In this paper we state a definition of $v_k$-Cyclic path covering to find the Cyclic path covering number of a digraph in which some of the vertex have privilege of being interior vertex of more than one path and we find $v_k$-Cyclic path Covering Number of simple digraph and we defined the $\{v_{k_1}, v_{k_2}, v_{k_3} \ldots v_{k_m}\}$- Cyclic path covering and have found the $\{v_{k_1}, v_{k_2}, v_{k_3} \ldots v_{k_m}\}$- Cyclic path covering number of such coverings. Also we defined $e_m$-Cyclic path Covering Number and finally found the $\{e_m, v_k\}$-Cyclic path covering number of any digraph.

**Keywords:** Cyclic path cover, Cyclic path covering number, $v_k$-Cyclic path cover, $e_m$-Cyclic path Covering Number.

**2010 Mathematics Subject Classification:** 05C38, 05C45, 05C70.

1. Introduction

If we consider the context of mobile traffic streams in a city road network, every vehicle driver would wish to encounter as many green signals at junctions as possible so as to minimize fuel consumption and travel time as also maximize unhindered distance so travelled. This necessitates relative optimization of signal times at various junctions in the road network in order to provide maximum possible average length mobile traffic streams-one may expect such an optimal traffic flow in the network to occur with maximum probability when the number mobile traffic streams in the network is minimum. Thus for any finite digraph $G$ the parameter $\psi(G) = \min \{|\psi| : \psi \in G_G\}$, where $G_G$ denotes the set of all distinct Cyclic path covers of $G$, is an ideal limit for any reasonably realistic measure of “mobility” of traffic flow in the city road network with $G$ as the underlying digraph. In our recent development of city in many places there are so many flyovers and over bridges at the road crossings to minimise the traffic signals. This motivates the development of $v_k$-Cyclic path cover which is being discussed elaborately in this paper. The concept of path decomposition and path covering number of a digraph was introduced by Harary[1]. The preliminary results on this paper were obtained by Harary and Schwenk[2], Perecohe[3] and Stanton etal.[4], [5]. Here the paths may intersect any number times only at the specified vertex of $G$. To get non-intersecting paths we impose some extra conditions in the definition of Path Covering. We follow the notations and terminology of Harary [6],[7]. All digraph considered in this paper are assumed to be connected digraphs without isolated points. Let $D = (V,E)$ be a digraph. We denote the number of vertices if $D$ by $n$ and the number of edges in $D$ by $e$.

**Definition 1.1.** A Path covering of a digraph $D$ is a collection $\psi$ of (not necessarily open) paths in $D$ such that every edge of $D$ is in exactly one path in $\psi$ and the minimum cardinality taken over all Path covers $\psi$ of $D$ with $|\psi| = \eta$ is called the Path covering number of $D$.

**Definition 1.2.** [8],[9]
A Cyclic Path covering of a digraph D is a collection $\Gamma$ of (not necessarily open) paths in D whose union is D satisfying the conditions for distinct paths $P_i$ and $P_j$ with terminal vertices $u$, $v$ and $w$, $z$ respectively,

$$P_i \hat{\cup} P_j = \begin{cases} \{A, A \text{ is the subset of the set } \{u, v, w, z\} \} & \text{if } P_i \text{ and } P_j \text{ are cyclic.} \\
\emptyset, & \text{if } \phi \end{cases}$$

**Definition 1.3. [8],[9],[10],[11]**
The Cyclic Path covering number of D is defined to be the minimum cardinality taken over all Cyclic Path covers of D. Any Cyclic Path cover $\Gamma$ of G with $|\Gamma| = \gamma$ is called a minimum Cyclic Path cover of D.

2. $v_k$ - Cyclic Path Cover

**Definition 2.1. [11]**
A $v_k$-Cyclic Path covering of a digraph D is a collection $\Gamma_k$ of paths (not necessarily open) in D whose union is D satisfying the conditions for distinct paths $P_i$ and $P_j$ with terminal vertices $u$, $v$ and $w$, $z$ respectively,

$$P_i \hat{\cup} P_j = \begin{cases} \{A, A \text{ is the subset of the set } \{u, v, w, z\} \} & \text{if } P_i \text{ and } P_j \text{ are cyclic.} \\
\emptyset, & \text{if } \phi \end{cases}$$

**Definition 2.2 [11]**
The $v_k$-Cyclic path covering number $\gamma_k$ of D is defined to be the minimum cardinality taken over all $v_k$-Cyclic path covers of D.

**Example 2.3.**
Consider the following digraph (Figure.1). The $v_k$-Cyclic path covers are

![Figure 1](image)

Here the path covers are

- $\psi_1 = \{v_1v_2v_4v_5, v_3v_2v_3\}$,
- $\psi_2 = \{v_1v_2v_3, v_2v_4v_5\}$,
- $\psi_3 = \{v_1v_2v_4, v_4v_5, v_3v_2v_3\}$,
- $\psi_4 = \{v_1v_2, v_2v_4, v_4v_5, v_3v_2, v_2v_3\}$

Then the path covering number = 2.
The $v_2$-Cyclic Path cover are $\{v_1v_2v_3,v_2v_4v_5v_2\}$, $\{v_1v_2v_4v_5v_2v_3\}$. Therefore the $v_2$- cyclic path covering number is $\gamma_2 = 1$.

**Example 2.4.** Consider the digraph $D$ given in the following figure.

![Figure 2](image)

Here the path covers are $\psi_1 = \{aebcda, ced\}$,
$\psi_2 = \{aeda, cebc, cd\}$,
$\psi_3 = \{aebcda, ced\}$
$\psi_4 = \{ae, eb, bc, ce, cd, ed, da\}$

Then the path covering number is $2$.

The Cyclic path covers are $\{ae, eb, bc, ce, cd, ed, da\}$, $\{ebcda, ce, cd\}$, $\{ebe, cdae, cd\}$, $\{edae, ebc, ce, cd\}$, $\{edae, ebd, ce\}$, $\{dae, ebc, ced, cd\}$. Here the Cyclic path covering number $\gamma = 3$.

The $v_3$-cyclic path cover is $\{aebcda, ced\}$. Therefore the $v_3$- cyclic path covering number is $\gamma_3 = 2$.

**Lemma 2.5.** In a digraph $D$ for a vertex $v_k$ with degree 1 or 2 the $v_k$ Cyclic path covering number is same as that of Cyclic path cover.

**Proof:** As there is almost one and only one path is possible through the vertex $v_k$, there is no change in the number of directed paths to cover the digraph $D$. With reference to the definitions 1.2 and 1.3 we have the $v_k$-Cyclic path covering number is same as that of Cyclic path covering number.

**Lemma 2.6.** In a digraph $D$ for a vertex of degree 3 the $v_k$-path covering number is $\gamma$.

**Proof:** Let $v_k$ be a vertex of degree 3 on a digraph $D$. There is only one path through $v_k$. Further there is no chance for more than one path through $v_k$. Therefore $\gamma_k = \gamma$.

**Lemma 2.7.** In a simple digraph $D$ with a vertex $v_k$ with $d'(v_k) = d'(v_k) = 2$ the $v_k$-Cyclic path cover number is $\gamma_k = \gamma - 1$. 

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**Proof:** Let $v_k$ be a vertex with total degree 4. Then there must be 4 adjacent vertices say $v_{k1}, v_{k2}, v_{k3}, v_{k4}$ and there are 5 different possible degree configuration at $v_k$. They are (i) $d^-(v_k) = 4$ and $d^+(v_k) = 0$ (ii) $d^-(v_k) = 3$ and $d^+(v_k) = 1$, (iii) $d^-(v_k) = 2$ and $d^+(v_k) = 2$, (iv) $d^-(v_k) = 1$ and $d^+(v_k) = 3$, and (v) $d^-(v_k) = 0$ and $d^+(v_k) = 4$. As more than one directed path is allowed through $v_k$, the $v_k$-Cyclic path cover $\psi_k$, allows two different directed paths through the same vertex $v_k$ as interior vertex. This is possible only in the case (iii). Actually, the Cyclic path cover $\psi$ admits only one path $P_1$ through the vertex $v_k$ as internal vertex. In the case (iii), there can be two directed paths in $\psi_k$ in which $v_k$ is the terminal vertex of two directed paths. These two paths create only one path $P_2$ in which $v_k$ is internal vertex and $P_1 \neq P_2$. The remaining vertices in D follows the conditions that are same in $\psi$ an $\psi_k$. Thus, we have $\psi_k$ with one path less than the number of paths in $\psi$. Thus we have $\gamma_k = \gamma - 1$.

**Lemma 2.8.** In a simple digraph D with a vertex $v_k$ with $d^-(v_k) = 2$ and $d^+(v_k) = 3$ or $d^-(v_k) = 3$ and $d^+(v_k) = 2$, the $v_k$-Cyclic path cover number is $\gamma_k = \gamma - 1$.

**Proof:** Let $v_k$ be a vertex with degree 5. Then there will be 5 adjacent vertices say $v_{k1}, v_{k2}, v_{k3}, v_{k4}$ and $v_{k5}$ and there are 6 different possible degree configuration at $v_k$. They are (i) $d^-(v_k) = 5$ and $d^+(v_k) = 0$ (ii) $d^-(v_k) = 4$ and $d^+(v_k) = 1$, (iii) $d^-(v_k) = 3$ and $d^+(v_k) = 2$, (iv) $d^-(v_k) = 2$ and $d^+(v_k) = 3$, (v) $d^-(v_k) = 1$ and $d^+(v_k) = 4$ and (vi) $d^-(v_k) = 0$ and $d^+(v_k) = 5$. As more than one path allowed through $v_k$, the $v_k$-Cyclic path cover $\psi_k$, allows two
The cyclic path cover $\psi$ admits only one path $P_1$ through the vertex $v_k$ as internal vertex. There are three paths in $\psi$ in which $v_k$ is one of terminal vertex in their respective paths. Any two of these paths can be joined at $v_k$ and create only one path $P_2$ in which $v_k$ is internal vertex and $P_1 \neq P_2$. The remaining vertices in $D$ follows the conditions as they are in $\psi$ and $\psi_k$. Thus in $\psi_k$ we have one path less than the number of paths in $\psi$.

Therefore we have $\gamma_k = \gamma - 1$.

**Lemma 2.9.** In a simple digraph $D$ with a vertex $v_k$ with $d^-(v_k) = 3$ and $d^+(v_k) = 3$, the $v_k$-cyclic path cover number is $\gamma_k = \gamma - 2$.

**Proof:** Let $v_k$ be a vertex with degree 6. Then there must be 6 adjacent vertices say $v_{k1}$, $v_{k2}$, $v_{k3}$, $v_{k4}$, $v_{k5}$ and $v_{k6}$ and there are 7 different possible degree configuration at $v_k$. They are (i) $d^-(v_k) = 6$ and $d^+(v_k) = 0$, (ii) $d^-(v_k) = 5$ and $d^+(v_k) = 1$, (iii) $d^-(v_k) = 4$ and $d^+(v_k) = 2$, (iv) $d^-(v_k) = 3$ and $d^+(v_k) = 3$, (v) $d^-(v_k) = 2$ and $d^+(v_k) = 4$, (vi) $d^-(v_k) = 1$ and $d^+(v_k) = 5$ and (vii) $d^-(v_k) = 0$ and $d^+(v_k) = 6$. As more than one path are allowed through $v_k$, the $v_k$ cyclic path cover $\psi_k$, allows three different paths through the same vertex $v_k$ as interior vertex. The cyclic path cover $\psi$ admits only one path $P_1$ through the vertex $v_k$ as internal vertex. There are four directed paths in $\psi_k$ in which $v_k$ is one of terminal vertex in each directed paths. These four directed paths combined at $v_k$ to create two different directed paths $P_2$ and $P_3$ different from $P_1$ in which $v_k$ is internal vertex. The remaining vertices in $D$ follows the conditions as they are in $\psi$ and $\psi_k$.

Thus in $\psi_k$ we have two paths less than the number of paths in $\psi$. (ie) $\gamma_k = \gamma - 2$. 


Theorem 2.10. Let $v_k$ be the vertex in $D$ with $d^-(v_k) = m$ and $d^+(v_k) = n$. Then the $v_k$-Cyclic path covering number $\gamma_k = \gamma - \left[ \min\{d^-(v_k),d^+(v_k)\} - 1 \right]$, where $\gamma$ is the Cyclic path covering number of $D$.

Proof: The vertex $v_k$ may have the degree $1, 2, 3, \ldots$. If $v_k$ has degree 1 or 2 then the result is obvious by using the lemma 2.5. If the degree of the vertex is 3 then we have $\gamma_k = \gamma$ (lemma 2.6). If the vertex $v_k$ is such that $d^-(v_k) = d^+(v_k) = 2$, the $v_k$-Cyclic path cover number is $\gamma_k = \gamma$ (lemma 2.7). For the vertex $v_k$ with total degree 5 such that with $d^-(v_k) = 2$ and $d^+(v_k) = 3$ or $d^-(v_k) = 3$ and $d^+(v_k) = 2$, the $v_k$-Cyclic path cover number is $\gamma_k = \gamma - 1$ (lemma 2.8). Even if the vertex $v_k$ with degree 6 such that $d^-(v_k) = 3$ and $d^+(v_k) = 3$, the $v_k$-Cyclic path cover number is $\gamma_k = \gamma - 2$ (lemma 2.9). Thus by considering all these results by induction we have the $v_k$-Cyclic path covering number $\gamma_k = \gamma - \left[ \min\{d^-(v_k),d^+(v_k)\} - 1 \right]$, where $\gamma$ is the Cyclic path covering number of $D$.

Definition 2.11. Let $v_{k1}, v_{k2}, v_{k3}, \ldots, v_{km}$ are some of the vertices of $D$. Then $\{v_{k1}, v_{k2}, v_{k3}, \ldots, v_{km}\}$-Cyclic path covering of the digraph $D$ is a collection $\Gamma_{v_{k1}, v_{k2}, v_{k3}, \ldots, v_{km}}$ of (not necessarily open) paths in $D$ satisfying the conditions for distinct paths $P_i$ and $P_j$ with terminal vertices $u, v, w, z$ respectively,

$$P_i \not\subset P_j$$

if $P_i$ and $P_j$ are cyclic.

$$V_{u1}, V_{u2}, \ldots, V_{um}$$

if $V_{ui}$ is the internal vertex of some $P_{ik}$ and $P_{jk}$.

Definition 2.12. The $\{v_{k1}, v_{k2}, v_{k3}, \ldots, v_{km}\}$-Cyclic path covering number $\hat{\gamma}_{v_{k1}, v_{k2}, v_{k3}, \ldots, v_{km}}$ of $D$ is defined to be the minimum cardinality taken over all $\{v_{k1}, v_{k2}, v_{k3}, \ldots, v_{km}\}$-Cyclic path covers of $D$.

Corollary 2.13. Let $v_{k1}, v_{k2}, v_{k3}, \ldots, v_{km}$ are some of the vertices of $D$. Then $\{v_{k1}, v_{k2}, v_{k3}, \ldots, v_{km}\}$-Cyclic path covering number $\gamma_{v_{k1}, v_{k2}, v_{k3}, \ldots, v_{km}}$ of the digraph $D$ is

$$\gamma_{v_{k1}, v_{k2}, v_{k3}, \ldots, v_{km}} = \gamma - \sum_{i=1}^{\infty} \left[ \min\{d^-(v_{ki}),d^+(v_{ki})\} - 1 \right]$$

3. $e_m$-Cyclic Path Cover

Definition 3.1. [11],[12],[13],[14]

A $e_m$-Cyclic directed path cover of a digraph $D$ is a collection $\psi_m$ of paths (not necessarily open) in $D$ whose union is $D$ satisfying the following conditions for distinct paths $P_i$ and $P_j$ with terminal vertices $u, v, w, z$ respectively,
\{e_m, v_h\}-Cyclic Path Covering Number of Digraphs

Pi \(\bar{E}P_j = \begin{cases} 
A, & \text{if } P_i \text{ and } P_j \text{ are cyclic.} \\
\phi, & \text{if } e_m \text{ is one of the arc in both } P_i \text{ and } P_j \\
e_m, & \text{if } e_m \text{ is the subset of the set } \{u, v, w, z\} 
\end{cases}\)

**Definition 3.2.** [11],[12],[13],[14]

The \(e_m\)-Cyclic path covering number \(\gamma_m\) of \(D\) is defined to be the minimum cardinality taken over all \(e_m\)-Cyclic directed path covers of \(D\).

**Theorem 3.3.** For a digraph \(D\) with edge \(e_m\) the \(e_m\)-Cyclic path covering number is

\[
\gamma_m = \gamma - \left[ \min \left\{ d^-(v'_m), \ d^- (v^h_m) \right\} - 1 \right]
\]

where \(\gamma\) is the Cyclic path covering number of \(D\) and \(v'_m\) and \(v^h_m\) are the tail and head vertex of the arc \(e_m\).

**Proof:** let \(\psi_m\) be the \(e_m\)-Cyclic path covering of the digraph \(D\) and \(\gamma_m\) be the \(e_m\) Cyclic path covering number of \(D\). Here \(e_m\) may be in any one of the following cases:

1) \(e_m\) may be the arc in a directed path \(P\) of length more than 1.
2) \(e_m\) may be a directed path of length 1.

Case (1): As \(e_m\) is an arc in \(P\) and let \(v'_m\) and \(v^h_m\) be the tail and head vertices of the arc \(e_m\).

Now arc \(e_m\) may be in one of the following 2 cases.

i) \(e_m\) is the intermediate arc in \(P\).
ii) \(e_m\) is the terminal arc in \(P\).

Case (ii) Then \(e_m\) has the following cases.

a) Degree of one vertex is 1.

Then there is nothing to prove.

b) the indegree and outdegree of both \(v'_m\) and \(v^h_m\) equal to 1.

Now also there is nothing to prove.

c) out degree of \(v^h_m\) one end vertex is 2.

Even in this case there is no change in the number of path coverings.

b) the in degree and out degree of both \(v'_m\) and \(v^h_m\) equal to 2.

As \(e_m\) is the intermediate arc and \(v'_m\) and \(v^h_m\) are the start and end vertices of \(e_m\), there must be two paths \(P_1\) and \(P_2\) whose end and start vertices are \(v'_m\) and \(v^h_m\) respectively.

As \(e_m\) allows more than one path, joining \(P_1\) and \(P_2\) as \(P^* = P_1 \lor \{e_m\} \lor P_2\) we have a new path. Thus in the Cyclic directed path cover one path has been minimized.

\(\gamma_m = \gamma - 1\).
e) \( d^+(v'_m) = 3 \) and \( d^-(v'_m) = 2 \).

Now in this case there must be two paths \( P^1_1 \) and \( P^2_1 \) are having end vertex \( v'_m \) and the directed path \( P^1_2 \) has start vertex at \( v^h_m \).

Now we get the new path in one of the following manner

\( P^1_1 \{e_m\} \rightarrow P^1_2 \) or \( P^2_1 \{e_m\} \rightarrow P^2_2 \)

Thus one path in the Cyclic path covering is reduced.

Therefore we have \( \gamma^m = \gamma - 1 \).

f) If \( d^+(v'_m) = d^-(v^h_m) = 3 \).

Now in this case there must be two paths \( P^1_1 \) and \( P^2_1 \) having end terminal vertex as \( v'_m \) and two paths \( P^1_2 \) and \( P^2_2 \) having start terminal vertex as \( v^h_m \).

Now we get two paths in the following manner.

(i) \( P^1_1 \{e_m\} \rightarrow P^1_2 \) or \( P^2_1 \{e_m\} \rightarrow P^2_2 \)

(ii) \( P^1_1 \{e_m\} \rightarrow P^1_2 \) or \( P^2_1 \{e_m\} \rightarrow P^2_2 \)

Thus two paths are being reduced from the total number of Cyclic path cover.

Therefore \( \gamma^m = \gamma - 2 \).

In general, \( \gamma^m = \gamma - \left[ \min \{d(v'_m), d(v^h_m)\} - 1 \right] \)

Case: (2) \( e_m \) is a directed path of length 1.

Let \( v'_m \) and \( v^h_m \) be the start and end vertices of \( e_m \).

a) If anyone of this vertex is of total degree \( \leq 2 \) then there is nothing to prove.
\{e_m, v_h\} - Cyclic Path Covering Number of Digraphs

(i.e there is no change in the minimal Cyclic path cover)

b) \(d(v'_m) = d(v^h_m) \geq 2\)

Then \(v'_m\) and \(v^h_m\) will be the intermediate vertex of some path \(P_{mt}\) and \(P_{mh}\). Now the reduction in the number of paths is possible only if \(d'(v'_m) \geq 2\) and \(d'(v^h_m) \geq 2\) (i). If \(d'(v'_m) = 2\) and \(d'(v^h_m) = 2\), then there is a path \(P^i_{mt}\) that is having \(v'_m\) as the end vertex and another path \(P^i_{mh}\) that is having \(v^h_m\) as the starting vertex apart from the paths \(P_{mt}\) and \(P_{mh}\) that are having \(v'_m\) and \(v^h_m\) as internal vertices respectively. Now there is a possibility to make a new path as \(P^i_{mh} \rightarrow \{e_m\} \rightarrow P^i_{mt}\)

Thus there is a reduction in the number of paths by one. \(\gamma_m = \gamma - 1\).

(i.e.) \(\gamma_m = \gamma - 3 +2 = \gamma - \min\{d(v_m), d(v_{2m})\} +2\)

c) If \(d'(v'_m) = 3\) and \(d'(v^h_m) = 2\)

Then \(v'_m\) and \(v^h_m\) will be the intermediate vertex of some path \(P_{mt}\) and \(P_{mh}\) respectively and as \(d'(v'_m) = 3\), there must be a paths \(P^i_{mt}\) and \(P^h_{mt}\) that is incident at \(v'_m\) and similarly as \(d'(v^h_m) = 2\) there is a path \(P^1_{mh}\) that has the start vertex at \(v^h_m\). Now there is possibility to get two new path as

(i) \(P^1_{mt} \rightarrow \{e_m\} \rightarrow P^i_{mh}\) or \(P^1_{mh} \rightarrow \{e_m\} \rightarrow P^i_{mt}\)

Thus there is a reduction in the number of paths by one. \(\gamma_m = \gamma - 1\). (i.e. \(\gamma_m = \gamma - [2 - 1] = \gamma - [\min\{d(v'_m), d(v^h_m)\} - 1]\). This same is the case for \(d'(v'_m) = 2\) and \(d'(v^h_m) = 3\)

d) If \(d'(v'_m) = 3\) and \(d'(v^h_m) = 4\)

As usual \(v'_m\) and \(v^h_m\) will be the intermediate vertices in the path \(P_{mt}\) and \(P_{mh}\) and as \(d'(v'_m) = 3\) and \(d'(v^h_m) = 4\) there must be paths \(P^i_{mt}\) and \(P^2_{mt}\) that are incident at \(v'_m\) and the paths \(P^i_{mh}\), \(P^2_{mh}\) and \(P^3_{mh}\) that are have the start vertex at \(v^h_m\). Now there is a possibility to get 2 new paths as

(i) \(P^i_{mt} \rightarrow \{e_m\} \rightarrow P^i_{mh}\) or \(P^i_{mh} \rightarrow \{e_m\} \rightarrow P^2_{mt}\)

(ii) \(P^2_{mt} \rightarrow \{e_m\} \rightarrow P^i_{mh}\) or \(P^2_{mh} \rightarrow \{e_m\} \rightarrow P^2_{mt}\)

This thus there is a reduction in the number of path covers by 2 and so the following results

\(\gamma_m = \gamma - 2\). (i.e., \(\gamma_m = \gamma - 3 + 1 = \gamma - \min\{d'(v'_m), d'(v^h_m)\} + 1\)

e) If \(d'(v'_m) = 4\) and \(d'(v^h_m) = 3\)

As usual \(v'_m\) and \(v^h_m\) will be the intermediate edges in the path \(P_{mt}\) and \(P_{mh}\) and as \(d'(v'_m) = 4\) and \(d'(v^h_m) = 3\) there must be paths \(P^i_{mt}\), \(P^2_{mt}\) and \(P^3_{mt}\) that are having end vertex at \(v'_m\) and the paths \(P^i_{mh}\) and \(P^2_{mh}\) that have the start vertex at \(v^h_m\). Now there is a possibility to get 2 new paths as

(i) \(P^i_{mt} \rightarrow \{e_m\} \rightarrow P^i_{mh}\) or \(P^i_{mh} \rightarrow \{e_m\} \rightarrow P^2_{mt}\)

(ii) \(P^i_{mt} \rightarrow \{e_m\} \rightarrow P^2_{mh}\)

\(\gamma_m = \gamma - 2\)
Thus there is a reduction in the number of path covers by 2 and so the following result $\gamma_m = \gamma - 2$.

(i.e., $\gamma_m = \gamma - 3 + 1 = \gamma - \lfloor \min\{d' (v_m^t), d' (v_m^h)\} \rfloor$)

In general we have for any digraph the $e_m$-Cyclic path covering is

$$\gamma_m = \gamma - \lfloor \min\{d' (v_m^t), d' (v_m^h)\} \rfloor - 1$$

**Definition 3.4.** A $\{e_m, v_k\}$-Cyclic path cover of a digraph $D$ is a collection $\psi_{mk}$ of directed paths (may be closed) in $D$ whose union is $D$ satisfying the conditions for distinct paths $P_i$ and $P_j$ with terminal vertices $u, v$ and $w, z$ respectively,

$$P_i \notin P_j = \begin{cases} A, A \text{ is the subset of the set } \{u, v, w, z\} \\ \phi, \text{ if } P_i \text{ and } P_j \text{ are cyclic.} \\ v_k, \text{if } v_k \text{ is internal vertex of both } P_i \text{ and } P_j, \\ e_m, \text{if } e_m \text{ is one of the arc in both } P_i \text{ and } P_j \end{cases}$$

**Definition 3.5.** $\{e_m, v_k\}$-Cyclic path covering number $\gamma_m$

The $\{e_m, v_k\}$-Cyclic path covering number $\gamma_{mk}$ of $D$ is defined to be the minimum cardinality taken over all $\{e_m, v_k\}$-Cyclic path covers of $D$.

**Theorem 3.6.** For a digraph $G$ with edge $e_m$ and vertex $v_k$ the $\{e_m, v_k\}$-Cyclic path covering number is

$$\gamma_{mk} = \gamma - \left[ \min\{d^-(v_k), \quad d^+(v_k)\} \right] - 1$$

$$- \left[ \min\{d^+(v_k^t), \quad d^-(v_k^h)\} \right] - 1$$

where $\gamma$ is the Cyclic path covering number of $D$ and $v_k^i$ and $v_k^h$ are tail and the head vertices of $e_m$.

**Proof:** Let $e_m$ and $v_k$ be any one edge and vertex of the digraph $D$. If $v_k$ is one of the terminal vertex of $e_m$ then $\gamma_{mk} = \gamma_m$. Otherwise if $v_k$ is different from the terminal vertices of $e_m$, then we have

$$\gamma_{mk} = \gamma - \left( \text{reduction of paths due to } v_k \right)$$

$$- \left( \text{reduction of path due to } e_m \right).$$

(i.e), $\gamma_{mk} = \gamma - \left[ \min\{d^-(v_k), \quad d^+(v_k)\} \right] - 1$

$$- \left[ \min\{d^+(v_k^t), \quad d^-(v_k^h)\} \right].$$

**4. Conclusion**

As the motivation behind the development of the cyclic path cover is mobile road traffic, some to avoid the congestion at the road junction, the junctions are made with flyovers. Even some of the roads are built with flyovers. Thus the $k^{th}$ junctions can be treated as $v_k$ vertex and the $m^{th}$ roads can be treated as $e_m$ edge. Thus if $k^{th}$ junction and $m^{th}$ road are made with flyovers these can be easily treated with $\{e_m, v_k\}$ cyclic path covers.
{\epsilon_m, v_k}\)-Cyclic Path Covering Number of Digraphs

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