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Induced Generalized Weighted Bonferroni Mean Operator for Intuitionistic Fuzzy Sets

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Abstract. By extending the Bonferroni mean operator to the intuitionistic fuzzy situation, we introduce a new kind of aggregating operator: the induced generalized weighted Bonferroni mean(I-GWBM) operator. The I-GWBM operator includes many famous aggregation operators for intuitionistic fuzzy sets(IFSs) as special cases. Then, we generalize the I-GWBM operator by using quasi-arithmetic means and propose a more general aggregating operator: the induced quasi-weighted Bonferroni mean(I-Quasi-WBM) operator. We also study their desirable properties, such as idempotency, commutativity, monotonicity and boundary, and examine their special cases.

Keywords. Group multiple attribute decision making; Intuitionistic fuzzy number; Induced generalized weighted Bonferroni mean operator

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1 Introduction

Intuitionistic fuzzy set(IFS), developed by Atanassov[2], is a powerful tool to deal with vagueness. To aggregate the information expressed by some IFSs into a collective one, many authors have proposed various aggregation operators. Yager[3] proposed the induced ordered weighted averaging(IOWA) operator, which represents an extension of the famous OWA operator. Xu[4] developed the intuitionistic fuzzy ordered weighted averaging(IFOWA) operator, and the intuitionistic fuzzy hybrid averaging(IFHA) operator. Recently, Xu et.al.[5] presented the induced generalized aggregation (I-GIFOWA) operator for IFSs, which provides a very general formulation that includes most aggregation operators for IFSs as special cases. However, like most arithmetic mean based aggregation operators, this type operator also cannot capture the interrelationship of the input arguments. On the other hand, Bonferroni mean(BM)

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which was proposed by Bonferroni[1] is another important mean, and its most fundamental characteristic is that it can capture the interrelationship of the input arguments, which makes it very useful in various application fields, such as decision making, information retrieval, pattern recognition, data mining, and medical diagnosis[4]. Therefore, it is meaningful to incorporate the BM into the I-GIFOWA proposed by Xu et.al.[5] and propose more general aggregation operator formulation for IVFs which can include the I-GIFOWA operator as special case.

In order to do so, this paper is structured as follows. In the next section, we introduce some basic concepts related to intuitionistic fuzzy numbers and some operational laws of intuitionistic fuzzy numbers. Then we develop the I-GWBM operator with intuitionisticfuzzy numbers and study its desirable properties, such as commutativity, boundary, idempotency and monotonicity in Section 3. We also introduce the I-Quasi-WBM operator in this section. We conclude the paper in the last section.

2 Preliminaries

In this section, we briefly describe some concepts and basic operational laws related to intuitionistic fuzzy numbers.

Definition 1[2] Let X be a universe of discourse. An intuitionistic fuzzy set(IFS) in X is an object having the form:

$$A = \{\langle x, \mu_A(x), v_A(x)\rangle | x \in X\},$$

where

$$\mu_A : X \to [0,1], v_A : X \to [0,1]$$

with the condition

$$0 \le \mu_A(x) + v_A(x) \le 1, \forall x \in X.$$

The numbers $\mu_A(x)$ and $v_A(x)$ denote the degree of membership and non-membership of x to A, respectively. For convenience, we call $\alpha = (\mu_\alpha, v_\alpha)$ an intuitionistic fuzzy number(IFN)[3], where $\mu_\alpha \in [0,1], v_\alpha \in [0,1]$ and $\mu_\alpha + v_\alpha \le 1$, and we let Ω be the set of all intuitionistic fuzzy numbers.

To aggregate intuitionistic preference information, Xu[6] defined the following operations:

Definition 2 [6]. Let $\alpha = (\mu_\alpha, v_\alpha)$ and $\beta = (\mu_\beta, v_\beta)$ be two intuitionistic fuzzy numbers, then

(1) $\alpha \oplus \beta = (\mu_\alpha + \mu_\beta - \mu_\alpha \cdot \mu_\beta, v_\alpha \cdot v_\beta)$.

(2) $\alpha \otimes \beta = (\mu_\alpha \cdot \mu_\beta, v_\alpha + v_\beta - v_\alpha \cdot v_\beta)$.

(3) $\lambda\alpha = (1 - (1 - \mu_\alpha)^\lambda, v_\alpha^\lambda), \lambda > 0$.

(4) $\alpha^\lambda = (\mu_\alpha^\lambda, 1 - (1 - v_\alpha)^\lambda), \lambda > 0$.

We can easily to get the following relationships among the operational laws (1-4):

Theorem 1.[6] Let α, β and γ be three IFNs, then

(1) $\lambda(\alpha \oplus \beta) = (\lambda\alpha) \oplus (\lambda\beta)$.

(2) $(\alpha \otimes \beta)^{\lambda} = \alpha^{\lambda} \otimes \beta^{\lambda}$.

(3) $(\lambda + \mu)\gamma = \lambda\gamma \oplus \mu\gamma$.

(4) $\gamma^{\lambda} \otimes \gamma^{\mu} = \gamma^{\lambda+\mu}$.

The induced ordered weighted averaging(IOWA) operator was developed by Yager in 1999, which can be defined in the following:

Definition 3 [3] An IOWA operator of dimension n is a mapping IOWA: $R^n \to R$, defined by an associated weighting vector $W = (w_1, w_2, \cdots, w_n)^{\top}$, such that $\sum_{i=1}^{n} w_i = 1$ and $w_i \in [0, 1]$, and a set of order-inducing variable d_i,according to the following formula:

$$\text{IOWA}(\langle d_1, a_1\rangle, \langle d_2, a_2\rangle, \cdots, \langle d_n, a_n\rangle) = \sum_{j=1}^{n} w_j b_j, \tag{1}$$

where $(b_1, b_2, \cdots, b_n)$ is simply $a_1, a_2, \cdots, a_n$ reordered in decreasing order of the values of the d_i.

The Bonferroni mean(BM) was originally proposed by Bonferroni[1] and was investigated intensively by Yager[7]. It is noted that original BM doesn't consider the situation that $i = j$, and the weight vector of the aggregated arguments is not also considered. Therefore, Xia et al.[8] defined the following weighted version of the BM with the arithmetic mean.

Definition 4 [8]. Let $p, q \geq 0$ and $a_i (i = 1, 2, \cdots, n)$ be a collection of nonnegative numbers with the weight vector $w = (w_1, w_2, \cdots, w_n)^{\top}$ such that $w_i \geq 0 (i = 1, 2, \cdots, n)$ and $\sum_{i=1}^{n} w_i = 1$. If

$$\text{WBM}^{p,q}(a_1, a_2, \cdots, a_n) = (\sum_{i,j=1}^{n} w_i w_j a_i^p a_j^q)^{\frac{1}{p+q}}, \tag{2}$$

then $\text{WBM}^{p,q}$ is called the weighted Bonferroni Mean(WBM) operator.

The generalize ordered weighted averaging(GOWA) operator proposed by Yager[6] is a generalization of the ordered weighted averaging(OWA) operator [7] by using generalized means. It can be defined as follows.

Definition 5 [7]. A GOWA operator of dimension n is a mapping GOWA: $R^n \to R$ that has an associated weight vector $w = (w_1, w_2, \cdots, w_n)^{\top}$ such that $w_j \geq 0$ and $\sum_{j=1}^{n} w_j = 1$. Furthermore,

$$\text{GOWA}(a_1, a_2, \cdots, a_n) = (\sum_{j=1}^{n} w_j a_{\sigma(j)}^{\lambda})^{\frac{1}{\lambda}},$$

where $\lambda \neq 0$, and $(\sigma(1), \sigma(2), \cdots, \sigma(n))$ is a permutation of $(1, 2, \cdots, n)$, such that $a_{\sigma(j-1)} \geq a_{\sigma(j)}$ for all $j = 2, \cdots, n$.

Note that, the above operators can be easily generalized to aggregate IFSs information.

3 The Induced Generalized Weighted Bonferroni Mean Operator

In this section, we are going to consider two generalizations of the BM operator where the input arguments are intuitionistic fuzzy information: the induced generalized weighted Bonferroni mean(I-GWBM) operator and the induced quasi generalized weighted Bonferroni mean(I-Quasi-GWBM) operator. The I-GWBM is an extension of the BM operator that uses generalized means and uncertain information that can be represented by using intuitionistic fuzzy information. With this generalization, we include in the same formulation a lot of aggregation operators such as the IOWA operator, the WBM operator, and the GOWA operator defined in Definitions 3-5. It can be defined as follows:

Definition 6. Let I-GWBM: $\Omega^n \to \Omega$, if I-GWBM

$$\mathrm{I-GWBM}^{p,q}(\langle d_1,\alpha_1\rangle,\langle d_2,\alpha_2\rangle,\cdots,\langle d_n,\alpha_n\rangle)=(\oplus_{i,j=1}^{n}w_iw_j(\alpha_{\sigma(i)}^{p}\alpha_{\sigma(j)}^{q})^{\lambda})^{\frac{1}{\lambda(p+q)}}, \quad (3)$$

where $w=(w_1,w_2,\cdots,w_n)^\top$ is the weight vector of $(\alpha_1,\alpha_2,\cdots,\alpha_n)$, with $w_i\geq 0$ and $\sum_{i=1}^{n}w_i=1$. The parameter $\lambda\neq 0$, and $\alpha_{\sigma(j)}=(\mu_{\alpha_{\sigma(j)}},v_{\alpha_{\sigma(j)}})$ is α_i reordered in decreasing order of the value of the d_i. d_i in $\langle d_i,\alpha_i\rangle$ is referred to as the order inducing variable, and $\alpha_i=(\mu_{\alpha_i},v_{\alpha_i})$ as the intuitionistic fuzzy numbers. Then I-GWBM is called the induced generalized weighted Bonferroni mean operator.

Theorem 1. Let $\alpha_j=(\mu_{\alpha_j},v_{\alpha_j})(j=1,2,\cdots,n)$ be a collection of intuitionistic fuzzy numbers, then their aggregated value by using the I-GWBM operator is also an intuitionistic fuzzy number, and

$$\mathrm{I-GWBM}^{p,q}(\langle d_1,\alpha_1\rangle,\langle d_2,\alpha_2\rangle,\cdots,\langle d_n,\alpha_n\rangle)=((1-\prod_{i,j=1}^{n}(1-\mu_{\alpha_{\sigma(i)}}^{\lambda p}\mu_{\alpha_{\sigma(j)}}^{\lambda q})^{w_iw_j})^{\frac{1}{\lambda(p+q)}},$$

$$1-(1-\prod_{i,j=1}^{n}(1-(1-v_{\alpha_{\sigma(i)}})^{\lambda p}(1-v_{\alpha_{\sigma(j)}})^{\lambda q})^{w_iw_j})^{\frac{1}{\lambda(p+q)}}), \quad (4)$$

where $\lambda\neq 0$, and $w=(w_1,w_2,\cdots,w_n)^\top$ is the weighting vector of $(\alpha_1,\alpha_2,\cdots,\alpha_n)$, with $w_i\geq 0$ and $\sum_{i=1}^{n}w_i=1$.

Proof. The proof is similar to that of the Theorem 2[8](Omitted).

Example 1. Assume the following collection of arguments with their respective order-inducing variable $\langle d_j,\alpha_j\rangle$:

$$\langle d_1,\alpha_1\rangle=\langle 0.5,(0.3,0.6)\rangle,\langle d_2,\alpha_2\rangle=\langle 0.4,(0.4,0.5)\rangle,$$

$$\langle d_3,\alpha_3\rangle=\langle(0.8,(0.7,0.1)\rangle,\langle d_4,\alpha_4\rangle=\langle 0.2,(0.1,0.6)\rangle.$$

If we assume that $w=(0.2,0.1,0.4,0.3)^\top$, and $\lambda=2,p=q=1$, then

$$\langle d_{\sigma(1)},\alpha_{\sigma(1)}\rangle=\langle 0.8,(0.7,0.1)\rangle,\langle d_{\sigma(2)},\alpha_{\sigma(2)}\rangle=\langle 0.5,(0.3,0.6)\rangle,$$

$$\langle d_{\sigma(3)}, \alpha_{\sigma(3)}\rangle = \langle 0.4, (0.4, 0.5)\rangle, \langle d_{\sigma(4)}, \alpha_{\sigma(4)}\rangle = \langle 0.2, (0.1, 0.6)\rangle.$$

This ordering includes the ordered intuitionistic fuzzy arguments:

$$\alpha_{\sigma(1)} = (0.7, 0.1), \alpha_{\sigma(2)} = \langle 0.5, (0.3, 0.6)\rangle, \alpha_{\sigma(3)} = \langle 0.4, (0.4, 0.5)\rangle, \alpha_{\sigma(4)} = \langle 0.2, (0.1, 0.6)\rangle$$

and then we get the aggregated value is:

$$\mathrm{I-GWBM}^{1,1}(\langle d_1, \alpha_1\rangle, \langle d_2, \alpha_2\rangle, \langle d_3, \alpha_3\rangle, \langle d_4, \alpha_4\rangle) = (0.4222, 0.4106)$$

If we analyze different values of the order inducing variable d_i, the parameter λ and p, q, we obtain a group of particular cases of the I-GWBM operator.

1. If $d_1 \geq d_2 \geq \cdots \geq d_n$, then the I-GWBM operator is reduced to

$$\mathrm{GWBM}^{p,q}(\langle d_1, \alpha_1\rangle, \langle d_2, \alpha_2\rangle, \cdots, \langle d_n, \alpha_n\rangle) = (\oplus_{i,j=1}^{n} w_i w_j (\alpha_i^p \alpha_j^q)^{\lambda})^{\frac{1}{\lambda(p+q)}}, \tag{5}$$

which is called the generalized weighted Bonferroni mean(GWBM) operator. Especially, if $\lambda = 1$, then the GWBM operator is further reduced to

$$\mathrm{WBM}^{p,q}(\langle d_1, \alpha_1\rangle, \langle d_2, \alpha_2\rangle, \cdots, \langle d_n, \alpha_n\rangle) = (\oplus_{i,j=1}^{n} w_i w_j \alpha_i^p \alpha_j^q)^{\frac{1}{p+q}}, \tag{6}$$

which is called the weighted Bonferroni mean(WBM) operator in [8].

2. If the 2-tuple $\langle d_i, \alpha_i\rangle$ can be expressed by $\langle f(\alpha_i), \alpha_i\rangle$, where f is an increasing function, then the I-GWBM operator is reduced to

$$\mathrm{GOWBM}^{p,q}(\langle d_1, \alpha_1\rangle, \langle d_2, \alpha_2\rangle, \cdots, \langle d_n, \alpha_n\rangle) = (\oplus_{i,j=1}^{n} w_i w_j \alpha_{\sigma(i)}^p \alpha_{\sigma(j)}^q)^{\frac{1}{p+q}}, \tag{7}$$

where $\alpha_{\sigma(i)}$ is the ith largest of α_i, which is called the generalized ordered weighted averaging operator.

3. If $p = 1, q = 0$, then the I-GWBM operator is reduced to the following:

$$\mathrm{I-GIFOWA}(\langle d_1, \alpha_1\rangle, \langle d_2, \alpha_2\rangle, \cdots, \langle d_n, \alpha_n\rangle) = (\oplus_{i,j=1}^{n} w_i w_j \alpha_{\sigma(i)}^{\lambda})^{\frac{1}{\lambda}} = (\oplus_{i=1}^{n} w_i \alpha_{\sigma(i)}^{\lambda})^{\frac{1}{\lambda}}, \tag{8}$$

which is called the induced generalized intuitionistic fuzzy ordered weighted averaging(I-GIFOWA) operator in [5]. Therefore, the proposed I-GWBM operator includes all the special cases of the I-GIFOWA operator.

Example 3. Assume the following collection of arguments with their respective order-inducing variable $\langle d_j, \alpha_j\rangle$:

$$\langle d_1, \alpha_1\rangle = \langle 0.5, (0.48, 0.32)\rangle, \langle d_2, \alpha_2\rangle = \langle 0.6, (0.73, 0.15)\rangle,$$

$$\langle d_3, \alpha_3\rangle = \langle(0.4, (0.23, 0.65)\rangle, \langle d_4, \alpha_4\rangle = \langle 0.3, (0.53, 0.32)\rangle.$$

If we assume that $w=(0.2,0.3,0.3,0.2)^\top$, and $\lambda=1,p=1,q=0$, then

$$\mathrm{I-GIFOWA}^{p,q}(\langle d_1,\alpha_1\rangle,\langle d_2,\alpha_2\rangle,\langle d_3,\alpha_3\rangle,\langle d_4,\alpha_4\rangle)=(0.5290,0.3153).$$

By considering different types of weighting vector w, the GOWBM operator also has the following special cases:

4. If $w=(1,0,\cdots,0)^\top$, then

$$\mathrm{I-GWBM}^{p,q}(\langle d_1,\alpha_1\rangle,\langle d_2,\alpha_2\rangle,\cdots,\langle d_n,\alpha_n\rangle)=(\alpha_{\sigma(j)}^{\lambda(p+q)})^{\frac{1}{\lambda(p+q)}}=\alpha_{\sigma(j)},$$

where j is the lower index of d_j, and $d_j=\max_i\{d_i\}$. Especially, if $\alpha_j=\max_i\{\alpha_i\}$, then

$$\mathrm{I-GWBM}^{p,q}(\langle d_1,\alpha_1\rangle,\langle d_2,\alpha_2\rangle,\cdots,\langle d_n,\alpha_n\rangle)=\max_i\{\alpha_i\}.$$

On the contrary, if $\alpha_j=\min_i\{\alpha_i\}$, then

$$\mathrm{I-GWBM}^{p,q}(\langle d_1,\alpha_1\rangle,\langle d_2,\alpha_2\rangle,\cdots,\langle d_n,\alpha_n\rangle)=\min_i\{\alpha_i\}.$$

5. If $w=(0,0,\cdots,1)^\top$, then

$$\mathrm{I-GWBM}^{p,q}(\langle d_1,\alpha_1\rangle,\langle d_2,\alpha_2\rangle,\cdots,\langle d_n,\alpha_n\rangle)=(\alpha_{\sigma(j)}^{\lambda(p+q)})^{\frac{1}{\lambda(p+q)}}=\alpha_{\sigma(j)},$$

where j is the lower index of d_j, and $d_j=\min_i\{d_i\}$. Especially, if $\alpha_j=\max_i\{\alpha_i\}$, then

$$\mathrm{I-GWBM}^{p,q}(\langle d_1,\alpha_1\rangle,\langle d_2,\alpha_2\rangle,\cdots,\langle d_n,\alpha_n\rangle)=\max_i\{\alpha_i\}.$$

On the contrary, if $\alpha_j=\min_i\{\alpha_i\}$, then

$$\mathrm{I-GWBM}^{p,q}(\langle d_1,\alpha_1\rangle,\langle d_2,\alpha_2\rangle,\cdots,\langle d_n,\alpha_n\rangle)=\min_i\{\alpha_i\}.$$

6. If $w=(1/n,1/n,\cdots,1/n)^\top$, then

$$\mathrm{I-GWBM}^{p,q}(\langle d_1,\alpha_1\rangle,\langle d_2,\alpha_2\rangle,\cdots,\langle d_n,\alpha_n\rangle)=(\frac{1}{n^2}\bigoplus_{i,j=1}^{n}(\alpha_{\sigma(i)}^{p}\alpha_{\sigma(j)}^{q})^{\lambda})^{\frac{1}{\lambda(p+q)}}.$$

7. If $\lambda\to0$, we get the induced weighted Bonferroni geometric mean(I-WBGM) operator:

$$\mathrm{I-WBGM}^{p,q}(\langle d_1,\alpha_1\rangle,\langle d_2,\alpha_2\rangle,\cdots,\langle d_n,\alpha_n\rangle)=\otimes_{i,j=1}^{n}(\alpha_{\sigma(i)}^{p}\alpha_{\sigma(j)}^{q})^{\frac{w_iw_j}{p+q}} \tag{9}$$

$$=(1-(1-\prod_{i,j=1}^{n}(1-(1-\mu_{\alpha_{\sigma(i)}})^p(1-\mu_{\alpha_{\sigma(j)}})^q)^{w_iw_j})^{\frac{1}{p+q}},$$

$$(1-\prod_{i,j=1}^{n}(1-v_{\alpha_{\sigma(i)}}^{p}v_{\alpha_{\sigma(j)}}^{q})^{w_iw_j})^{\frac{1}{p+q}})$$

8. If $\lambda \to +\infty$, we get the max operator in the Bonferroni mean:

$$\max^{p,q}(\langle d_1, \alpha_1\rangle, \langle d_2, \alpha_2\rangle, \cdots, \langle d_n, \alpha_n\rangle) = \max_{i,j}(\alpha_{\sigma(i)}^p \alpha_{\sigma(j)}^q)^{\frac{1}{p+q}}.$$

The I-GWBM operator has the following properties:

Theorem 2. (Idempotency) If all $\alpha_i (i = 1, 2, \cdots, n)$ are equal, i.e., $\alpha_i = \alpha$ for all i, then

$$\mathrm{I-GWBM}^{p,q}(\langle d_1, \alpha_1\rangle, \langle d_2, \alpha_2\rangle, \cdots, \langle d_n, \alpha_n\rangle) = \alpha.$$

Proof. Since $\alpha_i = \alpha = (\mu_\alpha, v_\alpha)(i = 1, 2, \cdots, n)$, we have

$$\mathrm{I-GWBM}^{p,q}(\langle d_1, \alpha_1\rangle, \langle d_2, \alpha_2\rangle, \cdots, \langle d_n, \alpha_n\rangle) = (\sum_{i,j=1}^{n} w_i w_j (\alpha^p \alpha^q)^\lambda)^{\frac{1}{\lambda(p+q)}}$$

$$= ((\sum_{i=1}^{n} w_i)(\sum_{j=1}^{n} w_j)\alpha^{\lambda(p+q)})^{\frac{1}{\lambda(p+q)}} = \alpha,$$

which completes the proof.

Theorem 3. (Commutativity)

$$\mathrm{I-GWBM}^{p,q}(\langle d_1, \alpha_1\rangle, \langle d_2, \alpha_2\rangle, \cdots, \langle d_n, \alpha_n\rangle) = \mathrm{I-GWBM}^{p,q}(\langle d_1', \alpha_1'\rangle, \langle d_2', \alpha_2'\rangle, \cdots, \langle d_n', \alpha_n'\rangle),$$

where $(\langle d_1', \alpha_1'\rangle, \langle d_2', \alpha_2'\rangle, \cdots, \langle d_n', \alpha_n'\rangle)$ is any permutation of $(\langle d_1, \beta_1\rangle, \langle d_2, \beta_2\rangle, \cdots, \langle d_n, \beta_n\rangle)$.

Theorem 4. (Monotonicity) Let $\alpha_i = (\mu_{\alpha_i}, v_{\alpha_i})(i = 1, 2, \cdots, n)$ and $\beta_i = (\mu_{\beta_i}, v_{\beta_i})(i = 1, 2, \cdots, n)$ be two collections of IFNs, if $\mu_{\alpha_i} \le \mu_{\beta_i}$ and $v_{\alpha_i} \ge v_{\beta_i}$, for all i, then

$$\mathrm{I-GWBM}^{p,q}(\langle d_1, \alpha_1\rangle, \langle d_2, \alpha_2\rangle, \cdots, \langle d_n, \alpha_n\rangle) \le \mathrm{I-GWBM}^{p,q}(\langle d_1, \alpha_1\rangle, \langle d_2, \alpha_2\rangle, \cdots, \langle d_n, \alpha_n\rangle).$$

Theorem 5. (Boundary) Let $\alpha_i = (\mu_{\alpha_i}, v_{\alpha_i})(i = 1, 2, \cdots, n)$ be a collection of IFNs, and

$$\alpha^- = (\min_i\{\mu_{\alpha_i}\}, \max_i\{v_{\alpha_i}\}), \alpha^+ = (\max_i\{\mu_{\alpha_i}\}, \min_i\{v_{\alpha_i}\})$$

then

$$\alpha^- \le \mathrm{I-GWBM}^{p,q}(\langle d_1, \alpha_1\rangle, \langle d_2, \alpha_2\rangle, \cdots, \langle d_n, \alpha_n\rangle) \le \alpha^+.$$

As it is pointed in [9], it is possible to further generalize the I-GWBM operator by using quasi-arithmetic means. In other words, if we replace the arithmetic averaging with quasi-arithmetic averaging, then the I-Quasi-GWBM operator is obtained as follows.

Definition 7. An induced quasi-arithmetic weighted Bonferroni mean operator of dimension n is a mapping I-Quasi-WBM: $\Omega^n \to \Omega$ obtained by an associated weighting vector $w = (w_1, w_2, \cdots, w_n)^\top$, with $w_i \ge 0$ and $\sum_{i=1}^n w_i = 1$, and a continuous strictly monotonic function h, shown as follows:

$$\mathrm{I-Quasi-WBM}^{p,q}(\langle d_1, \alpha_1\rangle, \langle d_2, \alpha_2\rangle, \cdots, \langle d_n, \alpha_n\rangle) = (h^{-1}(\oplus_{i,j=1}^n w_i w_j h(\alpha_{\sigma(i)}^p \alpha_{\sigma(j)}^q)))^{\frac{1}{p+q}}, \quad (10)$$

where $\alpha_{\sigma(j)} = (\mu_{\alpha_{\sigma(j)}}, v_{\alpha_{\sigma(j)}})$ is α_i reordered in decreasing order of the value of the d_i. d_i in $\langle d_i, \alpha_i \rangle$ is referred to as the order inducing variable, and $\alpha_i = (\mu_{\alpha_i}, v_{\alpha_i})$ as the intuitionistic fuzzy numbers.

By considering different types of function h, the I-Quasi-WBM operator includes many special cases, shown as follows:

1. When $h(x) = x$, then the I-Quasi-WBM operator becomes the I-GWBM operator defined by (3).
2. When $h(x) = \ln x$, then

$$
\begin{aligned}
& \mathrm{I-Quasi-WBM}^{p,q}(\langle d_1, \alpha_1 \rangle, \langle d_2, \alpha_2 \rangle, \cdots, \langle d_n, \alpha_n \rangle) \\
= & (\exp(\oplus_{i,j=1}^{n} w_i w_j \ln(\alpha_{\sigma(i)}^{p} \alpha_{\sigma(j)}^{q})))^{\frac{1}{p+q}} \\
= & (\otimes_{i,j=1}^{n} (\alpha_{\sigma(i)}^{p} \alpha_{\sigma(j)}^{q})^{w_i w_j})^{\frac{1}{p+q}} \\
= & \otimes_{i,j=1}^{n} (\alpha_{\sigma(i)}^{p} \alpha_{\sigma(j)}^{q})^{\frac{w_i w_j}{p+q}}.
\end{aligned}
$$

Therefore the I-Quasi-WBM operator is reduced to the I-WBGM operator defined by (9).

Example 3. Assume the following collection of arguments with their respective order-inducing variable $\langle d_j, \alpha_j \rangle$:

$$\langle d_1, \alpha_1 \rangle = \langle 0.6, (0.4, 0.5) \rangle, \langle d_2, \alpha_2 \rangle = \langle 0.2, (0.8, 0.1) \rangle,$$

$$\langle d_3, \alpha_3 \rangle = \langle (0.3, (0.5, 0.4) \rangle, \langle d_4, \alpha_4 \rangle = \langle 0.8, (0.2, 0.5) \rangle.$$

If we assume that $w = (0.1, 0.4, 0.4, 0.1)^{\top}$, and $h(x) = \ln x, p = q = 1$, then

$$\mathrm{I-Quasi-WBM}^{p,q}(\langle d_1, \alpha_1 \rangle, \langle d_2, \alpha_2 \rangle, \langle d_3, \alpha_3 \rangle, \langle d_4, \alpha_4 \rangle) = (0.5335, 0.3381).$$

It must be pointed out that the I-Quasi-WBM operator is not just these special situations above. Other families could be constructed by choosing different types of function h, such as $h(x) = \sin(\pi x/2), h(x) = x^t, h(x) = \gamma^{1/x}$, and so on. The I-Quasi-WBM also has some desirable properties, such as idempotency, commutativity.

4 Conclusion

The Bonferroni mean can reflect the correlations of the aggregated arguments, and suitable for aggregating the information taking the form of numerical values and intuitionistic fuzzy numbers. In this paper, motivated by the idea of Bonferroni mean and the induced operator, we propose the induced generalized weighted Bonferroni mean(I-GWBM) operator. We also developed the induced quasi-weighted Bonferroni mean(I-Quasi-WBM) operator by using the quasi-arithmetic means. Some desirable properties of the above operators have been studied, such as idempotency, commutativity, monotonicity and boundary.

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