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Strongly Indexable Graphs: Some New Perspectives

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Abstract

Given any positive integer k, a (p,q)-graph $G=(V,E)$ is *strongly* k*-indexable* if there exists a bijection $f:V\to\{0,1,2,\ldots,p-1\}$ such that $f^+(E(G))=\{k,k+1,k+2,\ldots,k+q-1\}$ where $f^+(uv)=f(u)+f(v)$ for any edge $uv\in E$ and $f^+(E(G)=\{f^+(e):e\in E(G)\}$; f is called a *strong* k*-indexer* of G. In particular, G is said to be *strongly indexable* whenever it admits a 1-strong indexer (or, simply a *strong indexer*). Even though many graphs are not strongly indexable, their line graphs may still be strongly indexable. The paper expounds the connection of strong indexers of graphs with Sidon sequences and Fibonacci sequences and explores classes of strongly indexable chain graphs whose blocks are complete or, the so-called `Husimi trees'.

Keywords & Phrases: Strongly Indexable graph, super edge-magic labeling, line graph.

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1. Introduction

Unless mentioned otherwise, by a *graph* we shall mean in this paper a finite, undirected, connected graph without loops or multiple edges. Terms not defined here are used in the sense of Harary [10].

Acharya and Hegde [2] introduced the concept of an `indexer' of a finite graph as a special case of *arithmetic labelings* [1]. In the general setting, *labeling* of a graph $G=(V,E)$ is an assignment $f$ of distinct nonnegative integers to the vertices of $G$; it is an *indexer* of $G$ if the induced `edge function' $f^+:E(G)\to\mathbb{N}$, from $E(G)$ into the set $\mathbb{N}$ of natural numbers, defined by the rule: $f^+(uv)=f(u)+f(v),\ \forall\, uv\in E(G)$, is also injective. It is known that every finite graph G has an indexer [1]; hence, an indexer f of G is said to be *optimal* if $f[G]:=max_{v\in V(G)}\{f(v)\}$ has the least possible value $\upsilon(G)$ amongst all the indexers of G. Clearly, $\upsilon(G)\geq|V(G)|-1$ for any finite graph G. For any given positive integer k, an indexer f of G is said to be k*-arithmetic* and G a k*-arithmetic graph*, if $f^+(E(G)):=\{f^+(uv):uv\in E(G)\}=\{k,k+1,k+2,\ldots,\}$. Not every graph is k-arithmetic as indicated by the following theorem for finite graphs.

Theorem 1.1 [1]. *Let $G=(V,E)$ be any (p,q)-graph and f be any k-arithmetic indexer of G, where k is odd. Then, there exists an `equitable partition' of V into two subsets V_o and V_e such that there are exactly $\lceil \frac{q+k-1}{2} \rceil$ edges each of which joins a vertex of V_o with one of V_e, where $\lceil . \rceil$ denotes the least integer function.*

In the above theorem, an *equitable partition* of a nonempty finite set X is defined as a partition $\{X_1, X_2\}$ of X such that the cardinalities of X_1 and X_2 differ by at most one, that is, if $||X_1|-|X_2|| \le 1$.

Next, if G is a (p,q)-graph, an indexer f is called a k*-strong indexr* of G, if for some positive integer *k*,

$$f(V(G)) := \{f(v) : v \in V(G)\} = \{0,1,2,\ldots,p-1\}$$

and $f^+(E(G)) = \{k, k+1, k+2, \ldots, k+q-1\}$. Further, G is said to be k*-strongly indexable* if it admits a k*-strong indexer*. Again, as mentioned already, for any positive integer k, not every graph is k-strongly indexable. If $V(G)$ is countably infinite, then any bijection f from $V(G)$ onto the set $\mathsf{N} \cup \{0\}$ of nonnegative integers such that $f^+(E(G)) = \{k, k+1, k+2, \ldots,\}$ is defined as a k-strong indexer of G. In particular, if $k=1$ in these definitions, then f is called a *strong indexer* of G and the graph G is said to be *strongly indexable* if it admits a strong indexer. We will frequently use the following results which are proved in [1, 2].

Theorem 1.2[1, 2]. *For any indexable (p,q)-graph $G, q \le 2p-3$.*

Theorem 1.3[1]. *Every strongly indexable finite graph has at most one nontrivial component which is either a star or has a triangle.*

Kotzig and Rosa [13] called a (p,q)-graph $G=(V,E)$ *edge-magic* if it admits an *edge-magic labeling* of G which is defined as a bijection $f: V(G) \cup E(G) \to \{1,2,\ldots,p+q\}$ such that there exists a constant s (called the *magic number* of f) with $f(u)+f(v)+f(uv)=s, \forall uv \in E(G)$. Enomoto *et.al.* [7] called an edge-magic labeling f of G *super-edge-magic* if $f(V(G))=\{1,2,\ldots,p\}$ and $f(E(G))=\{p+1,p+2,\ldots,p+q\}$ and G *super-edge-magic* if there exists a super-edge-magic labeling of G. The following Lemma of Figueroa-Centeno *et.al.* [8] gives an interesting connection between k-strongly indexable graphs and super-edge-magic graphs.

Lemma 1.4[8]. *A (p,q)-graph G is super-edge-magic if and only if there exists a bijective function $f: V(G) \to \{1,2,\ldots,p\}$ such that the set $S=\{f(u)+f(v) : uv \in E(G)\}$ consists of q consecutive integers. In such a case, f can be extended to a super-edge-magic labeling of G with constant $c=p+q+s$ where $s=min(S)$ and $s=\{c-(p+1), c-(p+2), \ldots, c-(p+q)\}$.*

It has been recently proved [4] that the class of strongly indexable graphs are proper subclass of super-edge-magic graphs.

Remark 1.5[4]. *For any positive integer* k*, if a* (p,q)*-graph* G *is* k*-strongly indexable then every* k*-strong indexer of* G *extends to* G *a super-edge-magic graph with its magic number equal to* $p+q+k+2$*. Conversely, if* G *is super-edge-magic with magic number* c *then* G *has a* k*-strong indexer with* $k=c-2-p-q$.

Remark 1.6 [4]. *Let* f *be a super-edge-magic labeling of* G *and let* $g(u)=f(u)-1\ \forall u\in V(G)$*, so that* g *is a* k*-strong indexer of* G *with* $k=c-2-p-q$*. If* c *is the magic number of super-edge-magic labeling of the* (p,q)*-graph* G*, then* $c\geq 1+2+p+1=p+4$ *and* $c=6$ *results in isolated edge, which is the trivial case. Hence,* $c\geq 6$*. Also,* $min_{v\in V(G)}\{g(v)\}=0$ *and* $min_{e\in E(G)}\{g(e)\}=k$*. Hence,* $k=c-2-(p+q)\geq 6-2-k \Leftrightarrow 2k\geq 4 \Rightarrow k\geq 2$.

From Remark 1.6, we see that the converse of Remark 1.5 is likely to fail for $k=1$ and, in fact, it does in view of the result that the cycle C_n is strongly indexable if and only if $n=3$ (*see* [2]) and the following result with any value of $n\geq 4$.

Theorem 1.7 [8]. *A cycle* $C_n, n\geq 3$*, is super-edge-magic if and only if* n *is odd.*

Recently, an interesting application of strongly k-indexable cycles has been found in Euclidean plane geometry [12]. In this paper, we are inclined to bring out few newer aspects of strongly indexable graphs.

2. Strongly Indexable Line Graphs

Theorem 1.3 implies that no triangle-free graph other than the star $K_{1,n}, n\geq 1$ is strongly indexable. However, line graph $L(G)$ of such a graph G contains a triangle whenever G contains a vertex of degree greater than or equal to three and hence there is a possibility that $L(G)$ could be strongly indexable. In fact, $L(K_{1,n})=K_n$ and we know that K_n is strongly indexable if and only if $n\in\{1,2,3\}$. The cartesian product graph $P_n\times K_2$ is called the n *-ladder* with n steps. Since this graph is triangle-free it is not strongly indexable by Theorem 1.3. However, the line graph of the 3-ladder $P_3\times K_2$ and of the 4-ladder $P_4\times K_2$ are strongly indexable as shown in Figure 1. We surmise that the line graph of the n-ladder is strongly indexable for all integers $n\geq 5$.

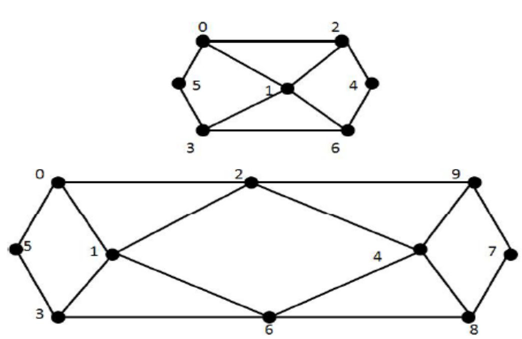

Figure 1:

The following result gives a necessary condition for a graph to have a strongly indexable line graph.

Theorem 2.1. *If* G *is a* (p,q)*-graph such that its line graph* $L(G)$ *is strongly indexable then*

$$\sum_{i=1}^{p}(d(v_i))^2 \le 6(q-1), \quad (1)$$

where $d(v_i)$ *denotes the degree of the vertex* v_i *in* G*, and the bound in (1) is best possible.*

Proof. This can be easily deduced from Theorem 1.2 and the fact that there are exactly $-q+\sum_{i=1}^{p}(d(v_i))^2$ edges in $L(G)$ (see [10], Theorem 8.1, p.72). That the bound in (1) is attainable can be seen from the graph C_3^+, where H^+ denotes the graph obtained from the given graph H by augmenting a new vertex v' as also the new edge vv' for each vertex $v \in V(H)$.

Problem 1. *Characterize* (p,q)*-graphs that attain the bound in (1).*

Conjecture 2.2. *For every* (p,q)*-graph* G *for which the bound in (1) is attained,* $L(G)$ *is strongly indexable.*

However, one can easily find counter-examples to the converse of Theorem 2.1 and what Conjecture 2.2 claims is that for such a graph there is a strict inequality in (1).

3. Co-strongly Indexable Graphs

From Theorem 1.2 it is easy to deduce that for no (p,q)-graph G, both G and its complement $\overline{G}$ can be strongly indexable. However, both G and its line graph $L(G)$ can be strongly indexable as, for instance, P_3 and $L(P_3) \cong P_2$ are both strongly indexable. Hence, we define a *co-strongly indexable graph* as a graph G such that both G and $L(G)$ are strongly indexable.

Proposition 3.1. *If* G *is a connected co-strongly indexable* (p,q)*-graph then either* $p \le 4$ *or* $\Delta(G) \ge 3$.

Proof. Let G be as given. Suppose G is triangle-free. Then, Theorem 1.3 implies that $G = K_{1,p-1}$, the star with $p-1$ pendant vertices and hence $L(G) \cong K_{p-1}$. Then, since $L(G)$ is also strongly indexable, Theorem 1.2 implies that $p-1 \le 3$, i.e., $p \le 4$.

Hence, let $p \ge 5$. Then, the above argument implies that G has a triangle. Now, if $\Delta(G) \le 2$ then we get $G \cong K_3$, a contradiction to our assumption that $p \ge 5$. ■

The *dot-composed graph* (c.f.: [10], p.23) $K_3 \bullet K_2$ and its line graph $K_4 - x$ (i.e., one edge x deleted from K_4) are both strongly indexable; thus, $K_3 \bullet K_2$ is co-strongly indexable graph of order 4 containing a triangle as well as a vertex of degree ≥ 3.

Problem 2. *Characterize co-strongly indexable graphs.*

4. Connection with Sidon sequences and Fibonacci sequences

For the vertices of a clique in a graph to be labeled so that no edge label is repeated, the labels must be chosen from a set of positive integers in which the sums of the pairs of distinct vertex labels are all distinct. Such a set is called a *Sidon set* (e.g., see [18]). When the members of the Sidon set are placed in ascending order, the resulting sequence is called a *Sidon sequence* or a *well-spread sequence*. Choose a Sidon sequence $(s_1, s_2, \ldots, s_r)$ in which the largest element s_r is as small as possible. If $s(r)$ denotes the smallest possible value of s_r taken over all Sidon sequences of length r then, a Sidon sequence of length r with largest element $s(r)$ must have 0 as the smallest element. Note that the Fibonacci sequence (f_n), defined by $f_1 = 1, f_2 = 2$ and $f_n = f_{n-1} + f_{n-2}$, is a Sidon sequence. Hence, Fibonacci numbers provide a reasonably good upper bound for the function σ whose values are all elements of a Sidon sequence. As a generalization of the concept of Sidon sets, a set of integers is called a (n_1, n_2) *-set* if every n_1-element subset determines at least n_2 distinct differences. Let $g(n)$ be the largest number such that any n-element (n_1, n_2)-set contains a g(n)-element Sidon set (i.e., a subset of g(n) elements with distinct sums).

Acharya and Germina [5] raised the following new problem and proved that `Given a Fibonacci sequence (f_r) with $f_1 = 1, f_2 = 2$ there exists a connected maximal strongly indexable graph of order $f_r + 1$ and size $2f_r - 1$ that contains a clique of order r'.

Problem 3 [4]. *Given the set $S_n(\mathsf{F})$ of the first $(n; a_i)$ terms of a given generalized Sidon set, determine the class of all non-isomorphic strongly indexable chain graphs whose blocks are all complete and for which $S_n(\mathsf{F}) \subset f(G)$ for some strong indexer f.*

The above problem can also be viewed in a different direction as: Given a Sidon sequence $(s_1, s_2, \ldots, s_r)$ in which the largest element s_r is as small as possible and a set of integers, called a (p_i, q_i) *-set* so that every p_i-element subset determines q_i distinct integers, determine the class of all non-isomorphic strongly indexable graphs for which $S_n(\mathsf{F}) \subset f(G)$ for some strong indexer f.

5. Strongly-indexable Husimi chains

Barrientos [6] defines a *chain graph* as one with blocks $B_1, B_2, \ldots, B_m$ such that for every i, B_i and B_{i+1} have a common vertex in such a way that the *block cut-point graph* (see [10]) is a path. Sin-Min Lee and J. Yun-Chin Wang [19] denote the chain graph with n blocks and the sequence of n blocks of complete graphs $K(a_1), K(a_2), \ldots K(a_n)$ by $CK(n; (a_1, a_2, \ldots, a_n))$, with $a_i \geq 2$. If $a_1 = a_2 = \ldots = a_n = 2$, then $CK(n; (2,2,\ldots,2)) = P_{n+1}$. In general, a separable graph in which every block is complete is well known as *Husimi tree* and hence we refer to $CK(n; (a_1, a_2, \ldots, a_n)), a_i \geq 2$ as a *Husimi chain*. In another view, note that the path is a member of a larger class of trees called *caterpillars*; a caterpillar is defined as a tree removal of whose *pendant vertices* (i.e., vertices of degree 1) results in a path called its *spine*. Clearly, the line graph $L(G)$ of a caterpillar G with $(u_0, u_1, u_2, \ldots, u_{n-1}, u_n)$ as its spine is obtained as a chain of *cliques* (i.e., maximal complete subgraphs; c.f.: [10]) of various orders `glued' to one another in a sequence such that the edge $e_i e_{i+1}, e_i = u_{i-1}u_i, 1 \leq i \leq n$, in $L(G)$ supports the clique Q_i whose other vertices correspond to the *pendant edges* (i.e., edges incident to pendant vertices) incident to u_i in G. In fact, it is this view that prompted us to study strong indexability of Husimi chains in detail.

It is well known that P_n is super-edge-magic (k-strongly indexable for $k = \lceil \frac{n}{2} \rceil$) and is strongly indexable if and only if $n = 3$. If $a_1 = a_2 = \ldots = a_n = 3$, then $CK(n;(3,3,\ldots,3))$ is the *triangular snake* which is graceful if $n \cong 1,2(mod\ 4)$ (see, [16]). Sin-Min Lee *et.al.* [19] investigated the existence of super-edge-magic labelings of certain classes of Husimi chains. Recently, the following general result has been established.

Theorem 5.1. [5]. *Let $G = (V,E)$ be any graph, not necessarily finite, f be an arbitrary assignment of integers to the vertices of G and let $f^+(uv) = f(u) + f(v)$ for each edge uv in G. Then in every cycle of G there are an even number of edges with odd f^+-values.*

Corollary 5.2 [5]. *Let $G = (V,E)$ be any Eulerian (p,q)-graph. If G is strongly k-indexable then $q \not\equiv 2(mod\ 4)$. Further, exactly one of the following congruences holds:*

$q \equiv 0(mod\ 4)$,
$q \equiv 1(mod\ 4)$ *and* $k \equiv 0(mod\ 2)$ *and*
$q \equiv 3(mod\ 4)$ *and* $k \equiv 1(mod\ 2)$.

Remark 5.3 [5]. *Corollary 5.2 can be used to rule out the possibility of certain classes of Eulerian graphs from their being strongly indexable. In this way, for instance, the cycles C_n for values of $n \equiv 1$ or $2(mod\ 4)$ cannot be strongly indexable. Of course, in general, C_n is not strongly indexable for any value of $n \geq 4$ by virtue of Theorem 1.1, thus demonstrating that the converse of Corollary 5.2 does not hold. A more complex example of an Eulerian graph that is not strongly indexable by this argument is the complement of K_3^+ (see [5]). An infinite class of Eulerian graphs that are not strongly indexable is the class of Husimi trees in which the number of blocks is $m \equiv 3(mod\ 4)$ since, in such a graph $H, |E(H)| \equiv 1(mod\ 4)$. One such well known class is that of ``friendship graphs'' $F_t := tK_2 + K_1$ (which consists of t triangles glued at one common vertex whence F_t consists of $q = 3t$ edges, so that $t \equiv 2$ or $3(mod\ 4)$ yielding $q \equiv 2$ or $1(mod\ 4)$ in the respective cases).*

If $n = 2$, then $CK(2;(a_1,a_2)) \cong K_{a_1} \bullet K_{a_2}$, the dot-composition (c.f. [3]), or more commonly called one-point union, of two complete graphs.

Theorem 5.4. *For $n = 2$ and $a_1 = 2$, $CK(2;(a_1,a_2))$ is strongly indexable if and only if $a_2 \leq 4$.*

Proof. A strong indexer of $CK(2;(a_1,a_2))$ in each of the cases $a_2 = 2,3,4$ is shown in Figure 2. If $a_2 \geq 5$ then $q(CK(2;(2,a_2))) > 2p(CK(2;(2,a_2))) - 3$, and hence $CK(2;(a_1,a_2))$ is not strongly indexable by virtue of Theorem 1.2.

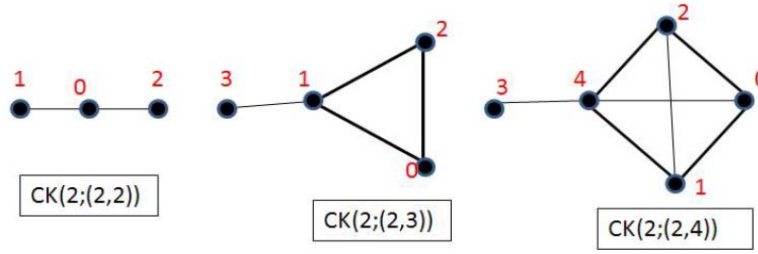

Figure 2:

Theorem 5.5. *For* $n = 2$ *and* $a_1 = 3$, $CK(2;(a_1, a_2))$ *is strongly indexable if and only if* $a_2 = 4$.

Proof. If $a_2 = 3$, then $CK(2;(3,3))$ is a triangular snake with two blocks and is an Eulerain graph with number of edges $q = 6 \cong 2(mod\ 4)$. Hence, by Corollary 5.2, $CK(2;(3,3))$ is not strongly indexable. A strong indexer of $CK(2;(3,a_2))$ for $a_2 = 4$ is depicted in Figure 3. If $a_2 \geq 5$ then $q(CK(2;(2,a_2))) \geq 2p(CK(2;(2,a_2))) - 3$, whence the graph is not strongly indexable by Theorem 1.2.

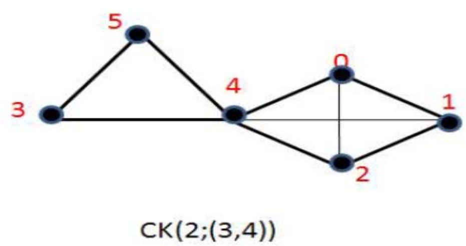

Figure 3:

Theorem 5.6. *For* $n = 2$ *and* $a_2 \geq 5$, $CK(2;(3,a_2))$ *is not strongly indexable.*

Proof. When $a_2 \geq 5$, we get $q(CK(2;(3,a_2))) > 2p((CK(2;(3,a_2))) - 3$, whence the result follows from Theorem 1.2.

Theorem 5.7. *For* $n = 2$ *and* $a_1 \geq 4$, $CK(2;(a_1,a_2))$ *is not strongly indexable for all* $a_2 \geq 4$.

Proof. In accordance with Theorem 1.2, the proof is immediate since $a_1 \geq 4$ implies $q(CK(2;(a_1,a_2))) > 2p(CK(2;(a_1,a_2))) - 3$ for all $a_2 \geq 4$.

Next, we consider Husimi chains with three blocks. There are several cases to consider. First, note that the graph $CK(3;(a_1,a_2,a_3)) \cong CK(3;(a_3,a_2,a_1))$.

Theorem 5.8. *For* $n = 3$, *the chain graph* $CK(3;(a_1,a_2,a_3))$ *is*

• strongly indexable if (a_1,a_2,a_3) is any of the triples $(2,4,3)$, $(2,3,3)$, $(3,4,4), (2,2,4), (2,3,4)$

$(2,3,2)$, $(4,2,4),(4,3,4)$ and $(2,4,4)$.

- 2-strongly indexable if $(a_1,a_2,a_3)=(2,2,2),(2,4,2)$
- not k-strongly indexable if (a_1,a_2,a_3) is one of $(2,2,3)$ and $(2,2,a_3), a_3 \geq 5$

$(a_1,a_2,a_3)=(2,3,a_3), a_3 \geq 5$
$(a_1,a_2,a_3)=(2,4,a_3), a_3 \geq 5$
(a_1,a_2,a_3) is one of $(3,2,3),(3,2,4)$ and $(3,2,a_3), a_3 \geq 5$ $(a_1,a_2,a_3)=(3,3,a_3), a_3 \geq 3$
(a_1,a_2,a_3) is one of $(3,4,3)$ and $(3,4,a_3), a_3 \geq 5$ $(a_1,a_2,a_3)=(4,2,a_3), a_3 \geq 5$
$(a_1,a_2,a_3)=(4,3,a_3), a_3 \geq 5$

Proof. A strong indexer and a 2-strong indexer of graphs cited in (a) and (b) of the theorem are given in Figure 4, Figure 5 and Figure 6. One can easily verify that these are the only choices for these strongly indexable graphs since, by virtue of Corollary 5.2, none of the graphs listed in (c) of the theorem can be 2-strongly indexable. ■

We now consider chain graphs with more than three blocks.

Theorem 5.9. *For* $n=4$*, the chain graph* $CK(4;(a_1,a_2,a_3,a_4))$*, is*

- strongly indexable if (a_1,a_2,a_3,a_4) is any one of $(2,3,2,4)$, $(2,3,3,4),(2,3,3,5)$ and $(2,3,4,4)$
- 2-strongly indexable if (a_1,a_2,a_3,a_4) is any one of $(2,2,2,2)$ $(2,3,2,3),(2,2,4,2),(2,2,3,3),(2,3,3,3)$ and $(2,3,4,2)$.
- 3-strongly indexable if (a_1,a_2,a_3,a_4) is one of $(2,2,2,4)$ and $(3,3,3,3)$.
- 4-strongly indexable if $(a_1,a_2,a_3,a_4)=(2,3,3,2)$.
- 5-strongly indexable if $(a_1,a_2,a_3,a_4)=(2,2,3,2)$
- not k-strongly indexable if (a_1,a_2,a_3,a_4) is any one of $(2,2,2,3)$, $(3,2,2,3),(3,2,3,3)$ and $(2,2,2,a_4), a_4 \geq 5$.

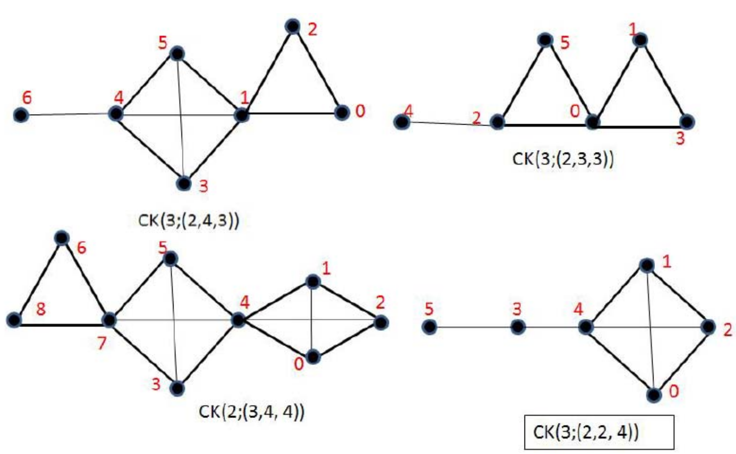

Figure 4:

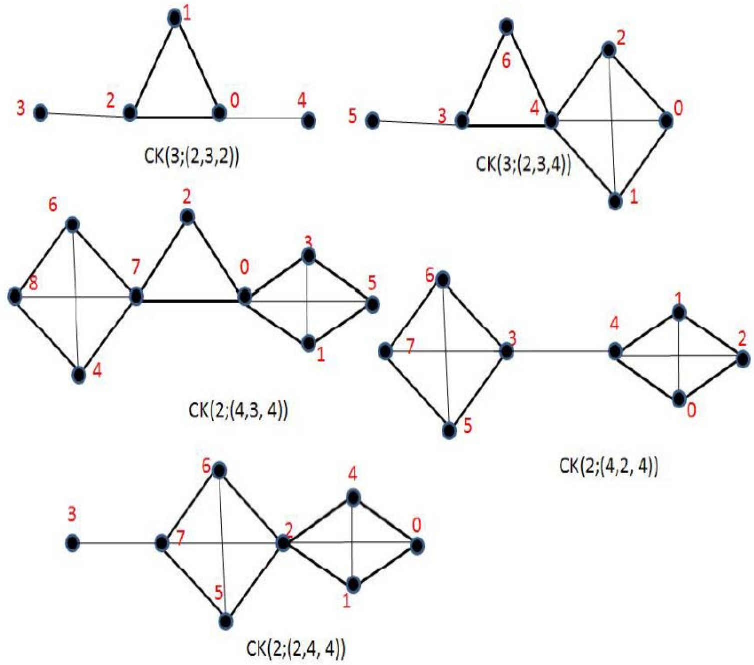

Figure 5:

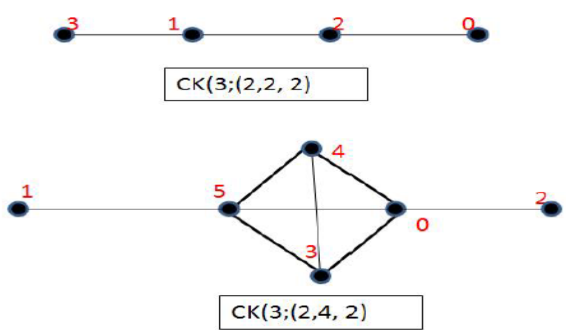

Figure 6:

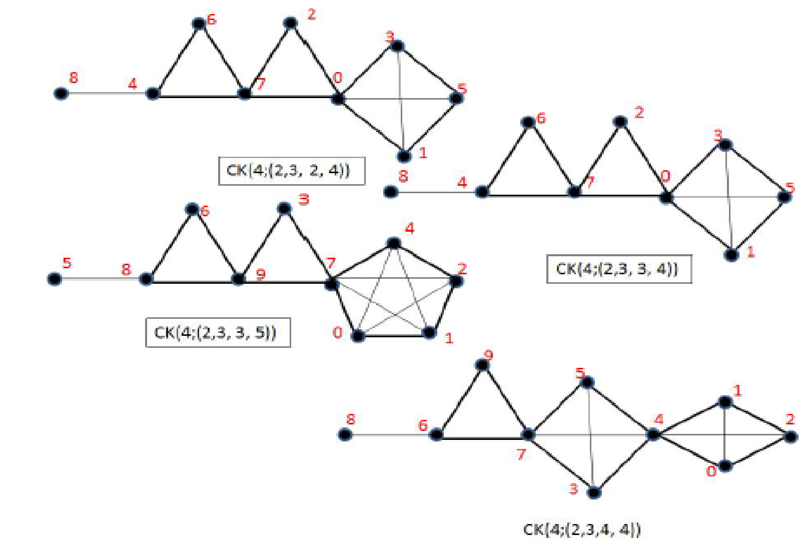

Figure 7:

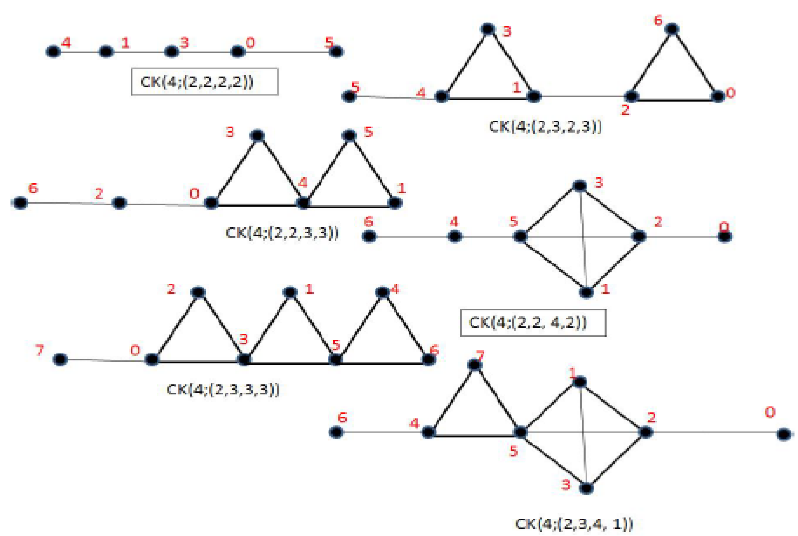

Figure 8:

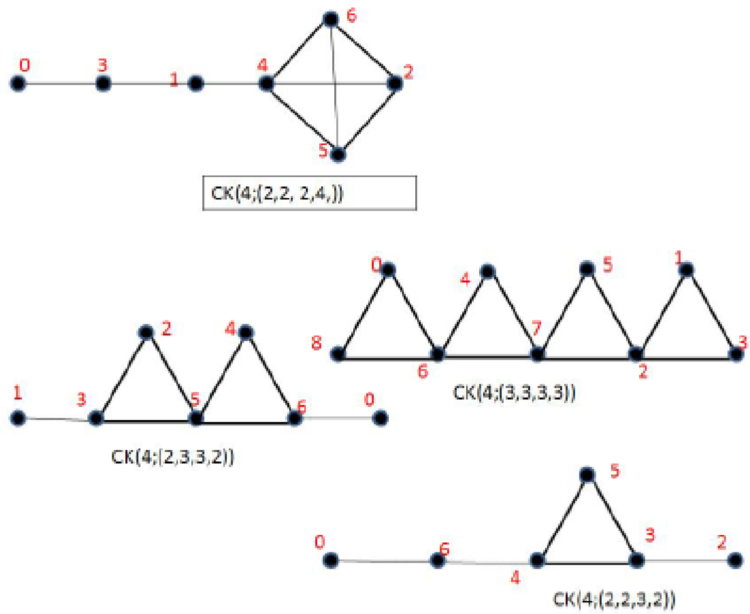

Figure 9:

Proof. A strong indexer, a 2-strong indexer, a 3-strong indexer, a 4-strong indexer and a 5-strong indexer of the chain graphs described in the parts (i), (ii), (iii), (iv) and (v) of the theorem, respectively, are depicted in Figure 7, Figure 8 and Figure 9. By virtue of Theorem 1.2 and Corollary 5.2, none of the graphs listed in (vi) of the theorem is k-strongly indexable for any value of k and in particular for $k=1$. ■

Remark 5.10. *For simplicity of presentation, we will represent a strong indexer of a Husimi chain (or, equivalently, `chain graph') whose blocks are complete graphs K_t, in the form of t-tuples and call each t-tuple a sequence and the strong indexer is written as a string of such sequences corresponding to the complete blocks in the Husimi chain. For example, consider the strongly indexable Husimi chain $CK(4;(2,3,4,4))$; we represent a strong indexer of it as the string:* $(8,6)(6,9,7)(7,5,3,4)(4,1,0,2)$.

Theorem 5.11. *For $n \geq 3$, the graphs $CK(n;(2,4,4,\dots,4))$ are strongly indexable, whereas $CK(n;(2,4,4,\dots,4,2))$ are 2-strongly indexable.*

Proof. We have a strong indexer of $CK(2;(2,4))$ as the string $(3,4)(4,2,1,0)$ and a 2-strong indexer of the chain graph $CK(3;(2,4,2))$ as $(1,5)(5,4,3,0)(0,2)$.

Now, add 2 to each of the numbers in the string and include $1,0$ in the last sequence of the string; we obtain a strong indexer of the chain graph $CK(3;(2,4,4))$ as $(3,7)(7,6,5,2)(2,4,1,0)$. To obtain a 2-strong indexer of the chain graph $CK(3;(2,4,4,2))$, add 1 to each of the numbers in the sequence and include $2,0$ in the last sequence of the string. Again, in the 2-strong indexer of the chain graph

$CK(4;(2,4,4,4))$, add 2 to each of the numbers in the string and include $1,0$ in the last sequence of the string. In general, given a 2- strong indexer of $CK(i;(2,4,4,\ldots,4,2))$, add 2 to each of the numbers in the string and include $1,0$ in the last sequence of the string; we obtain a strong indexer of the chain graph $CK(i;(2,4,4,\ldots,4,4))$. To obtain a 2-strong indexer of the chain graph $CK(i+1;(2,4,4,\ldots,4,4,2))$ from a similar string for $CK(i;(2,4,4,\ldots,4,4))$, add 1 to each of the numbers in the string and include $2,0$ in the last sequence of the string. Continuing like this, we get the series of chain graphs that are 2-strongly indexable and strongly indexable alternately and so on. Hence, we can take the block K_4 as many times as we wish so that the graphs $CK(n;(2,4,4,\ldots,4))$ are all strongly indexable and the graphs $CK(n;(2,4,4,\ldots,4,2))$ are all 2-strongly indexable. (See Figure 10). ■

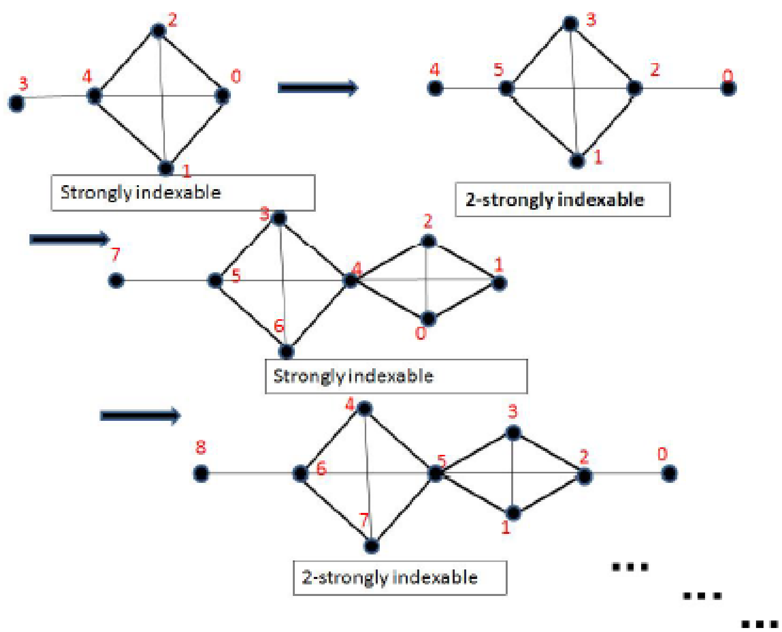

Figure 10:

Remark 5.12. *We start with a strongly indexable Husimi chain* $C_1 = CK(n;(2,4,4,\ldots,4))$, *for any* $n \geq 3$, *and can construct a* 2*-strongly indexable graph* $C_1' = CK(n;(2,4,4,\ldots,4,2)), n \geq 3$. *From* C_1', *we construct* $C_2 = CK(n;(2,4,4,\ldots,4)), n \geq 3$, *which is strongly indexable and from* C_2, *we construct* $C_2' = CK(n;(2,4,4,\ldots,4,2))$, $n \geq 3$, *which is* 2*-strongly indexable. Proceeding in this manner, we get an ascending sequence* $\mathsf{C} = C_1 \subset C_1' \subset C_2 \subset C_2' \subset \ldots$, *where* C_i *is strongly indexable and* C_i' *is* 2*-strongly indexable for each index* i. *Figure 10 illustrates this construction. The method of obtaining the sequences of* 2*-strongly indexable chain graphs (or strongly indexable chain graphs) can be started with a* 2*-strongly indexable chain graph* $CK(n;(2,4,4,\ldots,4,2))$, *instead of the strongly indexable chain graph* $CK(n;(2,4,4,\ldots,4))$.

Theorem 5.13. *For* $n \geq 3$ *, the graphs* $CK(n;(3,4,4,\ldots,4))$ *are strongly indexable and* $CK(n;(3,4,4,\ldots,4,2))$ *are* 2*-strongly indexable.*

Proof. Let us start with C_3, the cycle of length three, which is strongly indexable with the strong indexer $(1,2,0)$. Next, label the vertices of $CK(2;(3,2))$ as the string $(3,1,2)(2,0)$, which is a 2-strong indexer of the graph. Now, add 2 to each of the numbers in the string $(3,1,2)(2,0)$ and include $0,1$ in its last sequence to obtain a strong indexer of $CK(2;(3,4))$ given by the string $(5,3,4)(4,2,0,1)$. Next, add 1 to each of the numbers in this string and augment $(2,0)$ as the last sequence in the resulting string; this gives the 2-strong indexer of the chain graph $CK(3;(3,4,2))$ as the new string $(6,4,5)(5,3,1,2)(2,0)$. To obtain a strong indexer of the chain graph $CK(3;(3,4,4))$, add 2 to each of the numbers in this string and include the sequence $(1,0)$ as last sequence in the resulting string which is the strong indexer given by the new string $(6,4,5)(5,3,1,2)(2,0)$. Continuing in this manner, we get a series of chain graphs that are strongly indexable as well as another series of 2-strongly indexable graphs. Hence, we can have the block K_4 as many times as we wish yielding the chain graphs $CK(n;(3,4,4,\ldots,4))$ that are strongly indexable as also the chain graphs $CK(n;(3,4,4,\ldots,4,2))$ that are 2-strongly indexable. (See Figure 11). ■

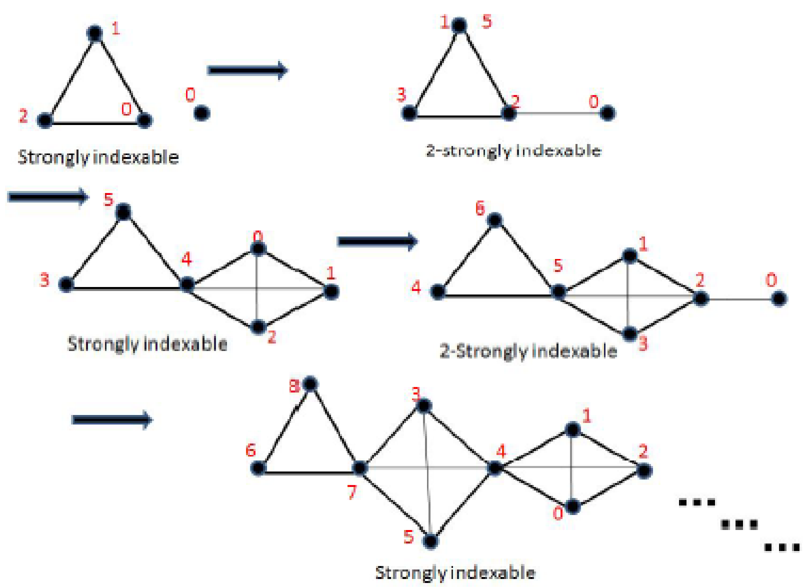

Figure 11:

Remark 5.14. *As stated in Remark 5.12, for* $n \geq 3$*, we can have similar construction of chain graphs* $CK(n;(3,4,4,\ldots,4))$ *,which are strongly indexable and then from* $CK(n;(3,4,4,\ldots,4))$ *one can obtain the chain graph* $CK(n;(3,4,4,\ldots,4,2))$ *which is* 2*-strongly indexable and then from* $CK(n;(3,4,4,\ldots,4,2))$ *a strongly indexable chain graph* $CK(n;(3,4,4,\ldots,4,4))$*, and so on. Figure 11 illustrates this construction. Proceeding like this, we can get an ascending chain*

$\mathsf{C}: C_1 \subset C_1' \subset C_2 \subset C_2' \subset \ldots, \ldots$ *where* C_i *is strongly indexable and* C_i' *is* 2*-strongly indexable graph for each positive integer* i. *The same construction can be done by taking a* 2*-strongly indexable chain graph* $CK(n;(3,4,4,\ldots,4,2))$ *as the initial graph, instead of the strongly indexable chain graph* $CK(n;(3,4,4,\ldots,4))$.

Theorem 5.15. *For* $n \geq 3$, *the graphs* $CK(n;(2,3,4,4,\ldots,))$ *are strongly indexable.*

Proof. We start with $CK(3;(2,3,4))$ together with one of its strong indexers given by the string $(5,3)(3,6,4)(4,2,0,1)$. Add 3 to each of the numbers in this string and append the sequence $(4,2,0,1)$ at the end of the string to obtain the strong indexer of $CK(4;(2,3,4,4))$ given by the string $(8,6)(6,9,7)(7,5,3,4)(4,2,0,1)$. Again, add 3 to each of the numbers and append the sequence $(4,2,0,1)$ at the end of the string to get a strong indexer of chain graph $CK(5;(2,3,4,4,4))$ given by the string $(11,9)\,(9,12,10)\,(10,8,6,7)\,(7,5,3,4)\,(4,2,0,1)$. Proceeding like this, we get a strong indexer of $CK(n;(2,3,4,4,\ldots,))$. (See, Figure 12) ■

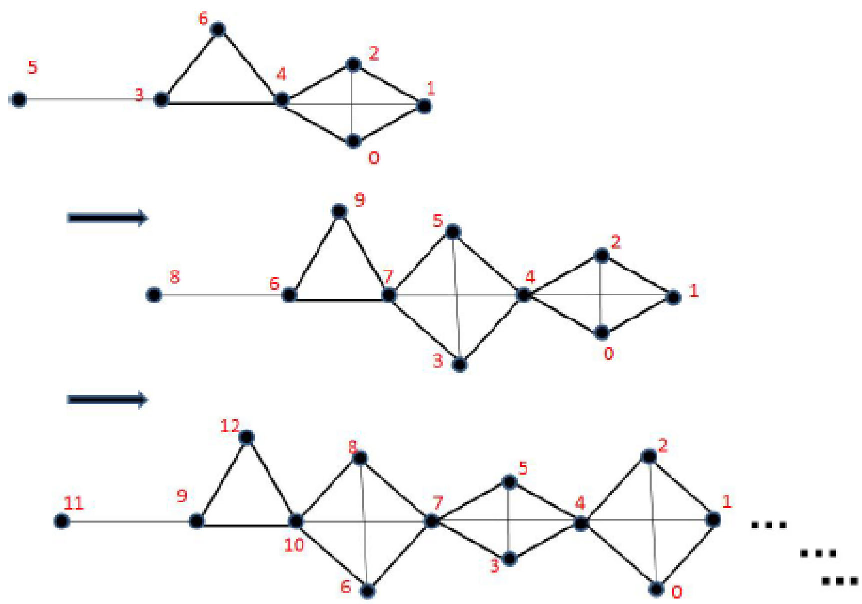

Figure 12:

Theorem 5.16. *For* $n \geq 3$, *the graphs* $CK(n;(4,2,4,4,\ldots,))$ *are strongly indexable.*

Proof. We start with the chain graph $CK(3;(4,2,4))$ together with one of its strong indexers given by the string $(7,6,5,3)(3,4)(4,2,0,1)$. Add 3 to each of the numbers in this string and append the sequence $(4,2,0,1)$ at its end to get the strong indexer of $CK(4;(4,2,4,4))$ as the new string $(10,9,8,6)(6,7)\ (7,5,3,4)(4,2,0,1)$. Again, adding 3 to each of the numbers in the new string and adjoining it with the last sequence $(4,2,0,1)$ we get the strong indexer for $CK(5;(4,2,4,4,4))$ given by the new string

$$(13,12,11,9)(9,10)(10,8,6,7)(7,5,3,4)(4,2,0,1).$$

Continuing in this manner, we get a sequence of strongly indexable chain graphs $CK(n;(4,2,4,4,\ldots,))$ such that for all $i \geq 3$,

$$CK(i;(4,2,4,4,4,\ldots)) \subset CK(i+1;(4,2,4,4,\ldots)).$$

(See, Figure 13). ■

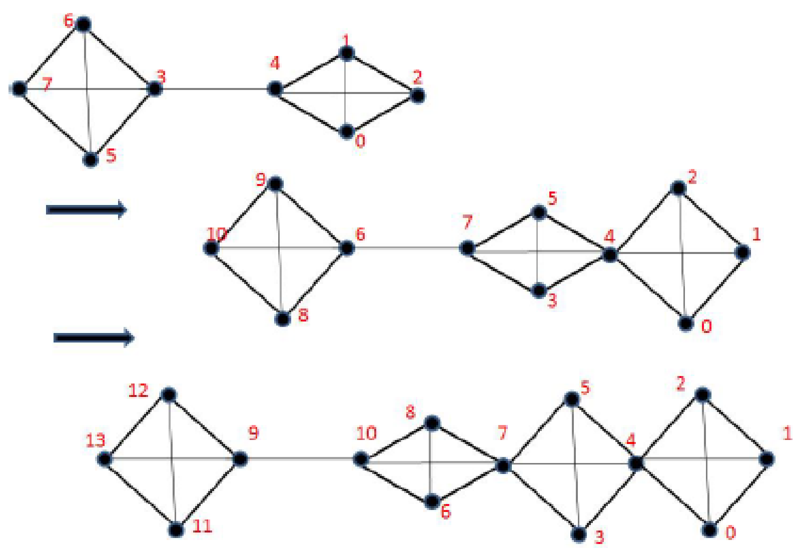

Figure 13:

Theorem 5.17. *For $n \geq 3$, the graphs $CK(n;(4,3,4,4,\ldots,))$ are strongly indexable.*

Proof. Consider the strong indexer of the chain graph $CK(3;(4,3,4))$ given by the string $(8,6,4,7)(7,2,0)\ \ (0,3,1,5)$. Add 3 to each of the numbers in this string and append the sequence $(4,2,0,1)$ at the end of it to get the strong indexer of $CK(4;(4,3,4,4))$ given by the new string $(11,9,7,10)(10,5,3)(3,8,6,4)(4,2,0,1)$. Hence, adding 3 to each of the numbers of the above new string and appending $(4,2,0,1)$ as the last sequence we get the strong indexer for $CK(5;(4,3,4,4,4))$ given by the new string $(14,12,10,13)(13,8,6),(6,11,9,7)(7,5,3,4)\ \ \ (4,2,0,1)$. We can continue this labeling procedure to obtain a strong indexer of $CK(n;(4,3,4,4,\ldots,))$ in general. (See Figure 14). ■

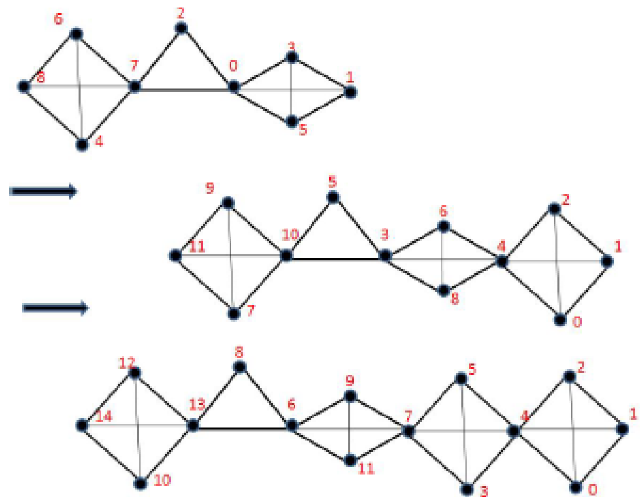

Figure 14:

Theorem 5.18. *For* $n > 4$*, the graphs* $CH(n;(2,3,3,5,4,\ldots,4))$ *are strongly indexable.*

Proof. Consider the strong indexer of the chain graph $CK(4;(2,3,3,5))$ given by the string

$$(5,8)(8,6,9)(9,3,7)(7,4,2,0,1).$$

Then add 3 to each of the numbers in this string and append $(4,2,0,1)$ as the last sequence to it so that we get a strong indexer for $CK(5;(2,3,3,5,4))$ as the new string

$$(8,11)(11,9,12)(12,6,10)(10,7,5,3,4)(4,2,0,1).$$

We can continue this procedure to get the strong indexer of $CK(n;(2,3,3,5,4,\ldots,4))$ in general as shown in Figure 15. ■

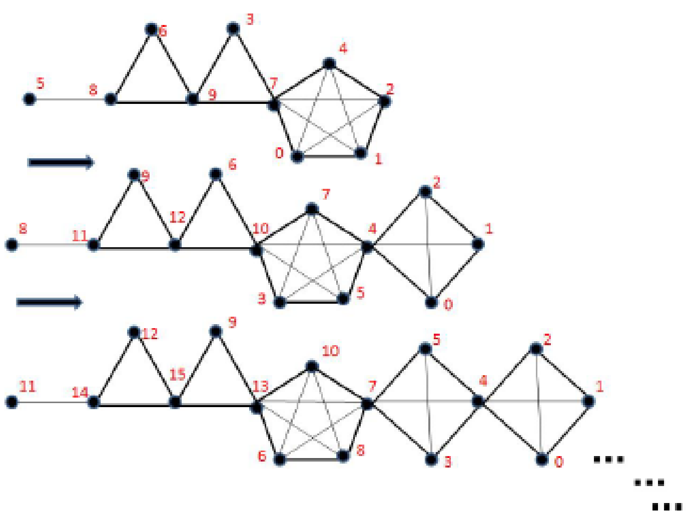

Figure 15:

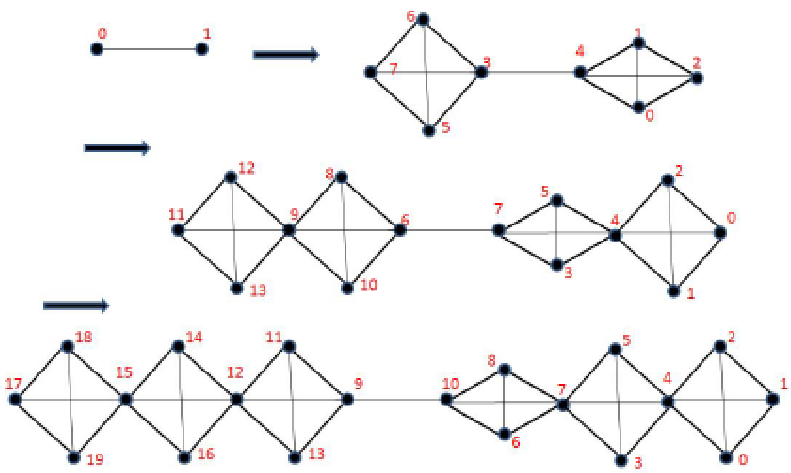

Figure 16:

Remark 5.19. *Another construction of classes of strongly indexable Husimi chains is as follows: Start with the strongly indexable graph* $C_1 = K_2$*; from* C_1 *construct* $C_2 = CK(2;(4,4))$ *and then from* C_2 *to* $C_3 = CK(5;(4,4,2,4,4))$*, and from* C_3 *construct* $C_4 = CK(7;(4,4,4,2,4,4,4))$*. In general, following the same pattern of construction, given* C_i *we can construct* C_{i+1}*. The construction is illustrated in Figure 16.*

6. Deficiency of strongly indexable graphs

Kotzig and Rosa [14] defined the magic deficiency of a graph G, denoted $\mu(G)$ as the minimum number of isolated vertices that we have to union with G so that the resulting graph is magic. If G is magic then $\mu(G)$ is defined to be zero. Figueroa *et.al* [9] extended this concept to super edge-magic labeling and calculated the deficiency of many classes of super edge-magic graphs. Hegde [11] defined this deficiency as *vertex dependent characteristic* of graphs for k-strongly indexable graphs.

Definition 6.1 [11]. *The* k*-vertex dependent characteristic of a graph* G*, denoted by* $d_c^k(G)$*, is the minimum number of isolated vertices needed to be added to* G *so that the resulting disconnected graph is strongly* k*-indexed. If a graph* G *is not strongly* k*-indexable by adding any number of isolated vertices then* $d_c^k(G)$ *is defined to be infinity and if* G *is strongly* k*-indexable then* $d_c^k(G)$ *is postulated to be* $=0$*. (In the notations here,* k *will be omitted if* $k=1$*.)*

The following results are due to Hegde [11].

Theorem 6.2[11]. *For any* $k>1$*, the* k*-vertex dependent characteristic of* C_n *is given by*

$$d_c^k(C_n) = \begin{cases} 1 & \text{if } n \equiv 0\ (mod\ 4) \\ 0 & \text{if } n \text{ is odd} \\ \infty & \text{if } n \equiv 2\ (mod\ 4). \end{cases}$$

Theorem 6.3 [11]. *For any $k > 1$, the k-vertex dependent characteristic of K_n is*

$$d_c^k(K_n) = \begin{cases} 0 & \text{if } n = 1,2,3 \\ 1 & \text{if } n = 4 \\ \infty & \text{if } n \geq 5. \end{cases}$$

Theorem 6.4 [11]. *For any $k > 1$, the k-vertex dependent characteristic of $K_{m,n} \leq (m-1)(n-1)$.*

Invoking Lemma 1.4 one can conclude that the deficiency of super-edge magic graphs and the vertex dependent characteristic of a k-strongly indexable graphs (for $k \geq 2$) are same. In this section, we initiate a study of the deficiency (or vertex dependent characteristic) of a strongly indexable graph G; we shall call it *strongly indexable deficiency* of G and denote it by $d_s(G)$. If a graph G is not strongly indexable by adding any number of isolated vertices then $d_s(G)$ is defined to be infinity (∞) and if G is strongly indexable then $d_s(G)$ is taken to be zero (0). Following observations are immediate.

Observation 6.5. *The strongly indexable deficiency of the cycle C_n is given by*

$$d_s(C_n) = \begin{cases} 0 & \text{if } n = 3 \\ \infty & \text{if } n \geq 4. \end{cases}$$

Observation 6.6. *The strongly indexable deficiency of the complete bipartite graph $K_{m,n}$ is given by*

$$d_s(K_{m,n}) = \begin{cases} 0 & \text{if } K_{m,n} \text{ is a star} \\ \infty & \text{if either } m \text{ or } n \geq 2. \end{cases}$$

Theorem 6.7. *The strongly indexable deficiency of the complete graph K_n is given by*

$$d_s(K_n) = \begin{cases} 0 & \text{if } n = 1,2,3 \\ 1 & \text{if } n = 4 \\ \infty & \text{if } n \geq 5. \end{cases}$$

Proof. Since K_1, K_2, K_3 and $K_4 \cup K_1$ are strongly indexable, $d_s(K_n) = 0 = d_s(K_4 \cup K_1)$ for $n = 1,2,3$. It is easy to verify that $d_s(K_4) = 1$.

Next, if possible let $K_n \cup mK_1$, $n \geq 5$ be strongly indexable for some m. Then, there exists a strong indexer

$$f : V(K_n \cup mK_1) \rightarrow \{0,1,2,3,\ldots,m+n-1\}$$

such that $f^+(E(K_n \cup mK_1)) = \{1,2,3,\ldots,nC_2\}$. Clearly $0,1,2 \in f(V(K_n))$. Without loss of generality, assume that $f(v_1) = 0, f(v_2) = 1$ and $f(v_3) = 2$. Now, $f(v_1v_2) = 3 \Rightarrow 3 \notin f(V(K_n))$, which in turn implies $4 \in f(V(K_n)) \Rightarrow 4,5,6 \in f^+(E(K_n))$. Hence $5,6 \notin f(V(K_n))$ so that 7 should necessarily be in $f(V(K_n))$, which implies $7,8,9,11 \in f^+(E(K_n))$ and hence $10 \notin f^+(E(K_n))$. Proceeding like this, we see that irrespective of how many number of isolated vertices are added to K_n, K_n cannot be made strongly indexable. Hence, for $n \geq 5, d_s(K_n) = \infty$.■

Since the only strongly indexable trees are stars, $K_{1,n}$, and we cannot have a strong indexer of a tree other than stars by adding any number of isolated vertices, we have the following theorem.

Theorem 6.8. *The strongly indexable deficiency of a tree* T *is is given by*

$$d_s(T) = \begin{cases} 0 & \textit{if } T \cong K_{1,n} \\ \infty & \textit{otherwise.} \end{cases}$$

7. Conclusions and scope

We have generated infinitely many classes of strongly indexable Husimi chains. It is worth investigating in general which Husimi trees are strongly indexable. Invoking Corollary 5.2, one can explore the classes of Eulerian graphs that are strongly indexable. An infinite class of Eulerian graphs that are not strongly indexable is the class of *Husimi trees* (viz., connected separable graphs in which every block is a triangle) in which the number of blocks is $m \equiv 3(mod\ 4)$ since, in such a graph $H, |E(H)| \equiv 1(mod\ 4)$. One such well known class is that of ``friendship graphs" $F_t := tK_2 + K_1$, which consists of t triangles glued at one common vertex whence F_t consists of $q = 3t$ edges, so that $t \equiv 2\ or\ 3(mod\ 4)$ yielding $q \equiv 2\ or\ 1(mod\ 4)$ in the respective cases. We conjecture that "Any Husimi tree consisting of $t \equiv 0(mod\ 4)$ triangle blocks is strongly indexable". One major difference between magic deficiency and strongly indexable deficiency is that magic deficiency is finite for almost all classes of graphs, while strongly indexable deficiency need not necessarily be finite for abundantly many graphs. It is worth studying the conditions that guarantee infinite strongly indexable deficiency and also to characterize the classers of graphs having finite strongly indexable deficiency.

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