}}, \tag{4.7}$$

where $\tilde{q} = 1 - \tilde{p} = (1 - p - \Delta_2, 1 - p, 1 - p + \Delta_1) = (q - \Delta_2, q, q + \Delta_1)$.

The fuzzy expectations of fuzzy random variables $\widetilde{TP}(w, y)$ and $\tilde{T}$ can be written as

$$\begin{aligned}\widetilde{ETP}(w,y) \;=\; & (v-c-d)y - K - \frac{hy^2}{x} + \Big(s - v + \frac{hy}{x}\Big)yE[\tilde{q}] + hwy\frac{E[\tilde{q}]}{\tilde{\lambda}} \\ & -\frac{hy^2}{2}\frac{E[\tilde{q}^2]}{\tilde{\lambda}} - \frac{(b+h)w^2}{2}\frac{1}{\tilde{\lambda}}\end{aligned} \tag{4.8}$$

$$\text{and} \quad \widetilde{ET} = y\frac{E[\tilde{q}]}{\tilde{\lambda}}. \tag{4.9}$$

The α-cut of $\widetilde{ETP}$ and $\widetilde{ET}$ are, respectively

$$ETP_\alpha(w,y) = [ETP_\alpha^-, ETP_\alpha^+] \tag{4.10}$$

$$\text{and } ET_\alpha = [ET_\alpha^-, ET_\alpha^+] \tag{4.11}$$

where

$$\begin{aligned}ETP_\alpha^- \;=\; & (v-c-d)y - K - \frac{hy^2}{x} + \Big(s - v + \frac{hy}{x}\Big)y(E[q] - \Delta_2 + \Delta_2\alpha) \\ & + hwy\frac{E[q] - \Delta_2 + \Delta_2\alpha}{\lambda + \Delta_4 - \Delta_4\alpha} - \frac{hy^2}{2}\frac{E[q + \Delta_1 - \Delta_1\alpha]^2}{\lambda - \Delta_3 + \Delta_3\alpha} - \frac{(b+h)w^2}{2}\frac{1}{\lambda - \Delta_3 + \Delta_3\alpha},\end{aligned} \tag{4.12}$$

$$\begin{aligned}ETP_\alpha^+ \;=\; & (v-c-d)y - K - \frac{hy^2}{x} + \Big(s - v + \frac{hy}{x}\Big)y(E[q] + \Delta_1 - \Delta_1\alpha) \\ & + hwy\frac{E[q] + \Delta_1 - \Delta_1\alpha}{\lambda - \Delta_3 + \Delta_3\alpha} - \frac{hy^2}{2}\frac{E[q - \Delta_2 + \Delta_2\alpha]^2}{\lambda + \Delta_4 - \Delta_4\alpha} - \frac{(b+h)w^2}{2}\frac{1}{\lambda + \Delta_4 - \Delta_4\alpha}\end{aligned} \tag{4.13}$$

$$\text{and} \quad ET_\alpha^- = \frac{y(E[q] - \Delta_2 + \Delta_2\alpha)}{\lambda + \Delta_4 - \Delta_4\alpha}, \tag{4.14}$$

$$ET_\alpha^+ = \frac{y(E[q] + \Delta_1 - \Delta_1\alpha)}{\lambda - \Delta_3 + \Delta_3\alpha} \tag{4.15}$$

The derivation of equations (4.12)–(4.15) are shown in Appendix A.

Now, our aim is to find the fuzzy expected profit per unit time. As we discussed in the introduction section, fuzzy random total profit per cycle and fuzzy random scheduling period generate a fuzzy renewal process with rewards. So, the fuzzy expected annual profit $\widetilde{ETPU}(w, y)$, can be obtained by applying the fuzzy random renewal reward theorem [see Hwang and Yang, 2011; Zhao et al., 2007].

$$\widetilde{ETPU}(w,y) = \frac{\widetilde{ETP}(w,y)}{\widetilde{ET}(y)}. \tag{4.16}$$

The fuzzy expectations $\widetilde{ETP}$ and $\widetilde{ET}$ are unique fuzzy numbers. Now, we employ the signed distance method to find the equivalent deterministic cost function (say $ETP(w,y)$) from the above fuzzy cost function.

$$ETP(w,y) = d\big(\widetilde{ETPU}(w,y),0\big) = \frac{1}{2}\int_0^1 \big[ETPU_\alpha^-(w,y) + ETPU_\alpha^+(w,y)\big]d\alpha$$

$$= (v-c-d)A - \frac{K}{y}A - \frac{hy}{x}A + \Big(s - v + \frac{hy}{x}\Big)B + hwl - \frac{hy}{2}m - \frac{(b+h)w^2}{2y}n, \tag{4.17}$$

where

$$A = \frac{1}{2}\int_0^1 \Big(\frac{\lambda - \Delta_3\alpha}{E[q]+\Delta_1\alpha} + \frac{\lambda+\Delta_4\alpha}{E[q]-\Delta_2\alpha}\Big)d\alpha, \tag{4.18}$$

$$B = \frac{1}{2}\int_0^1 \Big(\frac{(E[q]-\Delta_2\alpha)(\lambda-\Delta_3\alpha)}{E[q]+\Delta_1\alpha} + \frac{(E[q]+\Delta_1\alpha)(\lambda+\Delta_4\alpha)}{E[q]-\Delta_2\alpha}\Big)d\alpha, \tag{4.19}$$

$$l = \frac{1}{2}\int_0^1 \Big(\frac{(E[q]-\Delta_2\alpha)(\lambda-\Delta_3\alpha)}{(E[q]+\Delta_1\alpha)(\lambda+\Delta_4\alpha)} + \frac{(E[q]+\Delta_1\alpha)(\lambda+\Delta_4\alpha)}{(E[q]-\Delta_2\alpha)(\lambda-\Delta_3\alpha)}\Big)d\alpha, \tag{4.20}$$

$$m = \frac{1}{2}\int_0^1 \Big(\frac{E[q^2]+\Delta_1^2\alpha^2+2\Delta_1E[q]\alpha}{E[q]+\Delta_1\alpha} + \frac{E[q^2]+\Delta_2^2\alpha^2-2\Delta_2E[q]\alpha}{E[q]-\Delta_2\alpha}\Big)d\alpha \tag{4.21}$$

$$\text{and } n = \frac{1}{2}\int_0^1 \Big(\frac{1}{E[q]+\Delta_1\alpha} + \frac{1}{E[q]-\Delta_2\alpha}\Big)d\alpha. \tag{4.22}$$

Derivation of equation (4.17) to equation (4.22) are shown in Appendix B.

To find the optimum solution, we use classical method. For this we equate partial derivatives to zero

$$\frac{\partial ETP}{\partial y} = \frac{KA}{y^2} - \frac{hA}{x} + \frac{hB}{x} - \frac{hm}{2} + \frac{(b+h)nw^2}{2y^2} = 0$$

$$\text{and } \frac{\partial ETP}{\partial w} = hl - \frac{(b+h)nw}{y} = 0,$$

which give

$$y^* = \sqrt{\frac{KA}{h\Big[\frac{A}{x} - \frac{B}{x} + \frac{m}{2} - \frac{hl^2}{2(b+h)n}\Big]}} \tag{4.23}$$

$$\text{and } w^* = \frac{hly^*}{(b+h)n}. \tag{4.24}$$

(Concavity of $ETP(w,y)$ is shown in Appendix C).

Special case i: Only demand rate is in fuzzy sense

In this case, we assume that annual demand is fuzzy number and fraction of defective items is a random variable. Putting $\Delta_1 = \Delta_2 = 0$, then from equations (4.18)–(4.22), we have

$$A = \frac{1}{2}\int_0^1 \Big(\frac{\lambda-\Delta_3\alpha}{E[q]} + \frac{\lambda+\Delta_4\alpha}{E[q]}\Big)d\alpha = \frac{1}{E[q]}\Big(\lambda + \frac{\Delta_4-\Delta_3}{2}\Big), \tag{4.25}$$

$$
\begin{aligned}
B &= \frac{1}{2}\int_0^1 (\lambda - \Delta_3\alpha + \lambda + \Delta_4\alpha)d\alpha = \lambda + \frac{\Delta_4 - \Delta_3}{4}, && (4.26)\\
l &= \frac{1}{2}\int_0^1 \Big(\frac{\lambda - \Delta_3\alpha}{\lambda + \Delta_4\alpha} + \frac{\lambda + \Delta_4\alpha}{\lambda - \Delta_3\alpha}\Big)d\alpha \\
&= \frac{\lambda(\Delta_4 + \Delta_3)}{2\Delta_4^2}\log\Big(\frac{\lambda + \Delta_4}{\lambda}\Big) + \frac{\lambda(\Delta_4 + \Delta_3)}{2\Delta_3^2}\log\Big(\frac{\lambda}{\lambda - \Delta_3}\Big) - \frac{1}{2}\Big(\frac{\Delta_3}{\Delta_4} + \frac{\Delta_4}{\Delta_3}\Big), && (4.27)\\
m &= \frac{1}{2}\int_0^1 \Big(\frac{E[q^2]}{E[q]} + \frac{E[q^2]}{E[q]}\Big)d\alpha = \frac{E[q^2]}{E[q]} && (4.28)\\
\text{and } n &= \frac{1}{2}\int_0^1 \Big(\frac{1}{E[q]} + \frac{1}{E[q]}\Big)d\alpha = \frac{1}{E[q]}. && (4.29)
\end{aligned}
$$

The optimal order quantity and backorder quantity for this case can be easily obtained from equations (4.23) and (4.24), where $A,\ B,\ l,\ m$ and n are given in equations (4.25)–(4.29).

Special case ii: Only fraction of defective items is in fuzzy sense
In this case, it is assumed that the fraction of defective items is fuzzy random variable, while annual demand is crisp. If we put $\Delta_3 = \Delta_4 = 0$, then from equations (4.18)–(4.22), we have

$$
\begin{aligned}
A &= \frac{1}{2}\int_0^1 \Big(\frac{\lambda}{E[q] + \Delta_1\alpha} + \frac{\lambda}{E[q] - \Delta_2\alpha}\Big)d\alpha \\
&= \frac{\lambda}{2}\Big[\frac{1}{\Delta_1}\log\Big(\frac{E[q] + \Delta_1}{E[q]}\Big) + \frac{1}{\Delta_2}\log\Big(\frac{E[q]}{E[q] - \Delta_2}\Big)\Big], && (4.30)\\
B &= \frac{\lambda}{2}\int_0^1 \Big(\frac{E[q] - \Delta_2\alpha}{E[q] + \Delta_1\alpha} + \frac{E[q] + \Delta_1\alpha}{E[q] - \Delta_2\alpha}\Big)d\alpha \\
&= \frac{\lambda E[q](\Delta_1 + \Delta_2)}{2}\Big[\frac{1}{\Delta_1^2}\log\Big(\frac{E[q] + \Delta_1}{E[q]}\Big) + \frac{1}{\Delta_2^2}\log\Big(\frac{E[q]}{E[q] - \Delta_2}\Big)\Big] - \frac{\lambda}{2}\Big(\frac{\Delta_1}{\Delta_2} + \frac{\Delta_2}{\Delta_1}\Big), && (4.31)\\
l &= \frac{1}{2}\int_0^1 \Big(\frac{E[q] - \Delta_2\alpha}{E[q] + \Delta_1\alpha} + \frac{E[q] + \Delta_1\alpha}{E[q] - \Delta_2\alpha}\Big)d\alpha = \frac{B}{\lambda}, && (4.32)\\
m &= \frac{1}{2}\int_0^1 \Big(\frac{E[q^2] + \Delta_1^2\alpha^2 + 2\Delta_1 E[q]\alpha}{E[q] + \Delta_1\alpha} + \frac{E[q^2] + \Delta_2^2\alpha^2 - 2\Delta_2 E[q]\alpha}{E[q] - \Delta_2\alpha}\Big)d\alpha \ \text{ and} && (4.33)\\
n &= \frac{1}{2}\int_0^1 \Big(\frac{1}{E[q] + \Delta_1\alpha} + \frac{1}{E[q] - \Delta_2\alpha}\Big)d\alpha. && (4.34)
\end{aligned}
$$

The optimal order quantity and backorder quantity for this case can be easily obtained from equations (4.23) and (4.24), for that $A,\ B,\ l,\ m$ and n are given in equations (4.30)–(4.34).

5 Analysis

1. When $b \to \infty$, i.e., shortages are not allowed, then $\frac{h}{b+h} \to 0$. This implies backorder quantity $w^{**} = 0$ and lot size is,

$$
y^{**} = \sqrt{\frac{KA}{h[\frac{A}{x} - \frac{B}{x} + \frac{m}{2}]}}. \tag{5.1}
$$

2. It is obvious from equations (4.23) and (5.1), $y^* \geq y^{**}$.

3. When b increases, $\frac{h}{(b+h)}$ decreases; consequently both y^* and w^* will decrease.

4. When the annual demand rate is crisp and fraction of defective items is a random variable i.e., $\Delta_3 = 0 = \Delta_4$ and $\Delta_1 = 0 = \Delta_2$. Then from relation (4.18) to (4.22) we get, $A = \frac{\lambda}{E[q]}$, $B = \lambda$, $l = 1$, $m = \frac{E[q^2]}{E[q]}$ and $n = \frac{1}{E[q]}$. The lot size and backorder quantity becomes as

$$y^\dagger = \sqrt{\frac{2\lambda K}{h\Big[E[(1-p)^2] + \frac{2\lambda E[p]}{x} - \frac{h(E[1-p])^2}{(b+h)}\Big]}} \tag{5.2}$$

$$\text{and } w^\dagger = \frac{hE[1-p]y^\dagger}{b+h}. \tag{5.3}$$

This is [Chang and Ho, 2010] optimal policy.

Further, if we considering $q = 1$ (i.e., the fraction of defective items, $p = 0$), then from relation (5.2) and (5.3), we can obtain

$$y_{cl} = \sqrt{\frac{2K\lambda}{h}}\sqrt{\frac{b+h}{b}}$$

$$\text{and } w_{cl} = \frac{h}{b+h} y_{cl}.$$

This is the classical EOQ model with backorder.

6 Numerical examples and its sensitivity analysis

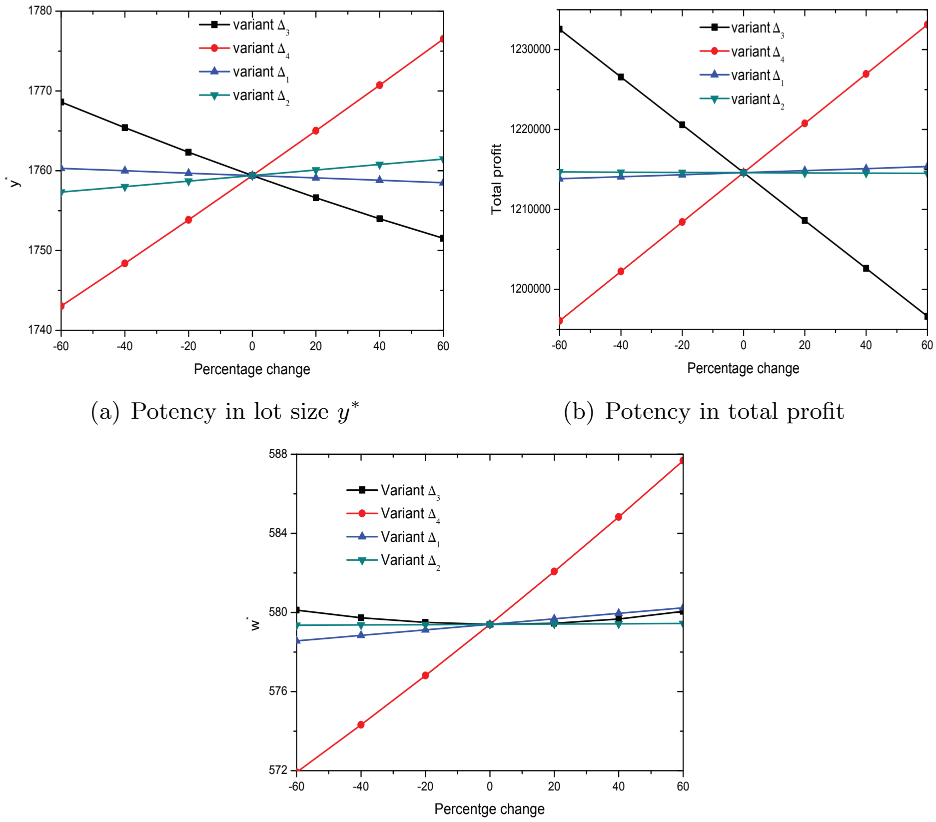
The preceding theory can be illustrated with some numerical examples. We adopt the data from [Wee et al., 2007] paper. The parameters are as follows:
λ = 50 000 units/year, K = \$100/order, h = \$5/unit/year, x = 175200, d = \$0.5/unit, c = \$25/unit, b = \$10/unit/year, s = \$50/unit, v = \$20/unit, The fraction of defective items p is uniformly distributed with probability density function, $f(p) = 25,\ 0 \leq p \leq 0.04$. So, $E[p] = 0.02$, i.e., $E[q] = E[1-p] = 0.98$ and $E[q^2] = E[(1-p)^2] = 0.9605$, $\Delta_1' = \Delta_2' = 0.01$, $\Delta_3' = \Delta_4' = 5000$.
Example 1. We first take an example in which shortages are allowed. If we take $\Delta_i = \Delta_i'$, $i = 1, 2, 3, 4$. Then the optimal solution for the above data is, $y^* = 1759.39$, $w^* = 579.396$ and the total annual profit, $ETP = 1214610$.
Sensitivity of fuzzy demand and fuzzy random fraction of defective items:
The sensitivity of fuzzy demand and fuzzy random fraction of defective items are shown in Fig. 2, where "percentage change" means, $\Delta_i = \Delta_i' + z\%$ of Δ_i', $z = -60, -40, -20, 0, 20, 40, 60$ and $i = 1, 2, 3, 4$. We can observed from Fig. 2,

1. When Δ_2 is fixed and Δ_1 increases, then the lot size y^* decreases marginally while backorder quantity w^* and total annual average profit $ETP(w^*, y^*)$ increase slightly, and as a result, fuzzy estimated fraction of defective items $d(E[\tilde{p}], 0) = E[p] + \frac{\Delta_2 - \Delta_1}{4}$ also decreases.

Conversely, if Δ_1 is fixed and Δ_2 increases, then y^* increases slightly, but w^* and ETP almost constant. Thus we can say that the left spread of fuzzy fraction of defective items is more sensitive than right spread.

2. When Δ_4 is fixed and Δ_3 increases, then y^*, $ETP(w^*, y^*)$ decrease, while w^* varies slightly; as a result estimated fuzzy annual demand $d(\tilde{\lambda}, 0) = \lambda + \frac{\Delta_4 - \Delta_3}{4}$ decreases. Conversely, if Δ_3 is fixed and Δ_4 increases, then y^*, ETP and w^* all increase and estimated demand $d(\tilde{\lambda}, 0)$ also increases. Fig. 2 indicates that the right spread Δ_4 of fuzzy demand is more sensitive then the left spread Δ_3.

3. Overall we can say that the fuzzy demand is more sensitive compared to fuzzy random fraction of defective item.



(a) Potency in lot size y^*

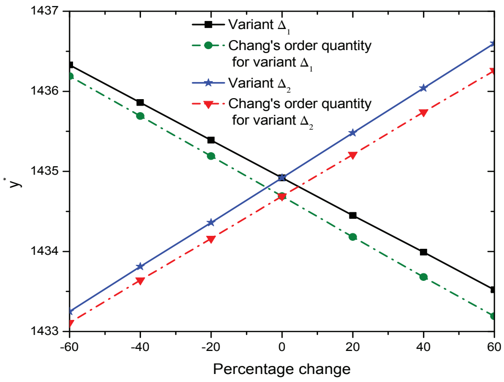
(b) Potency in total profit

(c) Potency in backorder quantity w^*

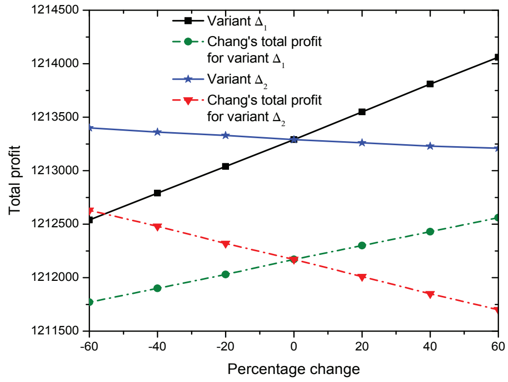
Fig. 2: Sensitivity of spreads of fuzzy demand and fuzzy random fraction of defective items when shortages are allowed

Example 2. In this example, it is assumed that shortages are not allowed. As discussed in the Analysis section, the backorder quantity $w^{**} = 0$ and lot size y^{**} is given in equation (5.1). For the data given in the beginning of this section, the optimal solution is, $y^{**} = 1434.92$ and total annual profit $ETP(y^{**}) = 1213290$.

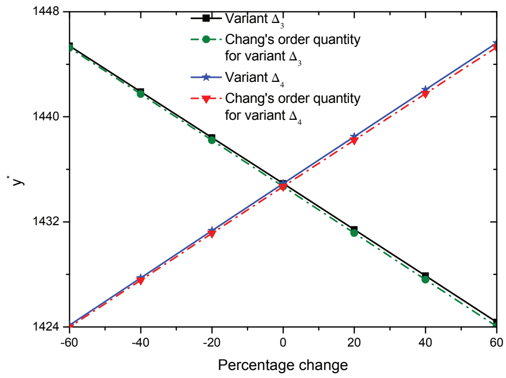
Let y_{chang} and ETP_{chang} denote the [Chang, 2004] order quantity and expected total annual profit, respectively. So, for the above data, $y_{chang} = 1434.69$ and $ETP_{chang} = 1212170$.



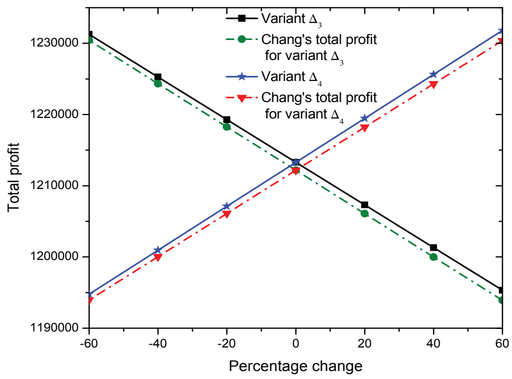
(a) Potency in lot size y^*



(b) Potency in total profit



(c) Potency in lot size y^*



(d) Potency in total profit

Fig. 3: Sensitivity of spreads of fuzzy demand and fuzzy random fraction of defective items when shortages are not allowed

Sensitivity of fuzzy demand and fuzzy random fraction of defective items: The sensitivity of fuzzy demand and fuzzy random fraction of defective items are shown in Fig. 3.

1. When Δ_2 is fixed and Δ_1 increases, then both of the order quantities y^{**} and y_{chang} decrease while $ETP(y^{**})$ and ETP_{chang} increase, and estimated fraction of defective items decreases. Conversely, if Δ_1 is fixed and Δ_2 increases, then y^{**} and y_{chang} increase, $ETP(y^{**})$ decreases slightly and ETP_{chang} decreases moderately and estimated fraction of defective items increases.

2. When Δ_4 is fixed and Δ_3 increases, then y^{**}, y_{chang}, $ETP(y^{**})$ and ETP_{chang} decrease and the estimated demand in fuzzy sense increases. Conversely, if Δ_3 is fixed and Δ_4 increases, then the result is just opposite.

3. The fuzzy demand is more sensitive than the fraction of defective items for both of the optimal policies.

4. Fig. 3 clearly show that the our model gives the better result than the [Chang, 2004] model i.e., fuzzy renewal reward theorem is better approach.

7 Conclusion

This paper deals with an EOQ inventory model for items with imperfect quality under fuzzy random environment. Shortages are allowed and backlogged. Due to unreliability in supply process, transportation etc., the received items are not cent percent perfect. The fraction of defective items may be imprecise (linguistic sense), and this linguistic information may varies randomly by experts' experiences. Thus the fraction of defective quality items is a fuzzy random variable. Similarly, considering a real life business transaction (due to imprecise nature of demand), the annual demand is represented by a fuzzy number. The fuzziness in demand and mixture of fuzziness and randomness in the fraction of defective items lead to fuzzy random total profit per cycle and scheduling period. We have applied fuzzy random renewal reward theorem to obtain the fuzzy expected profit per unit time. Further, signed distance method is used to derive the equivalent crisp total profit function from fuzzy function.

We analytically examined the effect of parameters on the optimal policies. When the backordering cost increases, then the order quantity and backorder quantity both decrease. If backordering cost tends to infinity, then the backorder quantity is zero, i.e., in this situation shortages are not allowed. The classical EOQ model with backorder quantity and [Chang and Ho, 2010] model are special case of our model.

The model is illustrated with the help of numerical examples, and sensitivity of the deferent parameters are also examined. Numerical examples show the significant differences between our model and others. Example 2 make a comparison of ours and [Chang, 2004] optimal policies. This example shows that the fuzzy random renewal reward theorem gives better results. This model further can be extended with realistic assumptions such as partial backlogging, fuzzy cost coefficients, etc.

Appendix A

The α-cut of $\tilde{q}$ and $\tilde{\lambda}$ are, respectively, $[q_\alpha^-, q_\alpha^+] = [q - \Delta_2 + \Delta_2\alpha, q + \Delta_1 - \Delta_1\alpha]$ and $[\lambda_\alpha^-, \lambda_\alpha^+] = [\lambda - \Delta_3 + \Delta_3\alpha, \lambda + \Delta_4 - \Delta_4\alpha]$.

Now, we use arithmetical operation of intervals as discussed in Preliminaries section [for detail see [Kaufmann and Gupta, 1991]], to find the α-cut of fuzzy $\frac{1}{\tilde{\lambda}}$, $\tilde{q}^2$, $\widetilde{ETP}(w,y)$, $\widetilde{ET}$ etc., and finally $\widetilde{ETUP}(w,y)$. The α-cut of $\frac{1}{\tilde{\lambda}}$ is

$$\Big[\frac{1}{\lambda_\alpha^+}, \frac{1}{\lambda_\alpha^-}\Big] = \Big[\frac{1}{\lambda + \Delta_4 - \Delta_4\alpha}, \frac{1}{\lambda - \Delta_3 + \Delta_3\alpha}\Big].$$

The α-cut of $\tilde{q}^2$ is $\big[\min\{q_\alpha^{-2}, q_\alpha^- q_\alpha^+, q_\alpha^{+2}\},\ \max\{q_\alpha^{-2}, q_\alpha^- q_\alpha^+, q_\alpha^{+2}\}\big]$, but $q_\alpha^- < q_\alpha^+$, hence

$$[q_\alpha^{2-}, q_\alpha^{2+}] = [q_\alpha^{-2}, q_\alpha^{+2}] = [(q - \Delta_2 + \Delta_2\alpha)^2, (q + \Delta_1 - \Delta_1\alpha)^2].$$

The α-cut of fuzzy expectations $E[\tilde{q}]$ and $E[\tilde{q}^2]$ are

$$\begin{aligned} \big[E[q_\alpha^-], E[q_\alpha^+]\big] &= \big[E[q] - \Delta_2 + \Delta_2\alpha, E[q] + \Delta_1 - \Delta_1\alpha\big] \\ \text{and } \big[E\big[q_\alpha^{2-}\big], E[q_\alpha^{2+}]\big] &= \big[E[q - \Delta_2 + \Delta_2\alpha]^2, E[q + \Delta_1 - \Delta_1\alpha]^2\big]. \end{aligned}$$

The α-cut of $\frac{E[\tilde{q}]}{\tilde{\lambda}}$ and $\frac{E[\tilde{q}^2]}{\tilde{\lambda}}$ are

$$\Big[\Big(\frac{E[q]}{\lambda}\Big)^-_\alpha,\ \Big(\frac{E[q]}{\lambda}\Big)^+_\alpha\Big] \;=\; \Big[\frac{E[q^-_\alpha]}{\lambda^+_\alpha},\ \frac{E[q^+_\alpha]}{\lambda^-_\alpha}\Big] = \Big[\frac{E[q]-\Delta_2+\Delta_2\alpha}{\lambda+\Delta_4-\Delta_4\alpha} + \frac{E[q]+\Delta_1-\Delta_1\alpha}{\lambda-\Delta_3+\Delta_3\alpha}\Big]$$

and

$$\Big[\Big(\frac{E[q^2]}{\lambda}\Big)^-_\alpha,\ \Big(\frac{E[q^2]}{\lambda}\Big)^+_\alpha\Big] = \Big[\frac{E[q^{2-}_\alpha]}{\lambda^+_\alpha},\ \frac{E[q^{2+}_\alpha]}{\lambda^-_\alpha}\Big] = \Big[\frac{E[q-\Delta_2+\Delta_2\alpha]^2}{\lambda+\Delta_4-\Delta_4\alpha} + \frac{E[q+\Delta_1-\Delta_1\alpha]^2}{\lambda-\Delta_3+\Delta_3\alpha}\Big].$$

Hence, the left and right end points of the α-cut of $\widetilde{ETP}(w,y)$ and $\widetilde{ET}$ are

$$\begin{aligned} ETP^-_\alpha(w,y) &= (v-c-d)y - K - \frac{hy^2}{x} + \Big(s-v+\frac{hy}{x}\Big)yE[q^-_\alpha] + hwy\frac{E[q^-_\alpha]}{\lambda^+_\alpha} \\ &\quad - \frac{hy^2}{2}\frac{E[q^{2+}_\alpha]}{\lambda^-_\alpha} - \frac{(b+h)w^2}{2}\frac{1}{\lambda^-_\alpha}, \end{aligned}$$

$$\begin{aligned} ETP^-_\alpha(w,y) &= (v-c-d)y - K - \frac{hy^2}{x} + \Big(s-v+\frac{hy}{x}\Big)yE[q^+_\alpha] + hwy\frac{E[q^+_\alpha]}{\lambda^-_\alpha} \\ &\quad - \frac{hy^2}{2}\frac{E[q^{2-}_\alpha]}{\lambda^+_\alpha} - \frac{(b+h)w^2}{2}\frac{1}{\lambda^+_\alpha}, \end{aligned}$$

$$ET^-_\alpha = y\frac{E[q^-_\alpha]}{\lambda^+_\alpha} \text{ and } ET^+_\alpha = y\frac{E[q^+_\alpha]}{\lambda^-_\alpha}.$$

Appendix B

The end points of the α-cut of $\widetilde{ETPU}(w,y)$ are

$$ETPU^-_\alpha = \frac{ETP^-_\alpha}{ET^+_\alpha} \text{ and } ETPU^+_\alpha = \frac{ETP^+_\alpha}{ET^-_\alpha}.$$

Hence,

$$ETP(w,y) = \frac{1}{2}\int_0^1 \Big(\frac{ETP^-_\alpha}{ET^+_\alpha} + \frac{ETP^+_\alpha}{ET^-_\alpha}\Big)d\alpha$$

$$= (v-c-d)A - \frac{K}{y}A - \frac{hy}{x}A + \Big(s-v+\frac{hy}{x}\Big)B + hwl - \frac{hy}{2}m - \frac{(b+h)w^2}{2y}n,$$

where

$$\begin{aligned} A &= \frac{1}{2}\int_0^1 \Big(\frac{\lambda-\Delta_3+\Delta_3\alpha}{E[q]+\Delta_1-\Delta_1\alpha} + \frac{\lambda+\Delta_4-\Delta_4\alpha}{E[q]-\Delta_2+\Delta_2\alpha}\Big)d\alpha, \\ B &= \frac{1}{2}\int_0^1 \Big(\frac{(E[q]-\Delta_2+\Delta_2\alpha)(\lambda-\Delta_3+\Delta_3\alpha)}{E[q]+\Delta_1-\Delta_1\alpha} + \frac{(E[q]+\Delta_1-\Delta_1\alpha)(\lambda+\Delta_4-\Delta_4\alpha)}{E[q]-\Delta_2+\Delta_2\alpha}\Big)d\alpha, \\ l &= \frac{1}{2}\int_0^1 \Big(\frac{(E[q]-\Delta_2+\Delta_2\alpha)(\lambda-\Delta_3+\Delta_3\alpha)}{(E[q]+\Delta_1-\Delta_1\alpha)(\lambda+\Delta_4-\Delta_4\alpha)} + \frac{(E[q]+\Delta_1-\Delta_1\alpha)(\lambda+\Delta_4-\Delta_4\alpha)}{(E[q]-\Delta_2+\Delta_2\alpha)(\lambda-\Delta_3+\Delta_3\alpha)}\Big)d\alpha, \\ m &= \frac{1}{2}\int_0^1 \Big(\frac{E[q^2]+\Delta_1^2(1-\alpha)^2+2\Delta_1E[q](1-\alpha)}{E[q]+\Delta_1-\Delta_1\alpha} + \frac{E[q^2]+\Delta_2^2(1-\alpha)^2-2\Delta_2E[q](1-\alpha)}{E[q]-\Delta_2+\Delta_2\alpha}\Big)d\alpha, \\ n &= \frac{1}{2}\int_0^1 \Big(\frac{1}{E[q]+\Delta_1-\Delta_1\alpha} + \frac{1}{E[q]-\Delta_2+\Delta_2\alpha}\Big)d\alpha. \end{aligned}$$

If we put, $1-\alpha=\beta$, in the above integrals, then we get the equations (4.18)–(4.22). Since, $0 \leq \Delta_1 \leq E[p]$, $0 \leq \Delta_2 < 1-E[p]$, $0 \leq \Delta_3 \leq \lambda$ and $\Delta_4 \geq 0$, hence A, B, l, m and n all are positive.

Appendix C

$$\begin{aligned} \frac{\partial^2 ETP}{\partial y^2} &= -\frac{2KA}{y^3}-\frac{(b+h)nw^2}{y^3}, \\ \frac{\partial^2 ETP}{\partial w^2} &= -\frac{(b+h)n}{y} \\ \text{and } \frac{\partial^2 ETP}{\partial w \partial y} &= \frac{(b+h)nw}{y^2}. \end{aligned}$$

Since $\frac{\partial^2 ETP}{\partial y^2} < 0, \ \forall\ y > 0$ and w, and

$$\frac{\partial^2 ETP}{\partial y^2}.\frac{\partial^2 ETP}{\partial w^2} - \Big(\frac{\partial^2 ETP}{\partial w \partial y}\Big)^2 = \frac{2KA(b+h)n}{y^4} > 0\ \forall\ y > 0.$$

Hence, the function of $ETP(w,y)$ is concave for $y > 0$ and $w \geq 0$.

References

Björk, K. M. [2009]. An analytical solution to a fuzzy economic order quantity problem, *International Journal of Approximate Reasoning.* **50**(3), 485–493.

Chang, H. C. [2004]. An application of fuzzy sets theory to the eoq model with imperfect quality items, *Computers & Operations Research.* **31**(12), 2079–2092.

Chang, H. C. and Ho, C. H. [2010]. Exact closed-form solutions for, *Omega.* **38**(3-4), 233–237.

Chang, H. C., Yao, J. S. and Ouyang, L. Y. [2006]. Fuzzy mixture inventory model involving fuzzy random variable lead time demand and fuzzy total demand, *European Journal of Operational Research.* **169**(1), 65–80.

Chang, S. C., Yao, J. S. and Lee, H. M. [1998]. Economic reorder point for fuzzy backorder quantity, *European Journal of Operational Research.* **109**(1), 183–202.

Chung, K. J., Her, C. C. and Lin, S. D. [2009]. A two-warehouse inventory model with imperfect quality production processes, *Computers & Industrial Engineering.* **56**(1), 193–197.

Das, D., Roy, A. and Kar, S. [2011]. A volume flexible economic production lot-sizing problem with imperfect quality and random machine failure in fuzzy-stochastic environment, *Computers & Mathematics with Applications.* .

Dey, O. and Chakraborty, D. [2009]. Fuzzy periodic review system with fuzzy random variable demand, *European Journal of Operational Research.* **198**(1), 113–120.

Dutta, P., Chakraborty, D. and Roy, A. [2007]. Continuous review inventory model in mixed fuzzy and stochastic environment, *Applied Mathematics and Computation.* **188**(1), 970–980.

Eroglu, A. and Ozdemir, G. [2007]. An economic order quantity model with defective items and shortages, *International Journal of Production Economics.* **106**(2), 544–549.

Gil, M. Á., López-Díaz, M. and Ralescu, D. A. [2006]. Overview on the development of fuzzy random variables, *Fuzzy Sets and Systems.* **157**(19), 2546–2557.

Hu, J. S., Zheng, H., Xu, R. Q., Ji, Y. P. and Guo, C. Y. [2010]. Supply chain coordination for fuzzy random newsboy problem with imperfect quality, *International Journal of Approximate Reasoning.* **51**(7), 771–784.

Hwang, C. M. [2000]. A theorem of renewal process for fuzzy random variables and its application, *Fuzzy Sets and Systems.* **116**(2), 237–244.

Hwang, C. M. and Yang, M. S. [2011]. On fuzzy renewal processes for fuzzy random variables and extended theorems, *International Journal of Intelligent Systems.* **26**(2), 115–128.

Kaufmann, A. and Gupta, M. M. [1991]. Introduction to fuzzy arithmetic: Theory and applications., *Van Nortrand Reinhold, New York.* .

Kazemi, N., Ehsani, E. and Jaber, M. Y. [2010]. An inventory model with backorders with fuzzy parameters and decision variables, *International Journal of Approximate Reasoning.* **51**(8), 964–972.

Kumar, R. S., De, S. K. and Goswami, A. [2012]. Fuzzy EOQ models with ramp type demand rate, partial backlogging and time dependent deterioration rate, *International Journal of Mathematics in Operational Research.* **4**(5), 473–502.

Kwakernaak, H. [1978]. Fuzzy random variables–i. definitions and theorems, *Information Sciences.* **15**(1), 1–29.

Lin, Y. J. [2008]. A periodic review inventory model involving fuzzy expected demand short and fuzzy backorder rate, *Computers & Industrial Engineering.* **54**(3), 666–676.

Liu, J. and Zheng, H. [2012]. Fuzzy economic order quantity model with imperfect items, shortages and inspection errors, *Systems Engineering Procedia.* **4**, 282–289.

Maddah, B. and Jaber, M. Y. [2008]. Economic order quantity for items with imperfect quality: Revisited, *International Journal of Production Economics.* **112**(2), 808–815.

Maddah, B., Salameh, M. K. and Moussawi-Haidar, L. [2010]. Order overlapping: A practical approach for preventing shortages during screening, *Computers & Industrial Engineering.* **58**(4), 691–695.

Papachristos, S. and Konstantaras, I. [2006]. Economic ordering quantity models for items with imperfect quality, *International Journal of Production Economics.* **100**(1), 148–154.

Popova, E. and Wu, H. C. [1999]. Renewal reward processes with fuzzy rewards and their applications to t-age replacement policies, *European Journal of Operational Research.* **117**(3), 606–617.

Porteus, E. L. [1986]. Optimal lot sizing, process quality improvement and setup cost reduction, *Operations Research.* pp. 137–144.

Puri, M. L. and Ralescu, D. A. [1986]. Fuzzy random variables, *Journal of Mathematical Analysis and Applications.* **114**(2), 409–422.

Rosenblatt, M. J. and Lee, H. L. [1986]. Economic production cycles with imperfect production processes, *IIE Transactions.* **18**(1), 48–55.

Salameh, M. K. and Jaber, M. Y. [2000]. Economic production quantity model for items with imperfect quality, *International Journal of Production Economics.* **64**(1), 59–64.

Shapiro, A. F. [2009]. Fuzzy random variables, *Insurance: Mathematics and Economics.* **44**(2), 307–314.

Wang, S. and Watada, J. [2009]. Fuzzy random renewal reward process and its applications, *Information Sciences.* **179**(23), 4057–4069.

Wee, H. M., Yu, J. and Chen, M. C. [2007]. Optimal inventory model for items with imperfect quality and shortage backordering, *Omega.* **35**(1), 7–11.

Yao, J. S., Chang, S. C. and Su, J. S. [2000]. Fuzzy inventory without backorder for fuzzy order quantity and fuzzy total demand quantity, *Computers & Operations Research.* **27**(10), 935–962.

Yao, J. S. and Wu, K. [2000]. Ranking fuzzy numbers based on decomposition principle and signed distance, *Fuzzy Sets and Systems.* **116**(2), 275–288.

Zhao, R., Tang, W. and Wang, C. [2007]. Fuzzy random renewal process and renewal reward process, *Fuzzy Optimization and Decision Making.* **6**(3), 279–295.

Zimmermann, H. J. [2001]. *Fuzzy set theory and its application*, Kluwer Academic Publishers, Dordrecht.