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Some New Families of Edge Product Cordial Graphs

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Abstract

For a graph $G=(V(G),E(G))$, an edge labeling function $f:E(G)\to\{0,1\}$ induces a vertex labeling function $f^*:V(G)\to\{0,1\}$ such that $f^*(v)$ is the product of the labels of the edges incident to v. This function f is called edge product cordial labeling of G if the edges with label 1 and label 0 differ by at most 1 and the vertices with label 1 & label 0 also differ by at most 1. In this paper we investigate some new families of edge product cordial graph.

***Keywords**:* Cordial graph, Product cordial graph, Edge Product cordial graph.

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1. Introduction

We begin with simple, finite, connected and undirected graph $G=(V(G),E(G))$. We will give brief summary of definitions and other information which are useful for the present investigations. The terms not defined here are used in the sense of Chartrand and Lesniak [1].

Definition 1.1. *A **graph labeling** is an assignment of integers to the vertices or edges or both subject to certain condition(s). If the domain of the mapping is the set of vertices (or edges) then the labeling is called **a vertex labeling (or an edge labeling)**.*

For an extensive survey on graph labeling and bibliographic references we refer to Gallian [2].

Most of the graph labeling techniques trace their origin to graceful labeling introduced independently by Rosa [3] and Golomb [4]. The famous Ringel-Kotzig graceful tree conjecture and illustrious work by Kotzig [5] brought a tide of labeling problems having graceful theme.

In 1987, Cahit [6] introduced the cordial labeling as a weaker version of graceful and harmonious labelings. Some labeling schemes are also introduced with minor variations in cordial theme. In 2004, Sundaram et al. [7] have introduced product cordial labeling in which the absolute difference in cordial labeling is replaced by product of the vertex labels.

The edge analogue of product cordial labeling was introduced by Vaidya and Barasara [8] and they named it as edge product cordial labeling which is defined as follows.

Definition 1.2. *For a graph $G=(V(G),E(G))$, an edge labeling function $f:E(G)\to\{0,1\}$*

induces a vertex labeling function $f^*: V(G) \to \{0,1\}$ *defined as* $f^*(v) = \prod f(e_i)$ *for* $\{e_i \in E(G) / e_i$ is incident to $v\}$.

Now denoting the number of vertices of G having label i under f^* as $v_f(i)$ and the number of edges of G having label i under f as $e_f(i)$. Then f is called **edge product cordial labeling** of graph G if $|v_f(0) - v_f(1)| \le 1$ and $|e_f(0) - e_f(1)| \le 1$. A graph G is called **edge product cordial** if it admits edge product cordial labeling.

Definition 1.3. *The **tadpole** is formed by joining the end point of a path* P_m *to a cycle* C_n. *It is denoted by* $C_n@P_m$.

Definition 1.4. *The **triangular snake*** T_n *is obtained from the path* P_n *by replacing every edge of a path by a triangle* C_3.

Definition 1.5. *The **double triangular snake*** DT_n *consists of two triangular snakes that have a common path.*

Definition 1.6. *The **quadrilateral snake*** Q_n *is obtained from the path* P_n *by replacing every edge of a path by a cycle* C_4.

Definition 1.7. *The **double quadrilateral snake*** DQ_n *consists of two quadrilateral snakes that have a common path.*

Definition 1.8. *The **double fan*** DF_n *is given by* $P_n + 2K_1$.

In this paper we investigate some new families of edge product cordial graphs.

2. Main Results

Theorem 2.1. *The tadpole* $C_n@P_m$ *is an edge product cordial graph for* $m+n$ *is even or* $m+n$ *is odd and* $m > n$ *while not an edge product cordial for* $m+n$ *odd and* $m < n$.

Proof. Let $e_1, e_2, \ldots, e_{m-1}$ be the edges of path P_m and $e_m, e_{m+1}, \ldots, e_{m+n-1}$ be the edges of cycle C_n. Consider tadpole $G = C_n@P_m$ having $m+n-1$ vertices and $m+n-1$ edges. Without loss of generality assume that e_{m-1}, e_m and e_{m+n-1} are adjacent edges. We consider following two cases.

Case 1: When $m+n$ is even.

Then the result holds as proved by Vaidya and Barasara [8] all unicyclic graph of odd size is edge product cordial.

Case 2: When $m+n$ is odd.

<u>Subcase 1:</u> When $m > n$.

$$f(e_i)=1; \quad 1 \le i \le \frac{m+n-1}{2}$$
$$f(e_i)=0; \qquad \text{otherwise.}$$

In view of the above defined labeling pattern we have

$$v_f(0)=v_f(1)=\frac{m+n-1}{2}$$
$$e_f(0)=e_f(1)=\frac{m+n-1}{2}$$

Subcase 2: When $m<n$.

In order to satisfy the edge condition for edge product cordial graph it is essential to assign label 0 to $\frac{m+n-1}{2}$ edges out of $m+n-1$ edges. The edges with label 0 will give rise at least $\frac{m+n+1}{2}$ vertices with label 0 and at most $\frac{m+n-3}{2}$ vertices with label 1 out of total $m+n-1$ vertices. Therefore $\left|v_f(0)-v_f(1)\right| \ge 2$. Thus the vertex condition for edge product cordial graph is violated.

Hence, the tadpole $C_n@P_m$ is an edge product cordial graph for $m+n$ is even or $m+n$ is odd and $m>n$ while not an edge product cordial for $m+n$ odd and $m<n$.

Example 2.2. *The tadpole $C_5@P_8$ and its edge product cordial labeling is shown in Figure 1.*

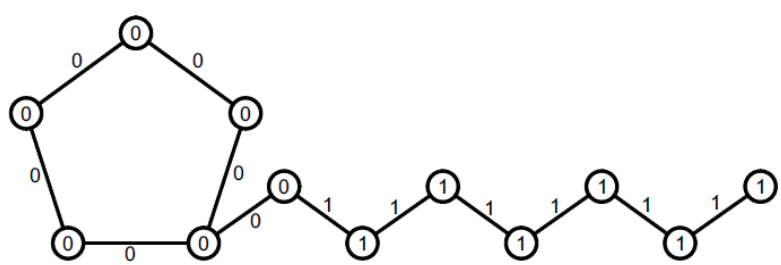

Figure 1

Theorem 2.3. *The graph T_n is edge product cordial graph.*

Proof. Let path P_n having vertices $v_1, v_2, \ldots, v_n$ and edges $e_1, e_2, \ldots, e_{n-1}$. To construct triangular snake T_n from path P_n join v_i and v_{i+1} to new vertex w_i by edges $e'_{2i-1}=v_iw_i$ and $e'_{2i}=v_{i+1}w_i$ for $i=1,2,\ldots,n-1$. $\left|V(T_n)\right|=2n-1$ and $\left|E(T_n)\right|=3n-3$. We consider following two cases.

Case 1: When n is odd.

$$f(e_i) = 1; \quad 1 \le i \le \frac{n-1}{2}$$
$$f(e_i) = 0; \quad \text{otherwise}$$
$$f(e_i') = 1; \quad 1 \le i \le n-1$$
$$f(e_i') = 0; \quad \text{otherwise}$$

In view of the above defined labeling patten we have

$$v_f(0) = v_f(1) + 1 = n$$
$$e_f(0) = e_f(1) = \frac{3n-3}{2}$$

Case 2: When n is even.

$$f(e_i) = 1; \quad 1 \le i \le \frac{n}{2}$$
$$f(e_i) = 0; \quad \text{otherwise}$$
$$f(e_i') = 1; \quad 1 \le i \le n-1$$
$$f(e_i') = 0; \quad \text{otherwise}$$

In view of the above defined labeling pattern we have

$$v_f(0) = v_f(1) + 1 = n$$
$$e_f(0) + 1 = e_f(1) = \frac{3n-2}{2}$$

Thus in all cases we have $\left|v_f(0) - v_f(1)\right| \le 1$ and $\left|e_f(0) - e_f(1)\right| \le 1$.

Hence, the graph T_n snake is edge product cordial graph.

Example 2.4. *The graph T_5 and its edge product cordial labeling is shown in Figure 2.*

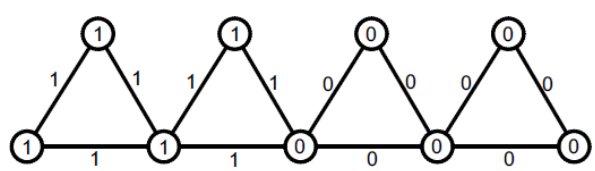

Figure 2

Theorem 2.5. *The graph DT_n is an edge product cordial graph for odd n and not an edge product cordial for even n.*

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices and $e_1, e_2, \ldots, e_{n-1}$ be the edges of path P_n. To construct double triangular snake DT_n from path P_n join v_i and v_{i+1} to two new vertices w_i and w_i' by edges $e_{2i-1}' = v_i w_i$, $e_{2i}' = v_{i+1} w_i$, $e_{2i-1}'' = v_i w_i'$ and $e_{2i}'' = v_{i+1} w_i'$ for $i = 1, 2, \ldots, n-1$. $\left|V(DT_n)\right| = 3n-2$ and $\left|E(DT_n)\right| = 5n-5$. We consider following two cases.

Case 1: When n is odd.

$$f(e_i)=0;\quad 1\le i\le \frac{n-1}{2}$$
$$f(e_i)=1;\quad \text{otherwise}$$
$$f(e_i')=0;\quad 1\le i\le n-1$$
$$f(e_i')=1;\quad \text{otherwise}$$
$$f(e_i'')=0;\quad 1\le i\le n-1$$
$$f(e_i'')=1;\quad \text{otherwise}$$

In view of the above defined labeling patten we have

$$v_f(0)=v_f(1)+1=\frac{3n-1}{2}$$
$$e_f(0)=e_f(1)=\frac{5n-5}{2}$$

Case 2: When n is even.

In order to satisfy the edge condition for edge product cordial graph it is essential to assign label 0 to at least $\left\lfloor \frac{5n-5}{2} \right\rfloor$ edges out of $5n-5$ edges. The edges with label 0 will give rise at least $\frac{3n}{2}$ vertices with label 0 and at most $\frac{3n}{2}-2$ vertices with label 1 out of total $3n-2$ vertices. Therefore $\left|v_f(0)-v_f(1)\right|\ge 2$. Thus the vertex condition for edge product cordial graph is violated.

Hence, the graph DT_n is an edge product cordial graph for odd n and not an edge product cordial for even n.

Example 2.6. *The graph DT_5 and its edge product cordial labeling is shown in Figure 3.*

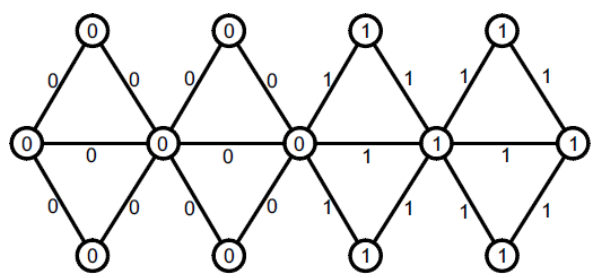

Figure 3

Theorem 2.7. *The graph Q_n is edge product cordial graph for odd n and not an edge product cordial graph for even n.*

Proof. Let $v_1,v_2,\ldots,v_n$ be the vertices and $e_1,e_2,\ldots,e_{n-1}$ be the edges of path P_n. To construct Q_n from path P_n we join v_i and v_{i+1} to two new vertices w_i and w_i' by edges $e_{2i-1}'=v_iw_i$, $e_{2i}'=v_{i+1}w_i'$, and $e_i''=w_iw_i'$ for $i=1,2,\ldots,n-1$. $\left|V(Q_n)\right|=3n-2$ and $\left|E(Q_n)\right|=4n-4$. We

consider following two cases.

Case 1: When n is odd.

$$f(e_i) = 0; \quad 1 \le i \le \frac{n-1}{2}$$
$$f(e_i) = 1; \quad \text{otherwise}$$
$$f(e_i') = 0; \quad 1 \le i \le n-1$$
$$f(e_i') = 1; \quad \text{otherwise}$$
$$f(e_i'') = 0; \quad 1 \le i \le \frac{n-1}{2}$$
$$f(e_i'') = 1; \quad \text{otherwise}$$

In view of the above defined labeling patten we have

$$v_f(0) = v_f(1) + 1 = \frac{3n-1}{2}$$
$$e_f(0) = e_f(1) = 2n-2$$

Case 2: When n is even.

In order to satisfy the edge condition for edge product cordial graph it is essential to assign label 0 to $2n-2$ edges out of $4n-4$ edges. The edges with label 0 will give rise at least $\frac{3n}{2}$ vertices with label 0 and at most $\frac{3n}{2}-2$ vertices with label 1 out of total $3n-2$ vertices. Therefore $\left|v_f(0) - v_f(1)\right| \ge 2$. Thus the vertex condition for edge product cordial graph is violated.

Hence, the graph Q_n is an edge product cordial graph for odd n and not an edge product cordial for even n.

Example 2.8. *The graph Q_5 and its edge product cordial labeling is shown in Figure 4.*

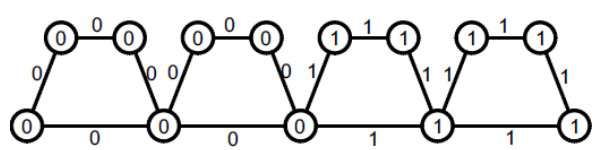

Figure 4

Theorem 2.9. *The graph DQ_n is edge product cordial for odd n and not an edge product cordial for even n.*

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices and $e_1, e_2, \ldots, e_{n-1}$ be the edges of path P_n. To construct DQ_n from path P_n we join v_i and v_{i+1} to four new vertices w_i, w_i', x_i and x_i' by edges $e_{2i-1}' = v_i w_i$, $e_{2i}' = v_{i+1} w_i'$, $e_i'' = w_i w_i'$, $e_{2i-1}^a = v_i x_i$, $e_{2i}^a = v_{i+1} x_i'$ and $e_i^b = x_i x_i'$ for $i = 1, 2, \ldots, n-1$.

$|V(DQ_n)| = 5n-4$ and $|E(DQ_n)| = 7n-7$. We consider following two cases.

Case 1: When n is odd.

$$f(e_i) = 0; \quad 1 \le i \le \frac{n-1}{2}$$
$$f(e_i) = 1; \quad \text{otherwise}$$
$$f(e_i') = 0; \quad 1 \le i \le n-1$$
$$f(e_i') = 1; \quad \text{otherwise}$$
$$f(e_i'') = 0; \quad 1 \le i \le \frac{n-1}{2}$$
$$f(e_i'') = 1; \quad \text{otherwise}$$
$$f(e_i^a) = 0; \quad 1 \le i \le n-1$$
$$f(e_i^a) = 1; \quad \text{otherwise}$$
$$f(e_i^b) = 0; \quad 1 \le i \le \frac{n-1}{2}$$
$$f(e_i^b) = 1; \quad \text{otherwise}$$

In view of the above defined labeling patten we have

$$v_f(0) = v_f(1) + 1 = \frac{5n-3}{2}$$
$$e_f(0) = e_f(1) = \frac{7n-7}{2}$$

Case 2: When n is even.

In order to satisfy the edge condition for edge product cordial graph it is essential to assign label 0 to at least $\left\lfloor \frac{7n-7}{2} \right\rfloor$ edges out of $7n-7$ edges. The edges with label 0 will give rise at least $\frac{5n-2}{2}$ vertices with label 0 and at most $\frac{5n-6}{2}$ vertices with label 1 out of total $5n-4$ vertices. Therefore $|v_f(0) - v_f(1)| \ge 2$. Thus the vertex condition for edge product cordial graph is violated.

Hence, the graph DQ_n is an edge product cordial graph for odd n and not an edge product cordial for even n.

Example 2.10. *The graph DQ_5 and its edge product cordial labeling is shown in Figure 5.*

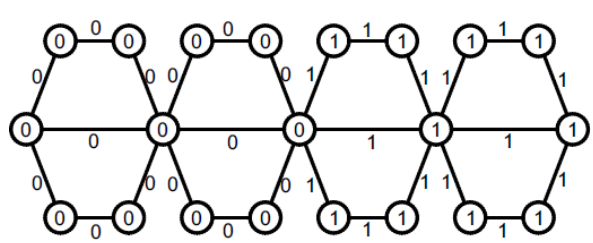

Figure 5

Theorem 2.11. *The graph* DF_n *is not an edge product cordial graph.*

Proof. We consider following two cases.

Case 1: When n is odd.

In order to satisfy the edge condition for edge product cordial graph it is essential to assign label 0 to $\frac{3n-1}{2}$ edges out of $3n-1$ edges. The edges with label 0 will give rise at least $\frac{n+5}{2}$ vertices with label 0 and at most $\frac{n-1}{2}$ vertices with label 1 out of total $n+2$ vertices. Therefore $\left|v_f(0)-v_f(1)\right| \geq 3$. Thus the vertex condition for edge product cordial graph is violated.

Case 2: When n is even.

In order to satisfy the edge condition for edge product cordial graph it is essential to assign label 0 to at least $\left\lfloor \frac{3n-1}{2} \right\rfloor$ edges out of $5n-5$ edges. The edges with label 0 will give rise at least $\frac{n+4}{2}$ vertices with label 0 and at most $\frac{n}{2}$ vertices with label 1 out of total $n+2$ vertices. Therefore $\left|v_f(0)-v_f(1)\right| \geq 2$. Thus the vertex condition for edge product cordial graph is violated.

Hence, the graph DF_n is not an edge product cordial graph.

3. Concluding Remarks

We contribute some new results on edge product cordial labeling. The labeling pattern is demonstrated by means of illustrations. To derive similar results for other graph families and in the context of different labeling problems is an open area of research.

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