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Cyclic Path Covering Number of Hypo Hamiltonian Graphs

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Abstract

In this paper the definition of subgraph H_G of G has been introduced and also the T-Hypohamiltonian graph of non-Hamiltonian graph is being defined. To find the Cyclic Path Covering Number of hypohamiltonian graphs, two theorems have been developed and based on this the Cyclic Path Covering Number of all non hamiltonian graphs are being found..

Keywords: Cyclic path covering, cyclic path covering number, hypohamiltonian graph, T-hypohamiltoniangrap, lamina graph, hypohamiltonian graph.

2010 Mathematics Subject Classification: 05C38, 05C45, 05C65, 05C70

1. Introduction

The concept of cyclic path covering is being developed with the concept of mobile traffic in city roads. If we consider the context of mobile traffic streams in a city road network, every vehicle driver would wish to encounter as many green signals at junctions as possible so as to minimize fuel consumption and travel time as also maximize unhindered distance so travelled. With the roads treated as edges and junctions as vertices we treat the whole city as a graph G and the Cyclic path covers of G, is an ideal limit for any reasonably realistic measure of "mobility" of traffic flow in the city road network with G as the underlying graph. The concept of path decomposition and path covering number of a graph was introduced by Harary[**1**]. The preliminary results on this paper were obtained by Harary and Schwenk[**2**] and Peroche[**3**]. In this the paths may intersect any number times and the intersection of any two paths is only vertices of G. To get non-intersecting paths we impose some extra conditions in the definition of Path Covering. All graph considered in this paper are assumed to be connected graphs without isolated points. Let G = (V,E) be a graph. We denote the number of vertices if G by n and the number of edges in G by e.

2. Cyclic path covering

Definition of Cyclic Path covering [3,4,5]

A Cyclic Path covering of a graph G is a collection Γ of paths in G whose union is G satisfying the conditions for distinct paths P_i and P_j with terminal vertices *u, v* and *w, z* respectively,

$$P_i \cap P_j = \begin{cases} \text{A, A is the subset of the set } \{u, v, w, z\} \\ \phi, \text{ if } P_i \text{ and } P_j \text{ are cyclic.} \end{cases}$$

Definition of Cyclic Path covering number γ [3,4,5]

The Cyclic Path covering number of G is defined to be the minimum cardinality taken over all Cyclic Path covers of G.

Any Cyclic Path cover Γ of G with |Γ| = γ is called a minimum Cyclic Path cover of G.

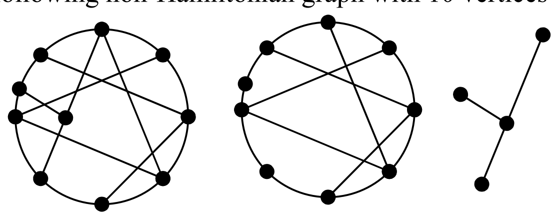
Definition: Let G be any graph and H be the sub graph of G. then the sub graph H_G is defined as $H_G = (V(H_G), E(H_G))$, where $E(H_G) = E(G) - E(H)$ and $V(H_G) = (V(G) - V(H)) \cap (V(G) \cup V(H))$.

Definition: The non-Hamiltonian graph G is said to be T-hypo Hamiltonian graph if for the sub tree T of G, T_G is a maximal Hamiltonian sub graph of G.

Theorem 1. Let G be a T-hypo Hamiltonian graph then $\gamma(G) = \gamma(T_G) + \gamma(T)$.

Proof: Let Γ_1 be the minimal path covering of T_G and Γ_2 be the minimal path covering of T. Let $\gamma(T_G)$ be the cyclic path covering number of the Hamiltonian graph T_G and in this minimal path cover all vertex in T_G are the internal vertex of some path, otherwise if v_1 be the terminal vertex of some path in Γ_1 and not the internal vertex of any path in Γ_1. Then as T_G is Hamiltonian and $v_1$cannot be the vertex of degree 1 and hence v_1 must be the terminal vertex of more than one path. Thus any two of these paths say P_i and P_j can be joined at v_1 and v_1 can be made into internal vertex of $P_i \cap P_j$. Let $v_1, v_2, \ldots, v_k$ be the terminal vertices of T such that $T_G \cap T = \{v_1, v_2, \ldots, v_k\}$. As $v_1, v_2, \ldots, v_k$ are all internal vertices of some path in T_G, no path in Γ_2 can be joined at any of the vertex $\{v_1, v_2, \ldots, v_k\}$ of T_G. Therefore γ (G) is the sum of $\gamma(T_G)$ and $\gamma(T)$.

Example 1. Consider the following non-Hamiltonian graph with 10 vertices and 17 edges.



Graph G Graph T_G Tree T

Fig.1

Here the subgraphT_G is a Hamiltonian graph with 9 vertices and 14 edges and hence $\gamma(T_G)$= 12 - 7 = 5 and T has 3 pendent vertices and thus $\gamma(T) = 3 - 1 = 2$. Thus by the consequence of the theorem.1 we have $\gamma(G) = \gamma(T_G) + \gamma(T) = 5 + 2 = 7$.

Theorem 2. Let G be any non-Hamiltonian graph and $H = \{T_1, T_2\}$ is a forest such that H_G is a Hamiltonian graph, then $\gamma(G) = \gamma(T_G) + \gamma(T_1) + \gamma(T_2)$.

Proof: Let Γ_1, Γ_2 and Γ_3 be the minimal path covering of H_G, T_1 and T_2 respectively. As H_G is a Hamiltonian graph, in the minimal cyclic path covering number of H_G, all the vertices are the internal vertex of some path, otherwise if v_1 be the terminal vertex of some path in $\Gamma_1$and not the internal vertex of any path in Γ_1. Then as H_G is Hamiltonian and v_1 cannot be the vertex of degree 1, and hence v_1 must be the terminal vertex of more than one path. Thus any two of these paths ay Pi and Pj can be joined at v_1 and v_1 can be made into internal vertex of $P_i \cap P_j$. Let $v_1, v_2, \ldots, v_k$ be the terminal vertices of T_1 or T_2 or both such that $[H_G \cup T_1] \cap [H_G \cup T_2] = \{v_1, v_2, \ldots, v_k\}$. As $v_1, v_2, \ldots, v_k$ are all internal vertices of some path in H_G, no path in Γ_2 or $\Gamma_3$can be joined at any of the vertices $\{v_1, v_2, \ldots, v_k\}$ of H_G. Therefore $\gamma(G)$ is the sum of $\gamma(H_G)$, $\gamma(T_1)$ and $\gamma(T_2)$.

Theorem 3. For the Graph $K_{m,n}$, we have $\gamma(K_{m,n}) = mn - m - n$.

Proof: Case (i): ($m = n$). If m = n then the Complete Bipartite graph $K_{m,n}$ is a Hamiltonian graph. Then $K_{m,n}= K_{n,n}$ have n^2 edges and $2n$ vertices.
Thus $\gamma(Km,n) = \gamma(Kn,n) = n^2 -2n= n(n-2)$.

Case (ii): $(m \neq n)$ .If $(m \neq n)$ then the Complete Bipartite graph $K_{m,n}$ is not a Hamiltonian graph. Let us take $m <n$. Then $H_{K_{m,n}}$ is the complete bipartite graph $K_{m,m}$, where H is the forest with n-m trees. (i.e.,) H= $\{T_1,T_2.T_3,\ldots T_{n\_m}\}$ and each of these n-m trees have m+1 vertices among them m vertices are of degree 1 and m edges.

Thus $\gamma(T_i) = m-1$, $i = 1,2,3,\ldots, n-m$. Then by the theorem: 2 we have

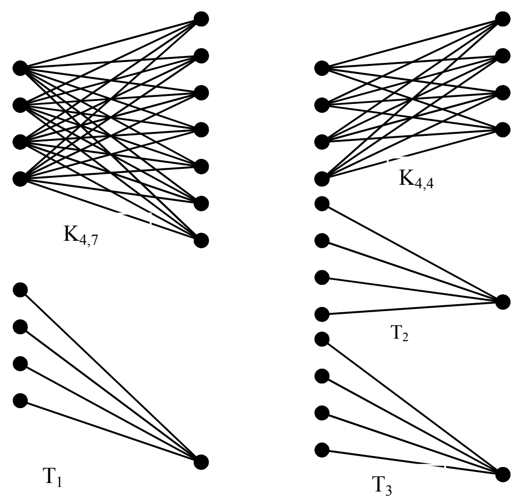


Fig.2

$$\gamma(K_{m,n}) = \gamma(K_{m,m}) + \gamma(T_1) + \gamma(T_2) + ... + \gamma(T_{n-m})$$
$$\gamma(K_{m,n}) = m(m-2) + (m-1) + (m-1) + ... + (m-1)$$
$$\gamma(K_{m,n}) = m(m-2) + (n-m)(m-1)$$
$$\gamma(K_{m,n}) = mn - m - n$$

Theorem 4. The cyclic path covering number of Honey comb graph of level n with n vertices and e edges is $\gamma(G) = e - n$.

Proof: Let G be the Honeycomb graph of level n. First let us prove that the Honey comb is a T-Hypo Hamiltonian graph by induction method.
Level 0: A Honeycomb of level 0 is simply a hexagon graph, which is C6 and hence $\gamma(G) = \gamma(C_6) = 1$.

Level 1: At level 1 The Honey comb graph has e = 30 and n = 24. This graph is a T-hypo Hamiltonian graph such that T_G is a Hamiltonian graph and T is a tree (i.e.,) T = P_3.

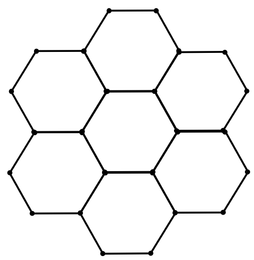


Fig.3

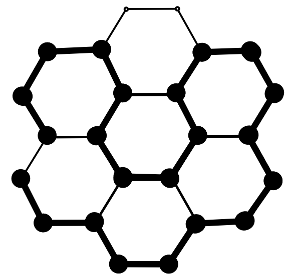


Fig.4

Level 2: At level 2, the Honey comb graph has e = 72 and n = 54. This graph is a T-hypo Hamiltonian graph with T_G is a Hamiltonian graph and T is a forest such that T = {T_1, T_2} in which $T_1 = P_2$ and T_2 = is the H graph.

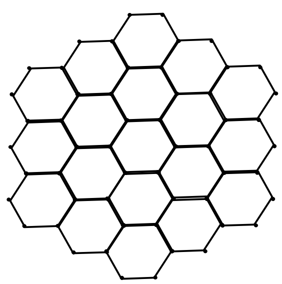


Fig.5

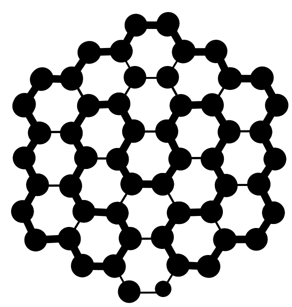


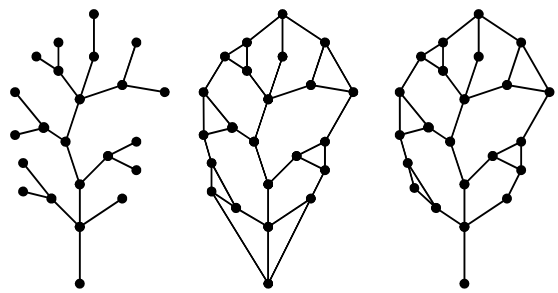
Fig.6

Level 3: At level 3 the graph G has $n = 96$ and $e = 132$. This graph is a T-hypohamiltonian graph such that H_G is a Hamiltonian graph and the forest H ={T_1,T_2, T_3}, where $T_1 = P_3$, T_2 and T_3 are H-graphs.
Level k (*k*> 2): Proceeding in the above manner, the Honey comb graph of level *k*> 2 , is a T-hypo

Hamiltonian graph, such that H_G is a Hamiltonian graph and H={T_1, T_2, T_3,…T_k} in which $T_1 = P_3$ and T_2, T_3,…, T_k are H-graphs. Thus

$$\gamma(G) = \gamma(H_G) + \gamma(T_1) + \gamma(T_2) + \gamma(T_3) + ... + \gamma(T_n)$$
$$\gamma(G) = ((e-(3+5(k-1)) - (n-2k)) + 1 + (k-1)3$$
$$\gamma(G) = e - n$$

Definition (Lamina graph). In a tree T with n pendent vertices, if all the adjacent pendent vertex are joined by edges then the resultant graph is graph is called n-*lamina graph* and if the pendent vertices except one pendent vertex (called the root vertex) are joined by edges then the resultant graph is called *n-petioleted lamina* .



Tree Lamina graph PetioletelaminaTree

Fig.9

Theorem 5. Let G be the n-lamina graph then $\gamma(G) = n$.

Proof: As G is n-lamina graph G can be written as $G = T_G \cap T$. Then $\gamma(G) = \gamma(T_G) + \gamma(T)$. That is $\gamma(G) = 1 + (n-1) = n$.

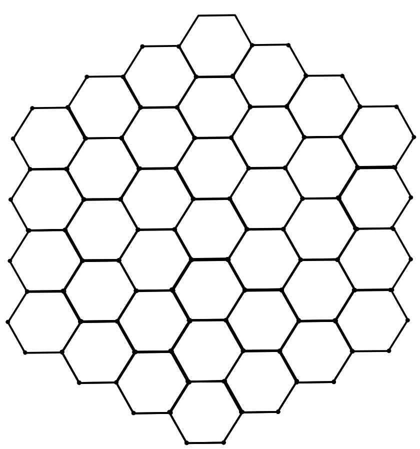


Fig.7

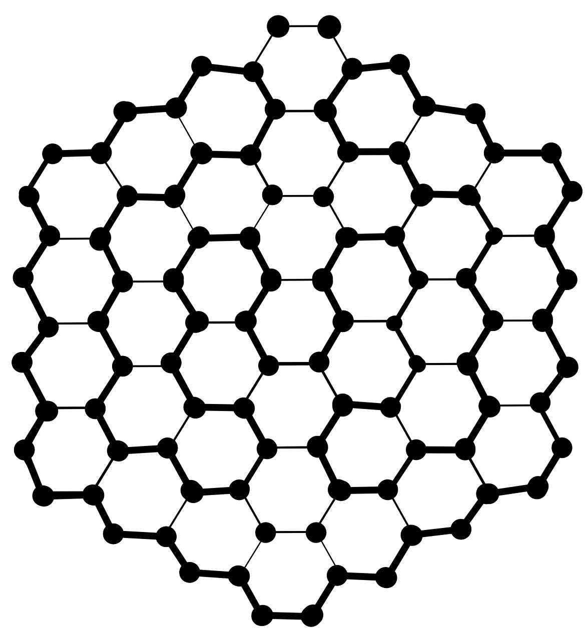


Fig.8

Theorem6. Let G be the n-petioleted graph then $\gamma(G) = n$.

Proof: As G is n-petioleted graph G can be written as $G = T_G \cap T$, Where T_G is simply a open path. Then $\gamma(G) = \gamma(T_G) + \gamma(T)$. That is $\gamma(G) = 1 + (n-1) = n$.

3. Hypohamiltonian graph

Definition. A graph G is hypohamiltonian if G is nonhamiltonian, but G - v is Hamiltonian for every $v \in V$ (Bondy and Murty 1976, p. 61).

Theorem 7. For any Hypohamiltonian graph G with n vertices and e edges,γ(G) = *e* - *n+1*, if G - v is a Cyclic Cyclomatic Graph, else γ(G) = *e – n.*

Proof: Let v∈V(G) be any vertex of G. Then G – v is the Hamiltonian graph and $(G\text{-}v)_G$ is a tree graph (actually a star graph) with d(v) number of pendent vertices. Then by our definition of T-hypo hamiltoniangraph we have G is a T-hypo Hamiltonian graph and hence we have γ(G) = γ(G-v) + $\gamma((G\text{-}v)_G)$. Now there are two cases.

Case (i) G - v is a cyclic cyclomatic graph. Then G - v will have e - d(v) edges and n-1 vertices. Therefore $\gamma(G-v) = e-d(v)-(n-1)+1 = e-n-d(v)+2$ and $\gamma((G-V)_G) = d(v)-1$. Hence $\gamma(G) = \gamma(G-v)+\gamma((G-v)_G) = e-n-d(v)+2+d(v)-1 = e-n+1$.

Case (ii) G - v is a non cycliccyclomatic graph. Then G - v will have e - d(v) edges and n-1 vertices. Therefore $\gamma(G-v) = e-d(v)-(n-1) = e-n-d(v)+1$ and $\gamma((G-v)_G) = d(v)-1$. Hence

$$\begin{aligned}\gamma(G) &= \gamma(G-v)+\gamma((G-v)_G) \\ &= e-n-d(v)+1+d(v)-1 \\ &= e-n\end{aligned}$$

4. Conclusion

As the motivation behind the development of the cyclic path cover is mobile road traffic, some to avoid the congestion at the road junction, the junctions are made with flyovers. During the peak hours of traffic one can able to find out which one is the right road to go fast so that one can reach the destination as quick as possible.

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