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A fast numerical method for solving calculus of variation problems

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ABSTRACT In the modeling of a large class of problems in science and engineering, the minimization of a functional is appeared. Finding the solution of these problems needs to solve the corresponding ordinary differential equations which are generally nonlinear. In recent years, differential transform method has been attracted a lot of attention of the researchers for solving nonlinear problems. This method finds the solution of the problem without any discretization of the equation. Since this method gives a closed form solution of the problem and avoids the round off errors, it can be considered as an efficient method for solving various kinds of problems. In this research, differential transform method (DTM) will be employed for solving some problems in calculus of variations. Some examples are presented to show the efficiency of the proposed technique.

Keywords: Calculus of variations; Euler–Lagrange equation; differential transform method.

1. Introduction

The calculus of variations is concerned with finding the maxima and minima of a certain functional [1, 2]. Functional minimization problems known as variational problems appear in engineering and science where minimization of functionals, such as Lagrangian, strain, potential, total energy, etc., give the laws governing the systems behavior. In optimal control theory, minimization of certain functionals gives control functions for optimum performance of the system [3]. The brachistochrone, geodesics and isoperimetric problems have played an important role in the development of the calculus of variations [1, 2].

Several methods have been used to solve variational problems. For example, the direct

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method of Ritz [1], Walsh functions [4], Laguerre series [5], shifted Legendre polynomial series [6], shifted Chebyshev series [7], and Fourier series [8] have been applied to solve variational problems. Legendre and Walsh wavelet functions are also used to solve variational problems in [9, 10], respectively. In [11, 12], rationalized Haar functions and Haar wavelets are proposed to solve variational problems. Recently, Adomian decomposition method [13], variational iteration method [14] and homotopy-perturbation method [15] have been intensively developed to obtain exact and approximate analytical solutions of this kind of variational problems.

Motivated by the above discussions, in the present work, we are concerned with the application of the differential transform procedure, for calculus of variational problems. The DTM was first proposed by Zhou [16] to solve nonlinear Genesio systems. It is a numerical method based on the Taylor series expansion which constructs an analytical solution in the form of a polynomial. The established high order Taylor series method requires only symbolic computation. Another side, the DTM obtains a polynomial series solution by means of an iterative procedure. The DTM is useful to obtain exact and approximate solutions of linear and non-linear differential equation systems. No necessity to linearization, discretization and large computational works. It has been used to solve efficiently, easily and accurately a large class of nonlinear problems with approximations. These approximations converge rapidly to exact solutions [17]-[45].

The paper is organized as follows. In Section 2, we introduce the general form of problems in calculus of variations, and their relations with ordinary differential equations are highlighted. In Section 3, theoretical aspects of the differential transform are discussed. In Section 4, effectiveness of the proposed approach is verified by solving several numerical examples. A conclusion is presented in Section 5.

2. Statement of the problem

Let us consider the simplest form of the variational problems

$$\eta\ [y(x)] = \int_{x_0}^{x_1} F(x, y(x), y'(x))dx, \tag{1}$$

where η is the functional that its extremum must be found. To find the extreme value of η, the boundary points of the admissible curves are known in the following form

$$y(x_0) = \theta, \qquad y(x_1) = \delta, \tag{2}$$

where θ and δ are known. The necessary condition for the solution of the problem (1) is to satisfy the Euler–Lagrange equation

$$F_y - \frac{d}{dx}F_{y'} = 0, \tag{3}$$

with boundary conditions given in (2). The boundary value problem (3) does not always have a solution and if the solution exists, it may not be unique. Note that in many variational problems the existence of a solution is obvious from the physical or geometrical meaning of the problem and if the solution of Euler's equation satisfies the boundary conditions, it is unique. Also this unique extremal will be the solution of the given variational problem [2].

The general form of the variational problem (1) is

$$\eta[y_1, y_2, ..., y_n] = \int_{x_0}^{x_1} F(x, y_1, y_2, ..., y_n, y_1', y_2', ..., y_n')dx, \tag{4}$$

with the given boundary conditions for all functions

$$y_1(x_0) = \theta_1, \quad y_2(x_0) = \theta_2, ..., y_n(x_0) = \theta_n, \tag{5}$$

$$y_1(x_1) = \delta_1, \quad y_2(x_1) = \delta_2, ..., y_n(x_1) = \delta_n. \tag{6}$$

Here the necessary condition for the extremum of the functional (4) is to satisfy the following system of second-order differential equations

$$F_{y_i} - \frac{d}{dx} F_{y_i'} = 0, \qquad i = 1, 2, ..., n, \tag{7}$$

with boundary conditions given in (5) and (6).

The Euler-Lagrange equation is generally nonlinear. In this manuscript we apply the DTM for solving Euler–Lagrange equations which arise from problems in calculus of variations. It is shown that this scheme is efficient for solving these kinds of problems.

3. Basic idea of differential transform method

For convenience of the reader, we will present a review of the DTM.

As in [46]-[49], the differential transform of the function $w(x)$ is in the form

$$W(k) = \frac{1}{k!} \left(\frac{d^k w(x)}{dx^k} \right)_{x=x_0}, \tag{8}$$

where $w(x)$ is the original function and $W(k)$ is the transformed function. The differential inverse transform of $W(k)$ is specified as follows

$$w(x) = \sum_{k=0}^{\infty} W(k)(x - x_0)^k. \tag{9}$$

From (8) and (9), we get

$$w(x) = \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{d^k w(x)}{dx^k} \right)_{x=x_0} (x - x_0)^k, \tag{10}$$

which implies that the differential transform is derived from Taylor series expansion, but the method does not evaluate derivatives symbolically. However, the corresponding derivatives

Table 1: The operations for the one-dimensional differential transform method.

Original function	Transformed function
$w(x) = u(x) \mp v(x)$,	$W(k) = U(k) \mp V(k)$
$w(x) = \alpha u(x)$	$W(k) = \alpha U(k)$
$w(x) = \frac{\partial^n u(x)}{\partial x^n}$	$W(k) = \frac{(k+n)!}{k!} U(k+n)$
$w(x) = u(x)v(x)$	$W(k) = \sum_{r=0}^{k} U(r)V(k-r)$
$w(x) = u(x)v(x)z(x)$	$W(k) = \sum_{r=0}^{k}\sum_{m=0}^{k-r} U(r)V(m)Z(k-r-m)$
$w(x) = x^n$	$W(k) = \delta(k-n)$, where $\delta(k-n) = \begin{cases} 1, & k=n \\ 0, & \text{otherwise} \end{cases}$
$w(x) = e^{\lambda t}$	$W(k) = \frac{\delta^k}{k!}$
$w(x) = t$	$W(k) = \delta(k-1)$

are calculated recursively, and are defined by the transformed equations of the original functions.

when x_0 is taken as 0, then the function $w(x)$ in (10) can be written as

$$w(x) = \sum_{k=0}^{\infty} W(k)x^k = \sum_{k=0}^{\infty} \frac{1}{k!} \left[\frac{d^k w(x)}{dx^k}\right]_{x=0} x^k. \tag{11}$$

In real applications, the function $w(x)$ can be expressed by a finite series as

$$w(x) = \sum_{k=0}^{n} W(k)x^k. \tag{12}$$

The solution of the Euler-Lagrange equation (3) with boundary conditions (2) is given in a series form that generally converges very rapidly in real physical problems. The fundamental mathematical operations performed by differential transform can readily be obtained and are listed in Table 1.

Here, we propose a new idea in order to use the DTM to solve optimization problem (1) with boundary conditions (2). We consider the the following initial value problem (IVP) of the Euler-Lagrange equation (3) as

$$\begin{cases} F_y - \frac{d}{dx}F_{y'} = 0, \\ y(x_0) = \theta, \\ y'(x_0) = \alpha, \end{cases} \tag{13}$$

where $\alpha \in I\!R$ is an unknown parameter. Using the DTM, we find the series solution of $y(x)$ consists of an unknown constant α. To find this constant, we impose the boundary condition $y(x_1) = \delta$ to the obtained approximate solution (12) which results in an equation in α. By solving this equation, we find α and then the optimal solution $y(.)$ is immediately given.

A similar procedure is done to solve problem (4) with respect to (5) and (6), where the imposed boundary condition is given by (6).

According to the above discussions, the following theorem can be stated:

Theorem 3.1 *Consider the calculus of variation problem (1) with boundary conditions (2). Employing the DTM, the optimal solution is given as*

$$y^*(x) = \sum_{k=0}^{\infty} Y(k)x^k, \qquad x \in [x_0, x_1]. \tag{14}$$

A similar theorem can be concluded for problem (4) with boundary conditions (5) and (6). It is clearly impossible to obtain the optimal trajectory law as in (14), since it contains infinite series. In practice, the Nth order suboptimal trajectory is obtained by replacing ∞ with a finite positive integer N in (14) as follows:

$$y(x) \simeq \sum_{k=0}^{N} Y(k)x^k, \qquad x \in [x_0, x_1]. \tag{15}$$

4. Simulation results

These examples are chosen such that there exist analytical solutions for them to give an obvious overview of the DTM.

Example 1: We consider the following variational problem [13]

$$\min \eta[y(x)] = \int_0^1 \left(y(x) + y'(x) - 4\exp(3x)\right)^2 dx, \tag{16}$$

with given boundary conditions

$$y(0) = 1, \qquad y(1) = e^3, \tag{17}$$

which has the following analytical solution

$$y(x) = \exp(3x). \tag{18}$$

The corresponding Euler-Lagrange equation is

$$\begin{cases} y'' - y - 8\exp(3x) = 0, \\ y(0) = 1, \\ y'(0) = \alpha. \end{cases} \tag{19}$$

Using the DTM we have

$$(k+1)(k+2)Y(k+2) - Y(k) - 8\frac{3^k}{k!} = 0, \tag{20}$$

$$Y(0) = 1, \quad Y(1) = \alpha. \tag{21}$$

Substituting (21) into (20) and by an iterative procedure, we achieve

$$Y(2)=\frac{9}{2},\ Y(3)=\frac{24+\alpha}{6},\ Y(4)=\frac{27}{8},\ Y(5)=\frac{1}{20}\left(36+\frac{24+\alpha}{6}\right),$$
$$Y(6)=\frac{81}{80},\ Y(7)=\frac{1}{42}\left(\frac{81}{5}+\frac{1}{20}(36+\frac{24+\alpha}{6})\right),\ Y(8)=\frac{729}{4480},$$
$$Y(9)=\frac{1}{72}\left(\frac{243}{70}+\frac{1}{42}(\frac{81}{5}+\frac{1}{20}(36+\frac{24+\alpha}{6}))\right),\ Y(10)=\frac{729}{44800},\ldots.$$

Substituting all $Y(k)$ into (15), the 11-term of the DTM series solution of $y(x)$ can be given by

$$y(x)\simeq\sum_{k=0}^{10}Y(k)x^k=1+\alpha x+\frac{9}{2}x^2+\frac{24+\alpha}{6}x^3+\frac{27}{8}x^4+\frac{1}{20}\left(36+\frac{24+\alpha}{6}\right)x^5+$$
$$\frac{81}{80}x^6+\frac{1}{42}x^7\left(\frac{81}{5}+\frac{1}{20}(36+\frac{24+\alpha}{6})\right)+\frac{729}{4480}x^8+$$
$$\frac{1}{72}x^9\left(\frac{243}{70}+\frac{1}{42}(\frac{81}{5}+\frac{1}{20}(36+\frac{24+\alpha}{6}))\right)+\frac{729}{44800}x^{10}. \tag{22}$$

This gives the approximation of the $y(x)$ in a series form. Now to find the constant α, the boundary condition at $x=1$ is imposed on the approximate solution of $y(x)$ in (22). We have

$$y(1)=e^3, \tag{23}$$

which results in

$$\alpha=3.0049963740455447321. \tag{24}$$

Replacing α into $y(x)$ in (22), an approximate solution is obtained for $y(x)$. Higher accuracy is also obtained using more components of $y(x)$; for example if $n=20$, we get $\alpha=3.0000000002015122890$. An absolute error between different values term of DTM solution $y(x)$ in (22) and the exact solution (18) is also depicted in Figure 1.

Example 2: In this example, consider the following variational problem [13]:

$$\min\ \eta[y(x)]=\int_0^1\frac{1+y^2(x)}{y'^2(x)}dx, \tag{25}$$

with given boundary conditions

$$y(0)=0,\qquad y(1)=0.5. \tag{26}$$

The exact solution of this problem is

$$y(x)=\sinh(0.4812118250x). \tag{27}$$

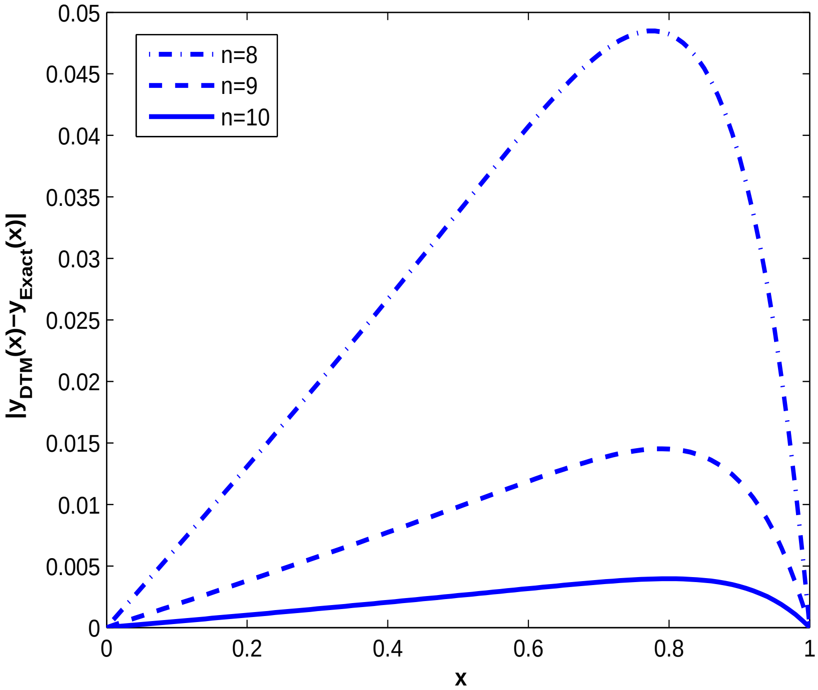

Figure 1: Error functions $|y_{DTM}(x) - y_{Exact}(x)|$ with different values of n for $0 \leq x \leq 1$ in Example 1.

The Euler-Lagrange equation of this problem is

$$\begin{cases} y'' + y''y^2 - yy'^2 = 0, \\ y(0) = 0, \\ y'(0) = \alpha. \end{cases} \tag{28}$$

Utilizing the DTM we get

$$(1+k)(2+k)Y(2+k) + \sum_{s=0}^{k}\sum_{m=0}^{k-s}(1+k-m-s)(2+k-m-s)Y(2+k-m-s)Y(m)Y(s) - \sum_{s=0}^{k}\sum_{m=0}^{k-s}(1+m)(1+k-m-s)Y(1+m)Y(1+k-m-s)Y(s) = 0, \tag{29}$$

$$Y(0) = 0, \qquad Y(1) = \alpha. \tag{30}$$

Substituting (30) into (29) we acquire

$$Y(k) = 0, \ \forall\ k = 2, 4, \ldots$$
$$Y(3) = \frac{\alpha^3}{6}, \quad Y(5) = \frac{\alpha^5}{120}, \quad , Y(7) = \frac{\alpha^7}{5040}, \quad Y(9) = \frac{\alpha^9}{362880}, \ldots$$

Substituting all $Y(k)$ into (15), the 5-term DTM series solution of $y(x)$ is obtained as following

$$y(x) \simeq \sum_{k=0}^{9} Y(k)x^k = x\alpha + \frac{x^3\alpha^3}{6} + \frac{x^5\alpha^5}{120} + \frac{x^7\alpha^7}{5040} + \frac{x^9\alpha^9}{362880}. \tag{31}$$

Implementing the boundary condition $y(1) = 0.5$ on $y(x)$ in (31), we obtain $\alpha = 0.48121182506679352167$. In Figure 2, the $|y'' + y''y^2 - yy'^2|$ is plotted.

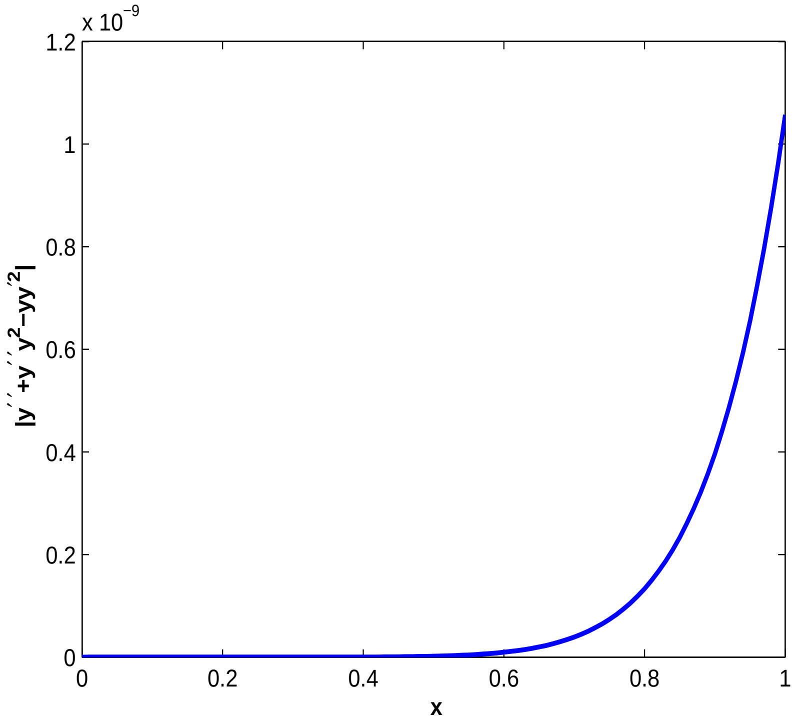

Figure 2: Error function $|y'' + y''y^2 - yy'^2|$ for $0 \leq x \leq 1$ in Example 2.

Example 3: We consider the following brachistochrone problem [13]

$$\min \eta[y(x)] = \int_0^1 \sqrt{\frac{1 + y'^2(x)}{1 - y(x)}}dx, \tag{32}$$

subject to the boundary conditions

$$y(0) = 0, \qquad y(1) = -0.5. \tag{33}$$

The analytical solution of this problem in the implicit form is

$$\begin{aligned}
&F(x, y(x)) = -\sqrt{-y^2 + 0.381510869y + 0.618489131} - \\
&0.8092445655 \times \arctan\left(\frac{y - 0.1907554345}{\sqrt{-y^2 + 0.381510869y + 0.618489131}}\right) \\
&-x + 0.5938731505 = 0.
\end{aligned}$$

The corresponding Euler-Lagrange equation is given by

$$\begin{cases} y'' - yy'' - \frac{1}{2} - \frac{y'^2}{2} = 0, \\ y(0) = 0, \\ y'(0) = \alpha. \end{cases} \tag{34}$$

According to the DTM, we have

$$2(k+1)(k+2)Y(k+2) = 2\sum_{s=0}^{k} Y(s)(k-s+2)(k-s+1)Y(k-s+2) + \sum_{s=0}^{k}(s+1)Y(s+1)(k-s+1)Y(k-s+1) + \delta(k), \tag{35}$$

$$Y(0) = 0, \quad Y(1) = \alpha. \tag{36}$$

Substituting (36) into (35) we get

$$\begin{aligned} Y(2) &= \frac{1}{4}(1+\alpha^2), \\ Y(3) &= \frac{1}{6}\alpha(1+\alpha^2), \\ Y(4) &= \frac{1}{48}(1+8\alpha^2+7\alpha^4) \\ Y(5) &= \frac{1}{240}t^5\alpha(11+46\alpha^2+35\alpha^4), \\ Y(6) &= \frac{11+237\alpha^2+681\alpha^4+455\alpha^6}{2880}, \\ Y(7) &= \frac{\alpha(73+696\alpha^2+1533\alpha^4+910\alpha^6)}{5040}, \\ &.... \end{aligned}$$

Substituting all $Y(k)$ into (15), the series solution of $y(x)$ is

$$y(x) \simeq \sum_{k=0}^{7} Y(k)x^k = \alpha x + \frac{1}{4}x^2(1+\alpha^2) + \frac{1}{6}x^3\alpha(1+\alpha^2) + \frac{1}{48}x^4(1+8\alpha^2+7\alpha^4) + \frac{1}{240}x^5\alpha(11+46\alpha^2+35\alpha^4) + \frac{x^6(11+237\alpha^2+681\alpha^4+455\alpha^6)}{2880} + \frac{x^7\alpha(73+696\alpha^2+1533\alpha^4+910\alpha^6)}{5040}. \tag{37}$$

Imposing the boundary condition $y(1) = -0.5$ on the DTM solution $y(x)$ in (37), we obtain $\alpha = -0.70774933732752545552 4125800640$. In Figure 3, the error function $|F(x, y_n)|$ is plotted for $n = 1, 3, 5$. The convergence of the iteration formula is clear in this figure.

Example 4: We consider the problem of finding the extremals of the functional [13]

$$\eta[y(x), z(x)] = \int_0^{\frac{\pi}{2}} \left[y'^2(x) + z'^2(x) + 2y(x)z(x)\right] dx, \tag{38}$$

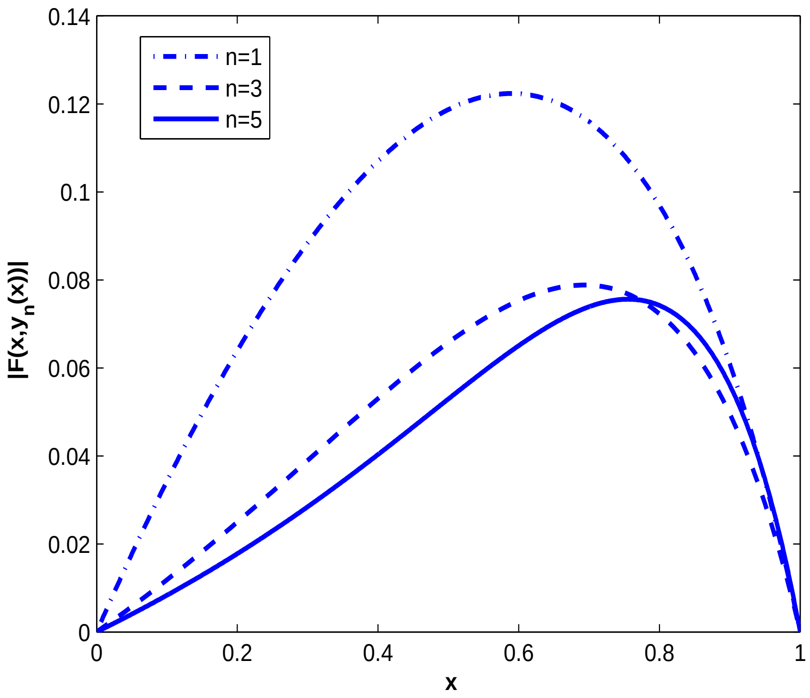

Figure 3: Error functions $F(x, y_1(x))$, $F(x, y_2(x))$ and $F(x, y_3(x))$ for $0 \le x \le 1$ in Example 3.

with the given boundary conditions as follows:

$$y(0) = 0, \qquad y(\frac{\pi}{2}) = 1, \tag{39}$$

$$z(0) = 0, \qquad z(\frac{\pi}{2}) = -1, \tag{40}$$

which has the following analytical solution

$$\begin{cases} y(x) = \sin(x), \\ z(x) = -\sin(x). \end{cases} \tag{41}$$

The system of Euler's differential equations is of the form

$$\begin{cases} y'' - z = 0, \\ z'' - y = 0, \\ y(0) = 0, \\ y'(0) = \alpha, \\ z(0) = 0, \\ z'(0) = \beta. \end{cases} \tag{42}$$

Implementing the DTM we have

$$(k+1)(k+2)Y(k+2) - Z(k) = 0, \tag{43}$$

$$(k+1)(k+2)Z(k+2) - Y(k) = 0, \tag{44}$$

$$Y(0) = 0, \qquad Y(1) = \alpha, \qquad Z(0) = 0, \qquad Z(1) = \beta. \tag{45}$$

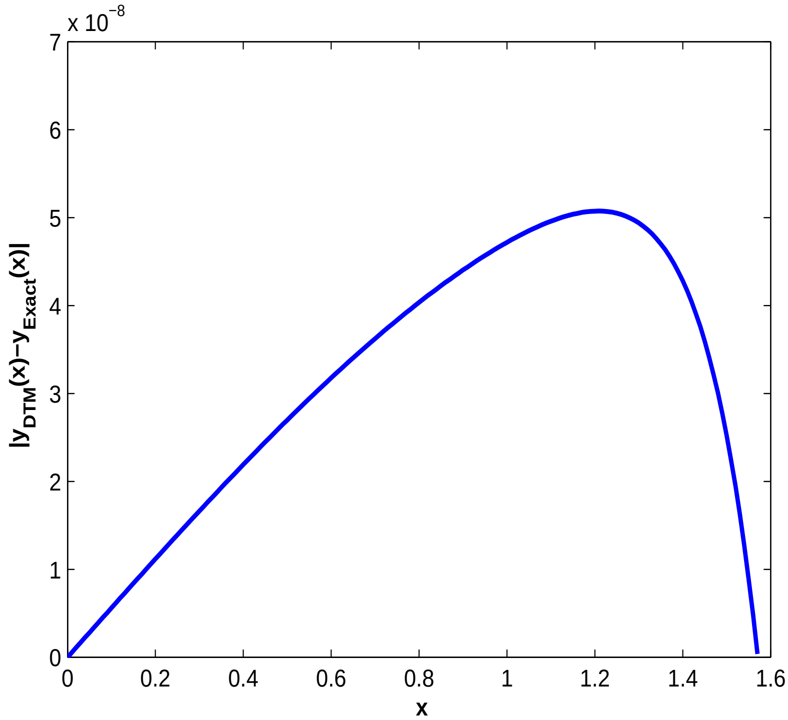

Figure 4: Comparison of the exact solution with the DTM solution.

Substituting (45) into (43) and (44) we get

$$\begin{aligned}
&Y(k) = 0, \ \forall\ k = 2, 4, \ldots \\
&Y(3) = \frac{\beta}{6}, \quad Y(5) = \frac{\alpha}{120}, \quad Y(7) = \frac{\beta}{5040}, \ldots \\
&Z(k) = 0, \ \forall\ k = 2, 4, \ldots \\
&Z(3) = \frac{\alpha}{6}, \quad Z(5) = \frac{\beta}{120}, \quad Z(7) = \frac{\alpha}{5040}, \ldots
\end{aligned}$$

Substituting all $Y(k)$ into (15), the 6-term DTM series solutions of $y(x)$ and $z(x)$ are given by

$$y(x) \simeq \sum_{k=0}^{11} Y(k)x^k = \alpha x + \frac{\beta x^3}{6} + \frac{\alpha x^5}{120} + \frac{\beta x^7}{5040} + \frac{\alpha x^9}{362880} + \frac{\beta x^{11}}{39916800}, \qquad (46)$$

$$z(x) \simeq \sum_{k=0}^{11} Z(k)x^k = \beta x + \frac{\alpha x^3}{6} + \frac{\beta x^5}{120} + \frac{\alpha x^7}{5040} + \frac{\beta x^9}{362880} + \frac{\alpha x^{11}}{39916800}. \qquad (47)$$

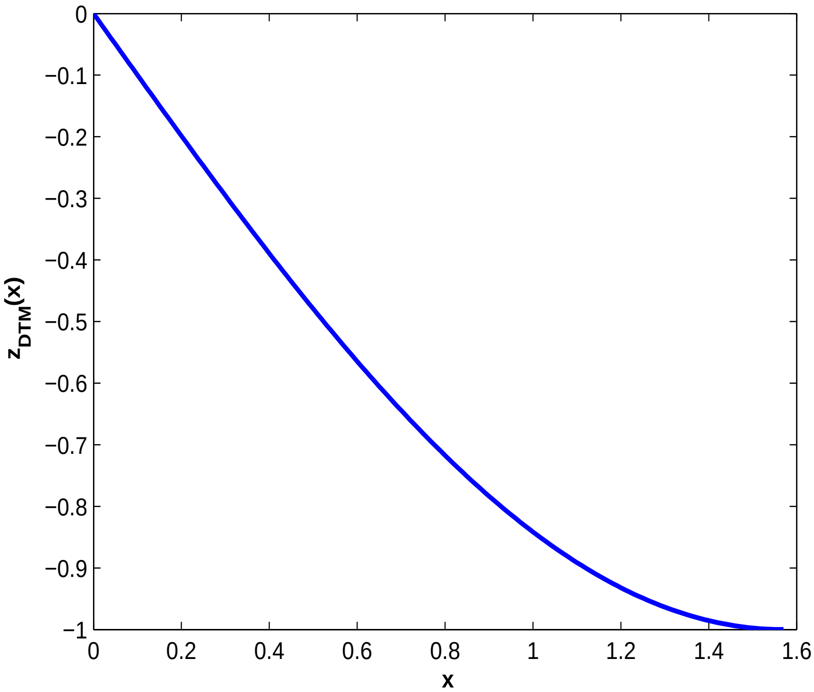

Figure 5: The DTM solution $z(t)$ in Example 4.

In order to find the unknown constants α and β, we use the boundary condition $y(\frac{\pi}{2}) = 1$ and $z(\frac{\pi}{2}) = -1$; we get

$$\begin{cases} y(\frac{\pi}{2}) = 1 \Longrightarrow \alpha = 1.0000000562589522947, \\ z(\frac{\pi}{2}) = -1 \Longrightarrow \beta = -1.0000000562589522947. \end{cases} \qquad (48)$$

An absolute error between $y(x)$ in (46) and the corresponding exact solution in (41) is depicted in Figures 4. The DTM solution of $z(x)$ in (47) is also shown in Figure 5. These graphs show that the proposed method has an appropriate convergence rate.

5. Conclusion

The DTM is employed for finding the solution of the ordinary differential equations which arise from problems of calculus of variations. The present study has confirmed that the DTM offers great advantages of straightforward applicability, computational efficiency and high accuracy. The DTM needs less work in comparison with the traditional methods. Therefore, this method can be applied to many complicated linear and non-linear problems and does not require linearization, discretization or perturbation. Mathematica and Matlab have been used for computations and simulations in this paper.

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