AMO - Advanced Modeling and Optimization, Volume 15, Number 1, 2013

Cordial labelling of Cactus Graphs

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Abstract

Suppose $G=(V,E)$ be a graph with vertex set V and edge set E. A vertex labelling $f:V\to\{0,1\}$ induces an edge labelling $f^*:E\to\{0,1\}$. For $i\in\{0,1\}$, let $v_f(i)$ and $e_f(i)$ be the number of vertices v and edges e with $f(v)=i$ and $f^*(e)=i$ respectively. A graph is cordial if there exists a vertex labelling f such that $|v_f(0)-v_f(1)|\le 1$ and $|e_f(0)-e_f(1)|\le 1$. In this paper, we label the vertices of cactus graph by cordial labelling and have shown that cactus graph is cordial under some restrictions.

Keywords: Cordial labelling; cactus graph; planar graph; tree

2010 Mathematics Subject Classification: 05C78

1. Introduction

We begin with cactus graph $G=(V,E)$ with n vertices. A cactus graph is a connected graph in which every block is either a cycle or an edge, in other words, no edge belongs to more than one cycle. This graph is one of the most useful discrete mathematical structure for modelling problem arising in the real world. It has many applications in various fields like computer scheduling, radio communication system, etc. Cactus graph have studied from both theoretical and algorithmic points of view. This graph is a subclass of planar graph and superclass of tree. Cactus graph has many intersecting subgraphs like edge, cycle, sun, star, triangle shape star graph, one point union of cycles, etc. These all are induced subgraphs of cactus graph.

If the vertices of a graph are assigned values subject to certain conditions is known as graph labelling. Most interesting graph labelling problems have three important characteristics.

1. A set of numbers from which the labels are chosen.
2. A rule that assigns a value to each edge.
3. A condition that these values must satisfy.

For detail survey on graph labelling one can refer Gallian [9]. Vast amount of literature is available on different types of graph labelling according to Beineke and Hegde [1] graph labelling serves as a frontier between number theory and structure of graphs.

Labelled graph have variety of applications in coding theory, particularly for missile guidance codes, design of good radar type codes and convolution codes with optimal autocorrelation properties. Labelled graph plays vital role in the study of X-Ray crystallography, communication network and to determine optimal circuit layouts. A detail study of variety of applications of graph labelling is given by Bloom and Golomb [2].

Some useful definitions are given below.

Definition 1. *Let* $G=(V,E)$ *be a graph. A mapping* $f:V(G)\rightarrow\{0,1\}$ *is called binary vertex labelling of* G *and* $f(v)$ *is called the label of the vertex* v *of* G *under* f.

For an edge $e=(u,v)$, the induced edge labelling $f^*:E(G)\rightarrow\{0,1\}$ is given by $f^*(e)=|f(u)-f(v)|$. Let $v_f(0)$, $v_f(1)$ be the number of vertices of G having labels 0 and 1 respectively under f and $e_f(0)$, $e_f(1)$ be the number of edges of G having labels 0 and 1 respectively under f^*.

Definition 2. *A binary vertex labelling of a graph* G *is called cordial labelling if* $|v_f(0)-v_f(1)|\leq 1$ *and* $|e_f(0)-e_f(1)|\leq 1$. *A graph is cordial if it admits cordial labelling.*

Cordial graphs were first introduced by Cahit [4] as a weaker version of both graceful graphs and harmonious graphs.

A fairly obvious reformulation of the definition of a cordial graph is given below.

Definition 3. *A graph* $G=(V,E)$ *is cordial if and only if there exists a partition* $\{V_1,V_2\}$ *of* V *such that two induced subgraphs* $G_1=(V_1,E_1)$ *and* $G_2=(V_2,E_2)$ *satisfy the following conditions:*

$||V_1|-|V_2||\leq 1$ and $||E|-2(|E_1|+|E_2|)|\leq 1$.

In this paper, we label the vertices of all possible induced subgraphs of cactus graph and cactus graph having finite number of vertices. We have shown that cactus graph is cordial under certain restrictions.

2. Review of previous work

Various work have been done on cordial labelling on different types of graphs. But to the best of our knowledge and believe, we solve this problem first time for cactus graph.

Cahit [3] proved the followings: every tree is cordial; K_n is cordial if and only if $n\leq 3$; $K_{m,n}$ is cordial for all m and n; the friendship graph $C_3^{(t)}$ (i.e., the one-point union of t-cycles) if and only if $t\equiv 2$ (mod 4); all fans are cordial; the wheel W_n is cordial if and only if $n\equiv 3$ (mod 4) (see also [6]); maximal outerplanar graphs are cordial; and an Eulerian graph is not cordial if its size is congruent to 2 (mod 4). Kuo, Chang and Kwong [15] determine all m and n for which mK_n is cordial. Liu and Zhu [16] proved that a 3-regular graph of order n is cordial if and only if $n\not\equiv 4$ (mod 8).

A k-angular cactus graph is connected graph all of whose blocks are cycles with k-vertices. In [3], Cahit proved that a k-angular cactus with t cycles is cordial if $kt\not\equiv 2$ (mod 4). This was improved by Kirchherr [13] and shown that any cactus whose blocks are cycles is cordial if and only if size of the graph is not congruent to 2 (mod 4). Kirchherr [14] also give a characterization of cordial graphs in terms of their adjacency matrices. Ho *et al.* [11] shown that a unicycle is cordial unless it is C_{4k+2} and that the generalized Petersen graph $P(n,k)$ if and only if $n\not\equiv 2$ (mod 4).

Seoud and Maqusoud [17] proved that if G is a graph with n vertices and m edges and every vertex has odd degree, then G is cordial if and only if $m+n \equiv 2$ (mod 4). They also prove the following: for $m \geq 2$, $C_n \times P_n$ is cordial except for the case $C_{4k+2} \times P_2$; P_n^4 is cordial if and only if $n \neq 4,5,6$. Seoud *et al.* [18] have proved that the following graphs are cordial: $P_n + P_m$ for all m and n except $(m,n)=(2,2)$; $C_n + C_m$ if $m \not\equiv 2$ (mod 4); $C_n + K_{1,m}$ for $m \not\equiv 3$ (mod 4) and odd m except $(n,m)=(3,1)$; $C_n + \overline{K_m}$ when n is odd, and when n is even and m is odd; $K_{1,m,n}$; the n-cube; books B_n if and only if $n \not\equiv 3$ (mod 4); $B(3,2,m)$ for all m; $B(4,3,m)$ if and only if m is even; and $B(5,3,m)$ if and only if $m \not\equiv 1$ (mod 4).

Diab [8] has proved that if G and H are cordial and one has even size, then $G \cup H$ is cordial; if G and H are cordial and both have even size, then $G+H$ is cordial; if G and H are cordial and one has even size and either one has even order, then $G+H$ is cordial; $C_n \cup C_m$ if and only if $m+n \not\equiv 2$ (mod 4); mC_n if and only if $(m,n) \neq (3,3)$ and $\{m \ (\mathrm{mod}\ 4),\ n \ (\mathrm{mod}\ 4)\} \neq \{0,2\}$; and if P_n^k is cordial, then $n \geq k+1+\sqrt{k-1}$.

Shee and Ho [19] have investigated the cordiality of the one-point union of n copies of various graphs.

A graph G of vertex set $V(G)$ and edge set $E(G)$ to be k-cordial if there is a vertex labelling f from $V(G)$ to Z_k, the group of integers modulo k, so that when each edge (x,y) is assigned the label $(f(x)+f(y))$ (mod k), the number of vertices (respectively, edges) labelled with i and the number of vertices (respectively, edges) labelled with j differ by at most one for all i and j in Z_k.

Du [7] proved that the disjoint union of $n \geq 2$ wheels is cordial if and only if n is even or n is odd and the number of vertices of each cycle is not 0 (mod 4), or n is odd and the number of vertices in each cycle is not 3 (mod 4). Hovey [10] has obtained the following: caterpillars are k-cordial; all trees are k-cordial for $k=3,4$ and 5; odd cycles with pendent edges attached are k-cordial for all k; cycles are k-cordial for all odd k; for even k C_{2mk+j} is cordial when $0 \leq j \leq \frac{k}{2}+2$ and when $k < j < 2k$; $C_{(2m+1)k}$ is not k-cordial; and for even k, K_{mk} is k-cordial if and only if $m=1$.

Cairnie and Edwards [5] have determined the computation complexity of cordial and k-cordial labellings. They proved the conjecture posed by Kirchherr [14] deciding whether a graph admits a cordial labelling if NP-complete.

In [20], Ramanjaneyulu proved that Pl_n, $n \geq 5$ is cordial if $n \not\equiv 0$ (mod 4); $Pl_{m,n}$, $m,n \geq 3$ is cordial; $Pl_{m,n}$ is total product cordial except for either m even and $n \not\equiv 2$ (mod 4), or m odd and $n \not\equiv 1$ (mod 4).

3. Cordial labelling of subclasses of cactus graph

Cycle is a subgraph of cactus graph. Ho *et al.* [11] proved that a unicycle is cordial except it is C_{4k+2}. We discuss this in the following lemma.

A proof is given of this lemma to prove other results of cactus graph.

Lemma 1. *[11] A cycle C_n of length n is cordial.*

Proof. Let the vertices and edges of C_n be $v_0, v_1, v_2, \ldots, v_{n-1}$; $e_0, e_1, e_2, \ldots, e_{n-1}$ respectively, where $e_i = (v_i, v_{i+1})$, $i = 0, 1, \ldots, n-2$ and $e_{n-1} = (v_0, v_{n-1})$. To label the vertices of C_n we classify the cycle into four groups, viz., C_{4k}, C_{4k+1}, C_{4k+2} and C_{4k+3}.

Case 1. If $n = 4k \equiv 0$ (mod 4).

In this case, $f(v_i)$ is defined as follows.

$$f(v_i) = \begin{cases} 0, \text{ if } i \equiv 0 \text{ (mod 4)}; \\ 0, \text{ if } i \equiv 1 \text{ (mod 4)}; \\ 1, \text{ if } i \equiv 2 \text{ (mod 4)}; \\ 1, \text{ if } i \equiv 3 \text{ (mod 4)}. \end{cases}$$

Here, $v_f(0) = \frac{n}{2} = 2k = v_f(1)$ and $e_f(0) = \frac{n}{2} = 2k = e_f(1)$, for $k = 1, 2, \ldots, \frac{n}{4}$. The vertex and edge condition is given in Table 1.

Case 2. If $n = 4k + 1 \equiv 1$ (mod 4).

Here we label the first $n - 1 = 4k$ vertices according to the same rule as in the above case. For the last vertex, f is defined as $f(v_{4k}) = 1$.

Here, $v_f(0) = \frac{n-1}{2} = \frac{4k+1-1}{2} = 2k$, $v_f(1) = \frac{n+1}{2} = \frac{4k+1+1}{2} = 2k+1$.

And $e_f(0) = \frac{n+1}{2} = \frac{4k+1+1}{2} = 2k+1$, $e_f(1) = \frac{n-1}{2} = \frac{4k+1-1}{2} = 2k$, for

$k = 1, 2, \ldots, \frac{n-1}{4}$.

Case 3. If $n = 4k + 2 \equiv 2$ (mod 4).

Here also we label first $n - 2 = 4k$ vertices as the same process as given in case 1. For the last two vertices v_{4k} and v_{4k+1}, f is defined as

$f(v_{4k}) = 0$ and $f(v_{4k+1}) = 1$ respectively.

Here, $v_f(0) = \frac{n}{2} = 2k+1 = v_f(1)$ and $e_f(0) = \frac{n-2}{2} = 2k$ and $e_f(1) = \frac{n+2}{2} = 2k+2$, for $k = 1, 2, \ldots, \frac{n-2}{4}$.

Case 4. If $n = 4k + 3 \equiv 3$ (mod 4).

The labelling of the vertices v_i's; $i = 0, 1, \ldots, 4k+1$, are same as given in case 1. Then we label the last vertex v_{4k+2} as

$f(v_{4k+2}) = 1$.

Here, $v_f(0) = \frac{n+1}{2} = 2k+2$, $v_f(1) = \frac{n-1}{2} = 2k+1$ and $e_f(0) = \frac{n-1}{2} = 2k+1$,

$e_f(1)=\dfrac{n+1}{2}=2k+2$, for $k=1,2,\ldots,\dfrac{n-3}{4}$.

Values of n	Condition on vertex	Condition on edge	Cordial
$n=4k$	$v_f(0)=v_f(1)$	$e_f(0)=e_f(1)$	Yes
$n=4k+1$	$v_f(0)+1=v_f(1)$	$e_f(0)=e_f(1)+1$	Yes
$n=4k+2$	$v_f(0)=v_f(1)$	$e_f(0)+2=e_f(1)$	Not
$n=4k+3$	$v_f(0)=v_f(1)+1$	$e_f(0)+1=e_f(1)$	Yes

Table 1: Vertex and edge conditions of all cases of Lemma 1

Thus in case 3 we see that $|e_f(0)-e_f(1)|\not\leq 1$. So a cycle is cordial except of length $4k+2$.

In [12], cordial labelling of one, two, three and four cycles, joined with a common cutvertex are discussed. And we can write the following result.

Lemma 2. *[12] Suppose G be a graph contains finite number of cycles of finite lengths, joined with a common cutvertex. Then the graph is cordial if and only if total number of edges of G is not congruent to 2 (mod 4).*

When some finite number of triangles (cycle of length 3) are joined with a common cutvertex then this graph is called triangle shape star graph or friendship graph. Cahit [3] proved the following result for a friendship graph $C_3^{(t)}$ (i.e., the one-point union of t 3-cycles).

Lemma 3. *A friendship graph $C_3^{(t)}$ is cordial if and only if $t\not\equiv 2$ (mod 4).*

For $C_m^{(n)}$, the one-point union of n copies of C_m, Shee and Ho [19] proved the following result.

Lemma 4. *For $C_m^{(n)}$,*

(i) if $m\equiv 0$ (mod 4), then $C_m^{(n)}$ is cordial for all n;

(ii) if $m\equiv 1$ (mod 4), then $C_m^{(n)}$ is cordial if and only if $n\not\equiv 2$ (mod 4);

(iii) if $m\equiv 2$ (mod 4), then $C_m^{(n)}$ is cordial if and only if n is even.

By combining Lemmas 1, 2, 3 and 4, we can conclude that if a graph G contains finite number of cycles of finite lengths then the graph is cordial having some restrictions.

Corollary 1. *Let a graph G contains finite number of cycles of finite lengths and finite number of edges, joined with a common cutvertex. Then G is cordial if and only if the total number of edges of all cycles is not congruent to 2 (mod 4).*

Now, if we add an edge to each of the vertex of a cycle of finite length, then we get a new subgraph of cactus graph called sun. The cordial labelling of this graph is discussed below.

Lemma 5. *A sun is cordial having finite number of vertices.*

Proof. Let us consider the length of the cycle be n. v_i; $i=0,1,\ldots,n-1$ are the vertices of C_n. Now we join an edge $e'_i=(v_i,v'_i)$ to each of the vertex of C_n and get the sun S_{2n}. To label the sun, first we label the cycle C_n according to the rule as given in Lemma 1. Then we label the pendent vertices in the following ways. Here total number of vertices and edges are $2n$.

Case 1. For $n=4k\equiv 0 \pmod 4$.

$$f(v'_i)=\begin{cases}0, & \text{if } i\equiv 0 \pmod 2;\\ 1, & \text{if } i\equiv 1 \pmod 2.\end{cases}$$

Here, $v_f(0)=\dfrac{8k}{2}=4k=v_f(1)$ and $e_f(0)=\dfrac{8k}{2}=4k=e_f(1)$ for $k=1,2,\ldots,\dfrac{n}{4}$.

Case 2. For $n=4k+1\equiv 1 \pmod 4$.

In this case, we label the vertices v'_i; $i=0,1,\ldots,4k-1$ as the same process as in the above case. Now we label the vertex v'_{4k} as $f(v'_{4k})=0$.

Here, $v_f(0)=\dfrac{8k+2}{2}=4k+1=v_f(1)$ and $e_f(0)=\dfrac{8k+2}{2}=4k+1=e_f(1)$ for $k=1,2,\ldots,\dfrac{n-1}{4}$.

Case 3. For $n=4k+2\equiv 2 \pmod 4$.

Here we label the pendent vertices as

$$f(v'_i)=\begin{cases}0, & \text{if } i\equiv 0 \pmod 2;\\ 1, & \text{if } i\equiv 1 \pmod 2.\end{cases}$$

So, $v_f(0)=\dfrac{8k+4}{2}=4k+2=v_f(1)$ and $e_f(0)=\dfrac{8k+4}{2}=4k+2=e_f(1)$ for $k=1,2,\ldots,\dfrac{n-2}{4}$.

Case 4. For $n=4k+3\equiv 3 \pmod 4$.

In this case we label first $n-2=4k+2$ vertices as given in case 3. And for v'_{4k+2}, $f(v'_{4k+2})=1$.

Here, $v_f(0)=\dfrac{8k+6}{2}=4k+3=v_f(1)$ and $e_f(0)=\dfrac{8k+6}{2}=4k+3=e_f(1)$ for $k=1,2,\ldots,\dfrac{n-3}{4}$.

Values of n	Condition on vertex	Condition on edge	Cordial
$n=4k$	$v_f(0)=v_f(1)$	$e_f(0)=e_f(1)$	Yes
$n=4k+1$	$v_f(0)=v_f(1)$	$e_f(0)=e_f(1)$	Yes
$n=4k+2$	$v_f(0)=v_f(1)$	$e_f(0)=e_f(1)$	Yes
$n=4k+3$	$v_f(0)=v_f(1)$	$e_f(0)=e_f(1)$	Yes

Table 2: Vertex and edge conditions of Cases 1, 2, 3 and 4 of Lemma 5

Thus, from all the above cases, we conclude that a sun of finite length is cordial.

Now another graph is obtained by adding a pendent edge to each of pendent vertex of a sun S_{2n}. Let G_1 be obtained by adding an edge $e''_i=(v'_i,v''_i)$ to each of the pendent vertex v'_i's of S_{2n}.

Now we label G_1 and get the following result.

Lemma 6. *A graph* G_1 *is cordial when is obtained by adding an edge to each of the pendent vertex of a sun* S_{2n}.

Proof. Here total number of vertices and edges are $3n$, where n is the length of C_n. First we label S_{2n} according to Lemma 5. Then we label the pendent vertices.

Case 1. For $n \equiv 0 \pmod 4$.

$$f(v_i'') = \begin{cases} 0, & \text{if } i \equiv 0 \ (\text{mod } 4) \text{ and } i \equiv 3 \ (\text{mod } 4); \\ 1, & \text{if } i \equiv 1 \ (\text{mod } 4) \text{ and } i \equiv 2 \ (\text{mod } 4). \end{cases}$$

Here, $v_f(0) = \frac{12k}{2} = 6k = v_f(1)$ and $e_f(0) = \frac{12k}{2} = 6k = e_f(1)$ for $k = 1, 2, \ldots, \frac{n}{4}$.

Case 2. For $n \equiv 1 \pmod 4$.

Here we label v_i'' as same process as the labelling of v_i (for $n = 4k+1$) of the cycle C_n.

So, $v_f(0) = \frac{12k+2}{2} = 6k+1 = v_f(1)$ and $e_f(0) = \frac{12k+2}{2} = 6k+1 = e_f(1)$ for $k = 1, 2, \ldots, \frac{n-1}{4}$.

Case 3. For $n \equiv 2 \pmod 4$.

For the vertices v_i''; $i = 0, 1, \ldots, 4k-1$, f is defined as

$$f(v_i'') = \begin{cases} 0, & \text{if } i \equiv 0 \ (\text{mod } 4) \text{ and } i \equiv 1 \ (\text{mod } 4); \\ 1, & \text{if } i \equiv 2 \ (\text{mod } 4) \text{ and } i \equiv 3 \ (\text{mod } 4). \end{cases}$$

And for the remaining vertices

$$f(v_i'') = \begin{cases} 0, & \text{if } i = 4k; \\ 1, & \text{if } i = 4k+1. \end{cases}$$

So, $v_f(0) = \frac{12k+6}{2} = 6k+3 = v_f(1)$ and $e_f(0) = \frac{12k+6}{2} = 6k+3 = e_f(1)$ for $k = 1, 2, \ldots, \frac{n-2}{4}$.

Case 4. For $n \equiv 3 \pmod 4$.

We label the pendent vertices as same rule as in case 2 except for the vertices v_{4k+1}'' and v_{4k+2}''. Now we label these vertices as

$$f(v_i'') = \begin{cases} 0, & \text{if } i = 4k+1; \\ 1, & \text{if } i = 4k+2. \end{cases}$$

Here, $v_f(0) = \frac{12k+9+1}{2} = 6k+5$, $v_f(1) = \frac{12k+9-1}{2} = 6k+4$ and

$$e_f(0) = \frac{12k+9+1}{2} = 6k+5, \quad e_f(1) = \frac{12k+9-1}{2} = 6k+4.$$

Values of n	Condition on vertex	Condition on edge	Cordial
$n=4k$	$v_f(0)=v_f(1)$	$e_f(0)=e_f(1)$	Yes
$n=4k+1$	$v_f(0)=v_f(1)$	$e_f(0)=e_f(1)$	Yes
$n=4k+2$	$v_f(0)=v_f(1)$	$e_f(0)=e_f(1)$	Yes
$n=4k+3$	$v_f(0)=v_f(1)+1$	$e_f(0)=e_f(1)+1$	Yes

Table 3: Vertex and edge conditions of all cases of Lemma 6

Thus from all the cases we conclude that the graph G_1 is cordial.

To prove a cactus graph is cordial, an important result is given below.

Lemma 7. *Let a graph G contains two cycles of finite lengths and they are joined by an edge. Then G is cordial*

Corollary 2. *If two suns having finite number of vertices are joined by an edge, then the graph is cordial.*

Lemma 8. *If a graph G contains a cycle of finite length and each vertex contains another cycle of finite length, then G is cordial*

Proof. Let length of the main cycle is n. Again let us consider v_i, e_i; $i=0,1,\ldots,n-1$ be the vertices and edges of C_n.

Case 1. Let each vertex of C_n contains cycle of length 3.

Let C_3^i's for $i=0,1,\ldots,n-1$ be the cycles of length 3. And v_j^i; $j=1,2$ and $i=0,1,\ldots,n-1$; e_j^i; $j=0,1,2$ and $i=0,1,\ldots,n-1$ are the vertices and edges of C_3^i's, where $e_0^i=(v_0,v_1^i)$, $e_1^i=(v_1^i,v_2^i)$ and $e_2^i=(v_0,v_2^i)$ for $i=0,1,\ldots,n-1$. Total number of vertices and edges are $3n$ and $4n$ respectively. Now we label the cycles of length 3 by considering the following four subcases.

Case 1.1. For $n=4k$.

If $i\equiv 0 \pmod 4$.

$f(v_j^i)=0$; for $j=1,2$.

If $i\equiv 1 \pmod 4$.

$f(v_j^i)=1$; for $j=1,2$.

If $i\equiv 2 \pmod 4$ and $i\equiv 3 \pmod 4$.

$f(v_1^i)=0$, $f(v_2^i)=1$.

Here, $v_f(0)=\frac{3n}{2}=\frac{12k}{2}=6k=v_f(1)$ and $e_f(0)=\frac{4n}{2}=\frac{16k}{2}=8k=e_f(1)$.

Case 1.2. For $n=4k$.

We label the vertices of C_3^i's according to the rule given in case 1.1 except the cycle C_3^{4k}. Now we label it as follows.

$f(v_j^{4k})=0$; for $j=1,2$.

Here, $v_f(0)=\frac{3n+1}{2}=\frac{12k+3+1}{2}=6k+2$, $v_f(1)=\frac{3n-1}{2}=\frac{12k+3-1}{2}=6k+1$ and $e_f(0)=\frac{4n}{2}=\frac{16k+4}{2}=8k+2=e_f(1)$.

Case 1.3. For $n=4k$.

The label of C_3^i; $i=0,1,\ldots,4k-2$ are same as the labelling of C_3^i; $i=0,1,\ldots,4k-1$ as given in case 1.2. For the remaining cycles f is defined as

$f(v_j^{4k-1})=0$ for $j=1,2$; $f(v_1^{4k})=0$ and $f(v_2^{4k})=1$; $f(v_j^{4k+1})=0$ for for $j=1,2$.

So, $v_f(0)=\frac{3n}{2}=\frac{12k+6}{2}=6k+3=v_f(1)$ and $e_f(0)=\frac{4n}{2}=\frac{16k+8}{2}=8k+4=e_f(1)$.

Case 1.4. For $n=4k$.

We label C_3^i; $i=0,1,\ldots,4k-1$ as the same given in case 1.1 of this lemma. For the remaining C_3^i's; $i=4k,4k+1,4k+2$, f is defined as

$f(v_j^{4k})=1$ for $j=1,2$; $f(v_j^{4k+1})=0$ for $j=1,2$; $f(v_1^{4k+2})=1$, $f(v_2^{4k+2})=0$.

Here, $v_f(0)=\frac{3n+1}{2}=\frac{12k+9+1}{2}=6k+5$, $v_f(1)\frac{3n-1}{2}=\frac{12k+9-1}{2}=6k+4$ and $e_f(0)=\frac{4n}{2}=\frac{16k+12}{2}=8k+6=e_f(1)$.

Values of n,m	Condition on vertex	Condition on edge	Cordial
$n=4k$, $m=3$	$v_f(0)=v_f(1)$	$e_f(0)=e_f(1)$	Yes
$n=4k+1$, $m=3$	$v_f(0)=v_f(1)+1$	$e_f(0)=e_f(1)$	Yes
$n=4k+2$, $m=3$	$v_f(0)=v_f(1)$	$e_f(0)=e_f(1)$	Yes
$n=4k+3$, $m=3$	$v_f(0)=v_f(1)+1$	$e_f(0)=e_f(1)$	Yes

Table 4: Vertex and edge conditions of Case 1 of Lemma 8

Case 2. Let each vertex of C_n contains cycle of length $4k\equiv 0$ (mod 4).

Let the cycles C_m^i's are joined to each of the vertex of C_n. Let v_j^i; $j=1,2,\ldots,m-1$ and $i=0,1,\ldots,n-1$; e_j^i; $j=0,1,\ldots,m-1$ and $i=0,1,\ldots,n-1$ are the vertices and edges of all C_m^i's, where $e_j^i=(v_j^i,v_{j+1}^i)$ for $j=1,2,\ldots,m-2$, $e_0^i=(v_i,v_1^i)$ and $e_{m-1}^i=(v_i,v_{m-1}^i)$ for $i=0,1,\ldots,n-1$. Total number of vertices and edges are nm and $(nm+n)$ respectively. Now we label the cycles of lengths m in the following four subcases.

Case 2.1. For $n=4k$.

If $i \equiv 0$ (mod 4) and $i \equiv 1$ (mod 4).

For v_j^i; $j = 4, 5, \ldots, m-1$

$$f(v_j^i) = \begin{cases} 0, & if\ j \equiv 0\ (mod\ 4); \\ 0, & if\ j \equiv 1\ (mod\ 4); \\ 1, & if\ j \equiv 2\ (mod\ 4); \\ 1, & if\ j \equiv 3\ (mod\ 4); \end{cases}$$

and $f(v_1^i) = 0$, $f(v_2^i) = f(v_3^i) = 1$.

If $i \equiv 2$ (mod 4) and $i \equiv 3$ (mod 4).

For v_j^i; $j = 4, 5, \ldots, m-1$

$$f(v_j^i) = \begin{cases} 1, & \text{if } j \equiv 0 \text{ (mod 4)}; \\ 1, & \text{if } j \equiv 1 \text{ (mod 4)}; \\ 0, & \text{if } j \equiv 2 \text{ (mod 4)}; \\ 0, & \text{if } j \equiv 3 \text{ (mod 4)}. \end{cases}$$

and $f(v_1^i) = 1$, $f(v_2^i) = f(v_3^i) = 0$.

Here, $v_f(0) = \dfrac{16k^2}{2} = 8k^2 = v_f(1)$ and $e_f(0) = \dfrac{16k^2 + 4k}{2} = 8k^2 + 2k = e_f(1)$.

Case 2.2. For $n = 4k+1$.

In this case, we label C_m^i's; $i = 0, 1, \ldots, 4k-1$ according to the case 1.1 of this lemma. Now we label C_m^{4k} as follows.

For v_j^{4k}; $j = 4, 5, \ldots, m-1$

$$f(v_j^i) = \begin{cases} 1, & \text{if } j \equiv 0 \text{ (mod 4)}; \\ 1, & \text{if } j \equiv 1 \text{ (mod 4)}; \\ 0, & \text{if } j \equiv 2 \text{ (mod 4)}; \\ 0, & \text{if } j \equiv 3 \text{ (mod 4)}. \end{cases}$$

and $f(v_1^{4k}) = 1$, $f(v_2^{4k}) = f(v_3^{4k}) = 0$.

Here, $v_f(0) = \dfrac{16k^2 + 4k}{2} = 8k^2 + 2k = v_f(1)$ and $e_f(0) = \dfrac{16k^2 + 8k + 1 + 1}{2} = 8k^2 + 4k + 1$, $e_f(1) = \dfrac{16k^2 + 8k + 1 - 1}{2} = 8k^2 + 4k$.

Case 2.3. For $n = 4k+2$.

The labelling of C_m^i's; $i = 0, 1, \ldots, 4k-1$ according to the case 2.1 of this lemma. We label C_m^{4k} as the same process of labelling of C_m^0 given in case 2.1. And also C_m^{4k+1} as same as C_m^2 respectively.

Here, $v_f(0)=\dfrac{16k^2+8k}{2}=8k^2+4k=v_f(1)$ and $e_f(0)=\dfrac{16k^2+12k+2-2}{2}=8k^2+6k$, $e_f(1)=\dfrac{16k^2+12k+2+2}{2}=8k^2+4k+2$.

Case 2.4. For $n=4k+3$.

We label C_m^i's according to the rule given in case 2.1 of this lemma.

Here, $v_f(0)=\dfrac{16k^2+12k}{2}=8k^2+6k=v_f(1)$ and

$e_f(0)=\dfrac{16k^2+16k+3-1}{2}=8k^2+8k+1$, $e_f(1)=\dfrac{16k^2+16k+3+1}{2}=8k^2+8k+2$.

Values of n,m	Condition on vertex	Condition on edge	Cordial
$n=4k$, $m=4k$	$v_f(0)=v_f(1)$	$e_f(0)=e_f(1)$	Yes
$n=4k+1$, $m=4k$	$v_f(0)=v_f(1)$	$e_f(0)=e_f(1)+1$	Yes
$n=4k+2$, $m=4k$	$v_f(0)=v_f(1)$	$e_f(0)+2=e_f(1)$	No
$n=4k+3$, $m=4k$	$v_f(0)=v_f(1)$	$e_f(0)+1=e_f(1)$	Yes

Table 5: Vertex and edge conditions of Case 2 of Lemma 8

Case 3. Let each vertex of C_n contains cycle of length $4k+1\equiv 1$ (mod 4).

Case 3.1. For $n=4k$.

If $i\equiv 0$ (mod 4).

For v_j^i; $j=4,5,\ldots,m-2$

$$f(v_j^i)=\begin{cases}1, & \text{if } j\equiv 0 \pmod 4;\\ 0, & \text{if } j\equiv 1 \pmod 4;\\ 0, & \text{if } j\equiv 2 \pmod 4;\\ 1, & \text{if } j\equiv 3 \pmod 4;\end{cases}$$

and $f(v_j^i)=1$ for $j=1,2$, $f(v_j^i)=0$ for $j=3,m-1$.

If $i\equiv 1$ (mod 4).

For v_j^i; $j=4,5,\ldots,m-3$

$$f(v_j^i)=\begin{cases}0, & \text{if } j\equiv 0 \pmod 4;\\ 0, & \text{if } j\equiv 1 \pmod 4;\\ 1, & \text{if } j\equiv 2 \pmod 4;\\ 1, & \text{if } j\equiv 3 \pmod 4;\end{cases}$$

and $f(v_j^i)=0$ for $j=1,m-2$, $f(v_j^i)=1$ for $j=2,3,m-1$.

If $i\equiv 2$ (mod 4).

For v_j^i; $j = 4,5,\ldots,m-2$

$$f(v_j^i) = \begin{cases} 1, & \text{if } j \equiv 0 \pmod 4; \\ 1, & \text{if } j \equiv 1 \pmod 4; \\ 0, & \text{if } j \equiv 2 \pmod 4; \\ 0, & \text{if } j \equiv 3 \pmod 4; \end{cases}$$

and $f(v_{m-1}^i) = 0$.

If $i \equiv 3 \pmod 4$.

The labelling of the vertices v_j^i; $j = 4,5,\ldots,m-4$ as same as the labelling of v_j^i; $j = 4,5,\ldots,m-2$ as in above (for $i \equiv 2 \pmod 4$) and then we label the remaining vertices as

$f(v_j^i) = 1$ for $j = 1, m-2, m-3$ and $f(v_j^i) = 0$ for $j = 2,3,m-1$.

Here, $v_f(0) = \dfrac{16k^2 + 4k}{2} = 8k^2 + 2k = v_f(1)$ and $e_f(0) = \dfrac{16k^2 + 8k}{2} = 8k^2 + 4k = e_f(1)$.

Case 3.2. For $n = 4k+1$.

In this case, we label the cycles C_m^i; $i = 0,1,\ldots,4k-1$ according to the procedure given in case 3.1. And we label C_m^{4k} as same as the labelling of C_m^{4k+2} given in case 3.1.

So, $v_f(0) = \dfrac{16k^2 + 8k + 1 + 1}{2} = 8k^2 + 4k + 1$, $v_f(1) = \dfrac{16k^2 + 8k + 1 - 1}{2} = 8k^2 + 4k$ and $e_f(0) = \dfrac{16k^2 + 12k + 2 + 2}{2} = 8k^2 + 6k + 2$, $e_f(1) = \dfrac{16k^2 + 12k + 2 - 2}{2} = 8k^2 + 6k$.

Case 3.3. For $n = 4k+2$.

In this case, we label the cycles C_m^i; $i = 0,1,\ldots,4k-1$ according to the procedure given in case 3.1. And we label C_m^{4k} as same as the labelling of C_m^{4k+3} given in case 3.1. And also C_m^{4k+1} as same as the labelling of C_m^{4k+2}.

So, $v_f(0) = \dfrac{16k^2 + 8k + 2}{2} = 8k^2 + 4k + 1 = v_f(1)$ and

$e_f(0) = \dfrac{16k^2 + 16k + 2 + 4}{2} = 8k^2 + 8k + 2 = e_f(1)$.

Case 3.4. For $n = 4k+3$.

The labelling of C_m^i; $i = 0,1,\ldots,4k$ are same as given in case 3.1. C_m^{4k+1} same as labelling of C_m^{4k+3} given in case 3.1 and also C_m^{4k+2} as same as the labelling of C_m^{4k+2}.

So, $v_f(0) = \dfrac{16k^2 + 16k + 3 + 1}{2} = 8k^2 + 8k + 2$, $v_f(1) = \dfrac{16k^2 + 16k + 3 - 1}{2} = 8k^2 + 8k + 1$ and $e_f(0) = \dfrac{16k^2 + 20k + 6}{2} = 8k^2 + 10k + 3 = e_f(1)$.

Values of n, m	Condition on vertex	Condition on edge	Cordial
$n = 4k$, $m = 4k+1$	$v_f(0) = v_f(1)$	$e_f(0) = e_f(1)$	Yes
$n = 4k+1$, $m = 4k+1$	$v_f(0) = v_f(1)+1$	$e_f(0) = e_f(1)+2$	Not
$n = 4k+2$, $m = 4k+1$	$v_f(0) = v_f(1)$	$e_f(0) = e_f(1)$	Yes
$n = 4k+3$, $m = 4k+1$	$v_f(0) = v_f(1)+1$	$e_f(0) = e_f(1)$	Yes

Table 6: Vertex and edge conditions of Case 3 of Lemma 8

Case 4. Let each vertex of C_n contains cycle of length $4k+2 \equiv 2$ (mod 4).

Case 4.1. For $n = 4k$.

If $i \equiv 0$ (mod 4).

The labelling procedure of v_j^i; $j = 4,5,\ldots,m-2$ are same as labelling of v_j^i; $j = 4,5,\ldots,m-2$ given in case 3.1 (for $i \equiv 0$ (mod 4)). Now we label remaining vertices as

$f(v_j^i) = 0$ for $j = 1,2$, $f(v_j^i) = 1$ for $j = 3, m-1$.

If $i \equiv 1$ (mod 4).

Here also the labelling procedure of v_j^i; $j = 4,5,\ldots,m-3$ are same as labelling of v_j^i; $j = 4,5,\ldots,m-2$ given in case 3.1 (for $i \equiv 0$ (mod 4)). Then we label other vertices as $f(v_j^i) = 0$ for $j = 1,2,m-2$, $f(v_{m-1}^i) = 1$.

If $i \equiv 2$ (mod 4).

Here we label the vertices v_j^i; $j = 4,5,\ldots,m-3$ as

$$f(v_j^i) = \begin{cases} 0, & \text{if } j \equiv 0 \text{ (mod 4)};\\ 1, & \text{if } j \equiv 1 \text{ (mod 4)};\\ 1, & \text{if } j \equiv 2 \text{ (mod 4)};\\ 0, & \text{if } j \equiv 3 \text{ (mod 4)}; \end{cases}$$

and then $f(v_j^i) = 1$ for $j = 1,2$, $f(v_j^i) = 0$ for $j = 3, m-2, m-1$.

If $i \equiv 3$ (mod 4).

The labelling rule of the vertices v_j^i; $j = 4,5,\ldots,m-3$ are same as the labelling of C_m^i (for $i \equiv 2$ (mod 4)). We label v_{m-2}^i, v_{m-1}^i as $f(v_{m-2}^i) = 1$, $f(v_{m-1}^i) = 0$.

Here, $v_f(0) = \dfrac{16k^2+8k}{2} = 8k^2+4k = v_f(1)$ and $e_f(0) = \dfrac{16k^2+12k}{2} = 8k^2+6k = e_f(1)$.

Case 4.2. For $n = 4k+1$.

First we label C_m^i's; $i = 0,1,\ldots,4k-1$ according to the same rule as given in case 4.1 of this lemma. Now we label v_j^{4k}; $j = 4,5,\ldots,m-2$ of C_m^{4k} as the same process of labelling of C_j^{4k+2}; $j = 4,5,\ldots,m-4$ as given in case 3.1 (for $i \equiv 2$ (mod 4)) of this lemma. Then we label the remaining vertices as

$f(v_1^{4k})=1$, $f(v_j^{4k})=0$ for $j=2,3,m-1$.

Here, $v_f(0)=\dfrac{16k^2+12k+2}{2}=8k^2+6k+1=v_f(1)$ and

$$e_f(0)=\frac{16k^2+16k+3-1}{2}=8k^2+8k+1,\ e_f(1)=\frac{16k^2+16k+3+1}{2}=8k^2+8k+2.$$

Case 4.3. For $n=4k+2$.

We label C_m^i's; $i=0,1,\ldots,4k-1$ as the same rule as given in case 4.1 of this lemma. Now we label C_m^{4k} as the same process of labelling of C_m^{4k+3} as given in case 4.1 (for $i\equiv 3 \pmod 4$)). And C_m^{4k+1} according to the rule of labelling of C_m^{4k+2} as given in case 4.1 (for $i\equiv 2 \pmod 4$)).

So, $v_f(0)=\dfrac{16k^2+16k}{2}=8k^2+8k=v_f(1)$ and

$$e_f(0)=\frac{16k^2+20k+6-2}{2}=8k^2+10k+2,\ e_f(1)=\frac{16k^2+20k+6+2}{2}=8k^2+10k+4.$$

Case 4.4. For $n=4k+3$.

The labelling of C_m^i's; $i=0,1,\ldots,4k$ are same as given in case 4.1 of this lemma. Now we label C_m^{4k+1} as the same process of labelling of C_m^{4k+3} as given in case 4.1 (for $i\equiv 3 \pmod 4$)). And C_m^{4k+2} according to the rule of labelling of C_m^{4k+2} as given in case 4.1 (for $i\equiv 2 \pmod 4$)).

So, $v_f(0)=\dfrac{16k^2+20k+6}{2}=8k^2+10k+3=v_f(1)$ and

$$e_f(0)=\frac{16k^2+24k+9+1}{2}=8k^2+12k+5,\ e_f(1)=\frac{16k^2+24k+9-1}{2}=8k^2+12k+4.$$

Values of n,m	Condition on vertex	Condition on edge	Cordial
$n=4k$, $m=4k+2$	$v_f(0)=v_f(1)$	$e_f(0)=e_f(1)$	Yes
$n=4k+1$, $m=4k+2$	$v_f(0)=v_f(1)$	$e_f(0)+1=e_f(1)$	Yes
$n=4k+2$, $m=4k+2$	$v_f(0)=v_f(1)$	$e_f(0)+2=e_f(1)$	Not
$n=4k+3$, $m=4k+2$	$v_f(0)=v_f(1)$	$e_f(0)=e_f(1)+1$	Yes

Table 7: Vertex and edge conditions of Case 4 of Lemma 8

Case 5. Let each vertex of C_n contains cycle of length $4k+3\equiv 3 \pmod 4$.

Case 5.1. For $n=4k$.

If $i\equiv 0 \pmod 4$ and $i\equiv 1 \pmod 4$.

We label the vertices of C_m^i according to the rule of labelling of C_n (for $n=4k+2$) as given in case 4 of Lemma 1.

If $i\equiv 2 \pmod 4$ and $i\equiv 3 \pmod 4$.

Here we label the vertices v_j^i; $j = 4,5,\ldots,m-2$ as

$$f(v_j^i) = \begin{cases} 1, & \text{if } j \equiv 0 \pmod 4; \\ 1, & \text{if } j \equiv 1 \pmod 4; \\ 0, & \text{if } j \equiv 2 \pmod 4; \\ 0, & \text{if } j \equiv 3 \pmod 4; \end{cases}$$

and then $f(v_j^i) = 1$ for $j = 1, m-1$, $f(v_j^i) = 0$ for $j = 2,3$.

Here, $v_f(0) = \dfrac{16k^2 + 12k}{2} = 8k^2 + 6k = v_f(1)$ and

$$e_f(0) = \frac{16k^2 + 16k}{2} = 8k^2 + 8k = e_f(1).$$

Case 5.2. For $n = 4k+1$.

The labelling of C_m^i; $i = 0,1,\ldots,4k-1$ are same as given in the above case. Now we label the vertices of C_m^{4k} as follows.

For v_j^{4k}; $j = 4,5,\ldots,m-2$

$$f(v_j^i) = \begin{cases} 1, & \text{if } j \equiv 0 \pmod 4; \\ 1, & \text{if } j \equiv 1 \pmod 4; \\ 0, & \text{if } j \equiv 2 \pmod 4; \\ 0, & \text{if } j \equiv 3 \pmod 4; \end{cases}$$

and then $f(v_1^{4k}) = 1$, $f(v_j^{4k}) = 0$ for $j = 2,3,m-1$ and $f(v_{m-1}^{4k}) = 1$.

Here, $v_f(0) = \dfrac{16k^2 + 16k + 3 - 1}{2} = 8k^2 + 8k + 1$, $v_f(1) = \dfrac{16k^2 + 16k + 3 + 1}{2} = 8k^2 + 8k + 2$ and $e_f(0) = \dfrac{16k^2 + 20k + 4}{2} = 8k^2 + 10k + 2 = e_f(1)$.

Case 5.3. For $n = 4k+2$.

The labelling of C_m^i's; for $i = 0,1,\ldots,4k-1$ are same as given in case 5.1 of this lemma. Now we label C_m^{4k} as the same process of labelling of C_m^{4k+3} (for $i \equiv 3 \pmod 4$). And C_m^{4k+1} according to the rule of labelling of C_m^{4k+2} (for $i \equiv 2 \pmod 4$).

So, $v_f(0) = \dfrac{16k^2 + 20k + 6}{2} = 8k^2 + 10k + 3 = v_f(1)$ and

$$e_f(0) = \frac{16k^2 + 24k + 8}{2} = 8k^2 + 12k + 4 = e_f(1).$$

Case 5.4. For $n = 4k+3$.

First we label C_m^i's; $i = 0,1,\ldots,n-3 = 4k$ according to the rule given in case 5.1 of this lemma. Then C_m^{4k+1} as the same process of labelling of C_m^{4k+3} (for $i \equiv 3 \pmod 4$). And C_m^{4k+2} according to the rule of labelling of C_m^{4k+2} (for $i \equiv 2 \pmod 4$).

Here, $v_f(0)=\dfrac{16k^2+24k+9+1}{2}=8k^2+10k+5$,

$v_f(1)=\dfrac{16k^2+24k+9-1}{2}=8k^2+10k+4$ and $e_f(0)=\dfrac{16k^2+28k+12}{2}=8k^2+14k+6=e_f(1)$.

Values of n,m	Condition on vertex	Condition on edge	Cordial
$n=4k$, $m=4k+3$	$v_f(0)=v_f(1)$	$e_f(0)=e_f(1)$	Yes
$n=4k+1$, $m=4k+3$	$v_f(0)+1=v_f(1)$	$e_f(0)=e_f(1)$	Yes
$n=4k+2$, $m=4k+3$	$v_f(0)=v_f(1)$	$e_f(0)=e_f(1)$	Yes
$n=4k+3$, $m=4k+3$	$v_f(0)=v_f(1)+1$	$e_f(0)=e_f(1)$	Yes

Table 8: Vertex and edge conditions of Case 5 of Lemma 8

There are only three cases in which the graph is not cordial. The cases are case 2.3, 3.2 and 4.3. Except these cases G is cordial.

One of the important subgraph of cactus graph is caterpillar graph, which is defined below.

Definition 4. *A caterpillar C is a tree where all vertices of degree ≥ 3 lie on a path, called the backbone of C. The hairlength of a caterpillar graph C is the maximum distance of a non-backbone vertex to the backbone.*

Lemma 9. *A caterpillar graph is cordial.*

A k-angular cactus is a connected graph all of whose blocks are cycles with k vertices. In [3], Cahit proved the following results.

Lemma 10. *A k-angular cactus with t cycles is cordial if and only if $kt \not\equiv 2$ (mod 4).*

Kirchherr [13] has improved the above result. He proved the result which is given below.

Lemma 11. *Let n is the size of a cactus whose blocks are cycles is cordial if and only if $n \not\equiv 2$ (mod 4).*

The cordial labelling of all possible subgraphs of cactus graph and their combination are discussed in the previous lemmas. Hence we have the following theorem.

Theorem 1. *A cactus graph is cordial with some restrictions those are discussed in above lemmas.*

4. Conclusion

The cordial labelling of all possible subgraphs of cactus graph like unicycle, one point union of finite number of cycles (that is, finite number of cycles of finite lengths are joined with a common cutvertex), sun have done here. And we have found that cactus graph is a cordial graph with some restrictions. As cactus graph is a treelike graph so, further we will do cordial labelling on other treelike graphs.

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