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Fractional Out-domination Numbers of Kautz and Generalised Kautz Digraphs

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Abstracts

Let $D=(V,A)$ be any digraph. An out-dominating function(ODF) of a digraph $D=(V,A)$ is a function $f:V\to[0,1]$ such that

$$\sum_{u\in N^+[v]} f(u)\ge 1 \text{ for all } v\in V\text{, where } N^+[v] \text{ consists of } v \text{ with all vertices adjacent from it.}$$

For a real-valued function $f:V\to \mathbb{R}$, the weight of f is $|f|=\sum_{v\in V} f(v)$. The fractional out-domination number of a digraph D, denoted $\gamma_{fo}(D)$, equals the minimum weright of an ODF of D. In this paper, we establish bounds on the fractional out-domination number for the generalised Kautz digraph and we obtain a condition for the fractional out-domination number attaining its lower bound. We also obtain the exact value of the fractional out-domination number for Kautz digraph.

Keywords: Digraphs, Out-dominating function, Minimal out-dominating function, fractional out-domination number $\gamma_{fo}(D)$, Kautz digraph, Generalised Kautz digraph.

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1. Introduction

We use Harary**[5]** for notation and terminology which are not defined here. The concept of dominating function and fractional domination number have been introduced in **[8]**. A dominating function (DF) of a Graph $G=(V,E)$ is a function $f:V\to[0,1]$ such that $\sum_{u\in N[v]} f(u)\ge 1$ for all $v\in V$,where $N[v]=\{u\in V/u$ is adjacent with $v\}\cup\{v\}$. A DF f is called a minimal dominating function(MDF) if there is no DF g of G such that $g(v)\le f(v)$ for all $v\in V$ and $g(v_0)\ne f(v_0)$ for some $v_0\in V$. For a real-valued function $f:V\to\mathbb{R}$, the weight of f is $|f|=\sum_{v\in V} f(v)$ and $S\subseteq V$, $f(S)=\sum_{v\in S} f(v)$ and so $|f|=f(V)$. For any DF f, the fractional domination number $\gamma_f(G)$ is defined by $\gamma_f(G)=\min\{|f|: f$ is a MDF of $G\}$.

Although domination and other related concepts have been extensively studied for undirected graphs, the respective analogue on digraphs have not received much attention. Of course, a survey of results on domination in directed graphs by Ghoshal, Lasker and Pillone is found in chapter 15 of Haynes et al.**[6]**, but most of the results in this survey chapter deals with the concepts of kernels and solutions in digraphs and on dominations in tournaments. For a survey of dominating functions, we also refer the research reviews by Haynes et al**[7]**. As an initiation of the present research work, we already transfered the concept of dominating function(DF) and fractional domination number $\gamma_f(G)$ to digraphs, called out-dominating function and fractional out-domination number $\gamma_{fo}(D)$ in **[13]**. In continuation, we study fractional out-domination number of generalised Kautz and Kautz digraphs.

2. Out-dominating function

In this paper, we deal with digraphs which possibly admit self-loops but no multiple arcs. Let D be a digraph with vertex set V and arc set A. Either v is adjacent from u or u is adjacent to v, if (u,v) is an arc of D. The out-degree od(v) of a vertex v is the number of vertices that are adjacent from it and the in-degree id(v) is the number of vertices adjacent to it. Let $N^+(v)$ denote the set of all vertices of D which are adjacent from v. Let $N^+[v] = N^+(v) \cup \{v\}$.

Definition 2.1. [13] *An out-dominating function(ODF) of a digraph* $D = (V, A)$ *is a function* $f : V \to [0,1]$ *such that* $\sum_{u \in N^+[v]} f(u) \geq 1$ *for all* $v \in V$.

Definition 2.2. [13] *An ODF* f *is called a minimal ODF if there is no ODF* g *of* D *such that* $g(v) \leq f(v)$ *for all* $v \in V$ *and* $g(v_0) \neq f(v_0)$ *for some* $v_0 \in V$.

Definition 2.3. [13] *The fractional out-domination number* $\gamma_{fo}(D)$ *is defined as* $\gamma_{fo}(D)$ *=min* $\{|f| : f$ *is a minimal out-dominating function of* $D\}$.

The Kautz digraph has been studied as interconnection networks because of various good properties**[1]**. Generalised Kautz digraph was introduced by Imase and Itoh **[9]**, **[10]**. It is well-known that this digraph as interconnection network topologies have good properties(see, for example,**[1]**,**[4]**, **[15]**,**[16]**,**[17]**). The generalization removes the restriction on the cardinality of vertex set and make the digraphs more genaral and valuable as network model. Thus this digraph has been widely studied as topologies for interconnection networks.

In recent years, some authors begin to study domination properties of the generalised Kautz digraph. Kikuchi and Shibata**[11]** investigated the domination number of this digraph. Tian and Xu **[14]** further consider their the distance domination number.

Definition 2.4. *For positive integers* $d \geq 2$ *and* $n \geq 1$, *the Kautz digraph* $K(d,n)$ *has* $d^n + d^{n-1}$ *vertics represented by* $x_1 x_2 \ldots x_n$ *such that* $0 \leq x_i \leq d$ *and* $x_i \neq x_{i+1}$ *for* $i = 1,2,\ldots,n-1$. *A vertex* $x = x_1 x_2 \ldots x_n$ *is adjacent to* d *vertices* $x_2 x_3 \ldots x_n x_{n+1}$ *for* $0 \leq x_{n+1} \leq d$ *and* $x_{n+1} \neq x_n$.

For every vertex x in $K(d,n)$, we have $od(x) = id(x) = d$. Hence $K(d,n)$ is d-regular.

Example 2.5. *The digraph* $K(2,1)$ *is given in fig 2.1.*

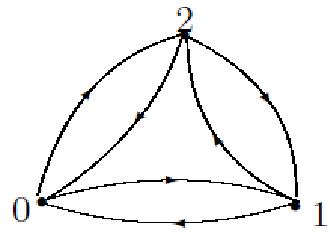

Fig 2.1

Definition 2.6. *The genaralized Kautz digraph* $G_K(n,d)$ *is given by*

$$\begin{cases} V(G_K(n,d)) = \{0,1,2,\ldots,n-1\} \\ A(G_K(n,d)) = \{(x,y)/y \equiv -dx - i(mod\ n), 0 < i \leq d\} \end{cases},$$

where n and d are positive integers such that $d \geq 2$ and $n \geq d$.

Example 2.7. *The digraph* $G_K(9,2)$ *is given in fig 2.2.*

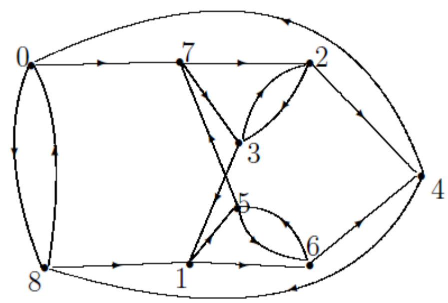

Fig 2.2

That is,
$V(G_K(9,2)) = \{0,1,2,3,4,5,6,7,8\}, A(G_K(9,2)) = \{(0,8),(0,7),(1,6),(1,5),(2,4),(2,3),(3,2),$
$(3,1),\ (4,0),(4,8),(5,7),(5,6),(6,5),(6,4),(7,3),(7,2),(8,1),(8,0)\}$.

It seems to be difficult to determine the fractional out-domination number for general generalised Kautz digraph. So, we begin by establishing bounds on the fractional out-dimination number in $G_K(n,d)$.

Theorem 2.8. $\dfrac{n}{d+1} \leq \gamma_{fo}(G_K(n,d)) \leq \dfrac{n}{d}$.

Proof: The vertex set V of $G_K(n,d) = \{0,1,2,\ldots,n-1\}$.

Then, by definition, $N^+(j) = \{-jd-1, -jd-2, \ldots, -d(j+1)\}(\text{modn}), j = 0,1,2,\ldots,n-1$. If a vertex j has a loop, then $|N^+[j]| = d$, otherwise $|N^+[j]| = d+1$.

An example in $G_K(6,3)$

j	$N^+(j)$
0	5, 4, 3
1	2, 0,

2	5, 4, 3
3	2, 1, 0
4	5, 3,
5	2, 1, 0.

Define $f: V \to [0,1]$ by $f(v) = \frac{1}{d}$ for all $v \in V$. Let $v \in V$

Case 1: v has a loop.

$|N^+[v]| = d$. Therefore, $f(N^+[v]) = d\frac{1}{d} = 1$.

Case 2: v does not have a loop.

$|N^+[v]| = d+1, f(N^+[v]) = \frac{d+1}{d} > 1$.

Hence, $f(N^+[v]) \geq 1$ for all $v \in V$. So, f is an out-dominating function of $G_K(n,d)$. Therefore, $\gamma_{fo}(G_K(n,d)) \leq |f| = \frac{n}{d}$.

To prove $\gamma_{fo}(G_K(n,d)) \geq \frac{n}{d+1}$, let f be any out-dominating function of $G_K(n,d)$. Since f is an out-dominating function of $G_K(n,d), f(N^+[j]) \geq 1, j = 0,1,2,\ldots,n-1$. Adding these n inequalities, we get

$$\sum_{j=1}^{n-1} f(N^+[j]) \geq n. \text{ That is, } \sum_{j=1}^{n-1} f(N^+(j)) + \sum_{j=1}^{n-1} f(j) \geq n.$$

$$\textit{Therefore, } \sum_{j=1}^{n-1} f(N^+(j)) + |f| \geq n. \tag{1}$$

If a vertex j has a loop, then $f(N^+(j))$ has $d-1$ terms and does not have the term $f(j)$. For each vertex j, having a loop, we add $f(j)$ to $\sum_{j=1}^{n-1} f(N^+(j))$ of (1). Since $f(j) \geq 0$, the inequality will not be changed. Now, we can see that each $f(j), j = 0, 1, 2, \ldots, n-1$ appears exactly d times in $\sum_{j=1}^{n-1} f(N^+(j))$ of (1). Therefore, $|f| + d(f(0) + f(1) + \ldots + f(n-1)) \geq n$. That is, $(d+1)|f| \geq n$. Hence, $|f| \geq \frac{n}{d+1}$. Taking minimum over f on both sides of this inequality, we get $\gamma_{fo}(G_K(n,d)) \geq \frac{n}{d+1}$.

Note 2.9. *Theorem 2.11 shows that the lower bound* $\gamma_{fo}(G_K(n,d)) \geq \frac{n}{d+1}$ *is sharp.*

Lemma 2.10. [16] $G_K(n,d)$ *has no self-loop if and only if* $(d+1) | n$.

Theorem 2.11. *If* $(d+1) | n, \gamma_{fo}(G_K(n,d)) = \frac{n}{d+1}$.

Proof: The vertex set V of $G_K(n,d) = \{0,1,2,\ldots,n-1\}$.

Then, by definition, $N^+(j) = \{-jd-1, -jd-2, \ldots, -d(j+1)\}(\text{modn}), j = 0,1,2,\ldots,n-1$.

Now, $N^+[j]=\{j\}\cup N^+(j), j=0,1,2,\ldots,n-1$. Since $(d+1)\mid n, G_K(n,d)$ has no seff-loop by Lemma 2.10. So, j is different from all the elements of $N^+(j), j=0,1,2,\ldots,n-1$. Therefore, $N^+[j]$ has $d+1$ elements, $j=0,1,2\ldots,n-1$. Since the vertices in $G_K(n,d)$ are consecutive integers, the vertices of $G_K(n,d)$ appear in turn in $\{N^+(0),N^+(1),\ldots,N^+(n-1)\}$. Also each vertex of $G_K(n,d)$ appears exactly $d+1$ times in $\{N^+[0],N^+[1],\ldots,N^+[n-1]\}$

An example in $G_K(12,3)$

j	$N^+(j)$	j	$N^+(j)$
0	11, 10, 9	6	5, 4, 3
1	8, 7, 6	7	2, 1, 0
2	5, 4, 3	8	11, 10, 9
3	2, 1, 0,	9	8, 7, 6
4	11, 10, 9,	10	5, 4, 3
5	8, 7, 6	11	2, 1, 0

Define $f:V\to[0,1]$ by $f(v)=\dfrac{1}{d+1}$ for all $v\in V$. Let $v\in V$.

Since $|N^+[j]|=d+1, f(N^+[v])=(d+1)\dfrac{1}{d+1}=1$. That is , $f(N^+[v])$=1 for all $v\in V$. So, f is a total out-dominating function of $G_K(n,d)$. Therefore, $\gamma_{fo}(G_K(n,d))\leq|f|=\dfrac{n}{d+1}$.

To prove $\gamma_{fo}(G_K(n,d))\geq\dfrac{n}{d+1}$, let f be any out-dominating function of $G_K(n,d)$. Since f is an out-dominating function of $G_K(n,d), f(N^+[j])\geq 1, j=0,1,2,\ldots,n-1$. Adding these n inequalities, we get $\sum_{j=1}^{n-1} f(N^+[j])\geq n$. We can see that each $f(j), j=0,1,2,\ldots,n-1$ appears exactly $d+1$ times in the sum. Therefore, $(d+1)(f(0)+f(1)+\ldots+f(n-1))\geq n$. That is, $(d+1)|f|\geq n$. Hence, $|f|\geq\dfrac{n}{d+1}$. Taking minimum over f on both sides of this inequality, we get $\gamma_{fo}(G_K(n,d))\geq\dfrac{n}{d+1}$.

We recall from **[12]** that if $n=d^{m-1}(d+1)$, then $G_K(n,d)$ is the Kautz digraph $K(d,m)$.

Corollary 2.12. $\gamma_{fo}(K(d,m))=d^{m-1}$.

Proof: We have $G_K(d^{m-1}(d+1),d)=K(d,m)$. Since $(d+1)\mid d^{m-1}(d+1)$, by Theorem 2.11, $\gamma_{fo}(G_K(d^{m-1}(d+1),d))=\dfrac{d^{m-1}(d+1)}{d+1}$. That is, $\gamma_{fo}(K(d,m))=d^{m-1}$.

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