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Three Families of Generalized Fuzzy Directed Divergence

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Abstract

In the present communication Four new measures of fuzzy directed divergence are proposed.The validity of these divergence measures is examined axiomatically. Relation of the proposed divergence measures with the cardinality of universe of discourse is established.Some properties of these divergence measures are established. Proposed generalized measures of fuzzy directed divergence are normalized.

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1 Introduction

The fuzzy set theory , proposed by [Zadeh,1965] gained vital interdisciplinary importance in many fields such as pattern recognition , image processing , fuzzy aircraft control , feature selection , Bioinformatics etc. Uncertainty and fuzziness are basic elements of the human perspective and of many real world objectives . The main use of information is to remove the uncertainty and fuzziness . The measure of uncertainty removed is information measure while the measure of vagueness is measure of fuzziness.

The entropy measure due to [Shannon,1948] changed the face of communication theory with extensive applications in several branches statistical studies found greater use of [Kullback *et al.*,1952] between two probability distributions (observed and proposed) of a random variable. [De Luca *et al.*,1972] defined fuzzy entropy corresponding to [Shannon,1948] for a fuzzy set A and many authors defined various divergence measures between fuzzy sets, some are cited in section 2.

In the present communication some preliminaries related to concept of fuzziness , probabilistic and fuzzy information and divergence are presented in section 2. In section 3 we introduce a Arithmetic -Geometric divergence measures between fuzzy sets and some properties of this divergence measures are presented .In section 4 a new one parametric measure of fuzzy directed divergence is introduced and characterized. In section 5 a unified (α, β) measure of fuzzy directed divergence is introduced and some properties are established.In section 6 generalized Arithmetic -Geometric measure of fuzzy directed divergence is introduced.In section 7 the measures of fuzzy directed divergence proposed in section 3,4,5 and 6 are normalized.In section 7 conclusion and discussion is presented.

2 Preliminaries and Review of Fuzzy Divergence Measures

2.1 Preliminaries On Fuzzy Set Theory

Definition Let a universe of discourse $X = \{x_1, x_2, x_3...x_n\}$ then a fuzzy subset of universe X is defined as

$$A = \{(x, \mu_A(x)) \mid x \in X, \mu_A(x) \in [0, 1]\}$$

Where $\mu_A(x) : X \to [0,1]$ is a membership function defined as follow:

$$\mu_A(x) = \begin{cases} 0 & \text{if x does not belong to A and there is no ambiguty} \\ 1 & \text{if x belong to A and there is no ambiguty} \\ 0.5 & \text{if there is maximum ambiguity whether x belongs to A or not} \end{cases}$$

In fact $\mu_A(x)$ associates with each $x \in X$ a grade of membership of the set A. Some notions related to fuzzy sets which we shall need in our discussion [Zadeh,1965].

Containment ; $A \subset B \Leftrightarrow \mu_A(x) \leq \mu_B(x)$ for all $x \in X$

Equality ; $A = B \Leftrightarrow \mu_A(x) = \mu_B(x)$ for all $x \in X$

Compliment ; $\bar{A}$ = Compliment of A $\Leftrightarrow \mu_{\overline{A}}(x) = 1 - \mu_A(x)$ for all $x \in X$

Union ; $A \cup B$ = Union of A and B $\Leftrightarrow \mu_{A \cup B}(x) = \max.\{\mu_A(x), \mu_B(x)\}$ for all $x \in X$

Intersection ; $A \cap B$ = Intersection of A and B $\Leftrightarrow \mu_{A \cap B}(x) = \min.\{\mu_A(x), \mu_B(x)\}$ for all $x \in X$

Product ; AB = Product of A and B $\Leftrightarrow \mu_{AB}(x) = \mu_A(x)\mu_B(x)$ for all $x \in X$

Sum; $A \oplus B$ = Sum of A and B $\Leftrightarrow \mu_{A \oplus B}(x) = \mu_A(x) + \mu_B(x) - \mu_A(x)\mu_B(x)$ for all $x \in X$

2.2 Probabilistic Measures of Directed Divergence and Corresponding Measures of Fuzzy Directed Divergence

The measure of information was defined Claude .E.Shannon in his treatise paper in 1948.

$$H(P) = \sum_{i=1}^{n} p_i \log p_i \tag{1}$$

where $\Gamma_n = \{P = (p_1, p_2, p_3 ... p_n) | p_i \geq 0, \sum_{i=1}^{n} p_i = 1, n \geq 2\}$ is the set of all complete finite discrete probability distributions.

The relative entropy is a measure of the distance between two probability distributions. In statistics, it arises as the expected logarithm of the likelihood ratio. The relative entropy $D(P, Q)$ is the measure of inefficiency of assuming that the distribution is q when the true distribution is p . For example, if we knew the true distribution of the random variable, then we could construct a code with average description length $H(P)$. If, instead, we used the code for a distribution q, we would need $H(P) + D(P,Q)$ bits on the average to describe the random variable[Kullback,1952,1959].

The relative entropy or Kullback Leibler distance [Kullback,1959] between two probability distributions is defined as

$$D(P,Q) = \sum_{i=1}^{n} p_i \log \frac{p_i}{q_i} \tag{2}$$

With the development in literature many parametric and non parametric measures had been suggested ,the famous among them concerning the present study are

Definition:[Couso *et al.*,2004] Let a universal set X and $F(X)$ be the set of all fuzzy subsets . A mapping $D : F(X) \times F(X) \to \mathbf{R}$ is called a divergence between fuzzy subsets if and only if the following axioms hold:

d_1 : $D(A,B) = D(B,A)$

d_2 : $D(A,A) = 0$

d_3 : max. $\{D(A \cup C, B \cup C), D(A \cap C, B \cap C)\} \leq D(B,A)$ for any $A, B, C \in F(X)$

Instead of axiom d_3 if D(A,B) is convex in A and B even then it a valid measure of divergence. Non-negativity of $D(A,B)$ is the natural assumption.

[Bhandari *et al.*,2004] Introduced a measure of divergence between fuzzy sets analogous to information theoretic divergence measure(2)as

$$D(A,B) = -\frac{1}{n} \sum_i \{\mu_A(x_i) \log \frac{\mu_A(x_i)}{\mu_B(x_i)} + (1-\mu_A(x_i)) \log \frac{(1-\mu_A(x_i))}{(1-\mu_B(x_i))}\} \tag{3}$$

Axioms d_1 ,d_2 ,d_3 are used to define a new divergence measure .Many divergence measures between fuzzy set had been defined in literature analogous to probabilistic divergence measures .

Corresponding to [Renyi,1961] generelized measure of directed divergence

$$D_\alpha(P,Q) = \frac{1}{\alpha - 1} \log(\sum_{i=1}^{n} p_i^\alpha q_i^{1-\alpha}), \alpha > 0, \alpha \neq 1 \tag{4}$$

and Sharma Mittal's generelized measure(1975) of directed divergence

$$D_{\alpha,\beta}(P,Q) = \frac{1}{1 - 2^{1-\beta}} [(\sum_{i=1}^{n} p_i^\alpha q_i^{1-\alpha})^{\frac{\beta-1}{\alpha-1}} - 1], \alpha > 0, \alpha \neq 1, \beta > 0, \beta \neq 1 \tag{5}$$

[Bajaj *et al.*, 2010] proposed the measure of fuzzy directed divergence

$$D_\alpha(A,B) = \frac{1}{\alpha-1}\sum_{i=1}^{n}\log[\mu_A(x_i)^\alpha\mu_B(x_i)^{1-\alpha} + (1-\mu_A(x_i))^\alpha(1-\mu_B(x_i))^{1-\alpha}], \tag{6}$$

$\alpha > 0, \alpha \neq 1$ and

$$D_{\alpha,\beta}(A,B) = \frac{1}{1-2^{1-\beta}}\sum_{i=1}^{n}[(\mu_A(x_i)^\alpha\mu_B(x_i)^{1-\alpha}+(1-\mu_A(x_i))^\alpha(1-\mu_B(x_i))^{1-\alpha})^{\frac{\beta-1}{\alpha-1}}-1] \tag{7}$$

,$\alpha > 0, \alpha \neq 1, \beta > 0, \beta \neq 1$ respectively.

[Taneja ,2005] introduced Arithmetic-Geometric divergence measure

$$T(P,Q) = \sum_{i=1}^{n}(\frac{p_i+q_i}{2})\log(\frac{p_i+q_i}{2\sqrt{p_iq_i}}) \tag{8}$$

In terms of Kullback-Leibler's divergence measure $D(P,Q)$ Arithmetic -Geometric divergence measure $T(P,Q)$ can be expressed as:

$$T(P,Q) = \frac{1}{2}[D(\frac{P+Q}{2},P) + D(\frac{P+Q}{2},Q)] \tag{9}$$

[Taneja, 2012] proposed a one parametric measure of directed divergence

$$\Delta_\alpha(P,Q) = \sum_{i=1}^{n}\frac{(p_i-q_i)^{2\alpha}}{(p_i+q_i)^{2\alpha-1}} \tag{10}$$

[Taneja ,2005] further proposed a unified (α,β) measure of directed divergence

$$D_\alpha^\beta(P,Q) = \begin{cases} D_\alpha^\beta(P,Q), & \alpha \neq 1, \beta \neq 1; \\ D_1^\beta(P,Q), & \alpha = 1, \beta \neq 1; \\ \\ D_\alpha^1(P,Q), & \alpha \neq 1, \beta = 1; \\ D(P,Q), & \alpha = 1, \beta = 1. \end{cases} \tag{11}$$

Where

$$D_\alpha^\beta(P,Q) = \frac{1}{1-\beta}[(\sum_{i=1}^{n}p_i^\alpha q_i^{1-\alpha})^{\frac{\beta-1}{\alpha-1}} - 1], \alpha > 0, \alpha \neq 1, \beta > 0, \beta \neq 1 \tag{12}$$

$$D_1^\beta(P,Q) = \frac{1}{1-\beta}[e^{(\beta-1)D(P,Q)} - 1], \alpha = 1, \beta \neq 1 \tag{13}$$

$$D_\alpha(P,Q) = \frac{1}{\alpha-1}\log(\sum_{i=1}^{n} p_i^{\alpha} q_i^{1-\alpha}), \alpha = \beta \neq 1 \tag{14}$$

(Renyi's Measure of Directed Divergence)

$$D(P,Q) = \sum_{i=1}^{n} p_i \log\frac{p_i}{q_i}, \alpha = \beta = 1 \tag{15}$$

(Kullback-Leibler's Measure of Directed Divergence)
Now replacing D with D_α^β in value of T(P,Q) We have,

$$T_\alpha^\beta(P,Q) = \frac{1}{2}[D_\alpha^\beta(\frac{P+Q}{2},P) + D_\alpha^\beta(\frac{P+Q}{2},Q)] \tag{16}$$

Where

$$T_\alpha^\beta(P,Q) = T(P,Q) \tag{17}$$

at $\alpha = \beta = 1$
In the subsequent sections the measures of fuzzy directed divergence corresponding to $T(P,Q)$,$\Delta_\alpha(P,Q)$, $D_\alpha^\beta(P,Q)$,$T_\alpha^\beta(P,Q)$ are proposed.

3 Arithmetic-Geometric Measure of Fuzzy Directed Divergence

We propose the measure of fuzzy directed divergence for two arbitrary fuzzy sets A and B corresponding to (8) as follow:

$$T(A,B) = \sum_{i=1}^{n}[\frac{(\mu_A(x_i)+\mu_B(x_i))}{2}\log\frac{(\mu_A(x_i)+\mu_B(x_i))}{2\sqrt{\mu_A(x_i)\mu_B(x_i)}}$$
$$+\frac{(2-\mu_A(x_i)-\mu_B(x_i))}{2}\log\frac{(2-\mu_A(x_i)-\mu_B(x_i))}{2\sqrt{(1-\mu_A(x_i))(1-\mu_B(x_i))}}] \tag{18}$$

Theorem 3.1 *$T(A,B)$ is a valid measure of fuzzy directed divergence.*

Proof. From (18),it is clear that
(a)$T(A,B) \geq 0$
(b)$T(A,B) = 0$ if $\mu_A(x_i) = \mu_B(x_i)$
(c)$T(A,B) = T(B,A)$
(d) Next, we examine the convexity of $T(A,B)$:
we have,$\frac{\partial T}{\partial \mu_A(x_i)} = \frac{\mu_A(x_i)-\mu_B(x_i)+\mu_B^2(x_i)}{4}$

$\frac{\partial^2 T}{\partial \mu_A^2(x_i)} = \frac{1}{4} > 0$
Similarly, $\frac{\partial^2 T}{\partial \mu_B^2(x_i)} = \frac{1}{4} > 0$
Thus $T(A,B)$ is convex function of A and B and hence in view of the definition of fuzzy directed divergence in section 2, $T(A,B)$ is a valid measure of fuzzy directed divergence.

Theorem 3.2 *(a)*$T(A \cup B, A \cap B) = T(A,B)$
(b)$T(A \cup B, A) + T(A \cap B, A) = T(A,B)$

Proof. consider the sets

$$W_1 = \{x | x \in X,\ \mu_A(x_i) \geq \mu_B(x_i)\}$$

and

$$W_2 = \{x | x \in X,\ \mu_A(x_i) < \mu_B(x_i)\}$$

Using definitions in section (2.1), In set W_1,
$A \cup B$ = Union of A and B $\Leftrightarrow \mu_{A \cup B}(x) = \max.\{\mu_A(x), \mu_B(x)\} = \mu_A(x)$ and
$A \cap B$ = Intersection of A and B $\Leftrightarrow \mu_{A \cap B}(x) = \min.\{\mu_A(x), \mu_B(x)\} = \mu_B(x)$
In set W_2,
$A \cup B$ = Union of A and B $\Leftrightarrow \mu_{A \cup B}(x) = \max.\{\mu_A(x), \mu_B(x)\} = \mu_B(x)$ and
$A \cap B$ = Intersection of A and B $\Leftrightarrow \mu_{A \cap B}(x) = \min.\{\mu_A(x), \mu_B(x)\} = \mu_A(x)$
we have

$$T(A \cup B, A \cap B) = \sum_{W_1} [\frac{(\mu_A(x_i) + \mu_B(x_i))}{2} \log \frac{(\mu_A(x_i) + \mu_B(x_i))}{2\sqrt{\mu_A(x_i)\mu_B(x_i)}}$$

$$+ \frac{(2 - \mu_A(x_i) - \mu_B(x_i))}{2} \log \frac{(2 - \mu_A(x_i) - \mu_B(x_i))}{2\sqrt{(1-\mu_A(x_i))(1-\mu_B(x_i))}}]$$

$$+ \sum_{W_2} [\frac{(\mu_A(x_i) + \mu_B(x_i))}{2} \log \frac{(\mu_A(x_i) + \mu_B(x_i))}{2\sqrt{\mu_A(x_i)\mu_B(x_i)}}$$

$$+ \frac{(2 - \mu_A(x_i) - \mu_B(x_i))}{2} \log \frac{(2 - \mu_A(x_i) - \mu_B(x_i))}{2\sqrt{(1-\mu_A(x_i))(1-\mu_B(x_i))}}]$$

$$= \sum_{X}[\frac{(\mu_A(x_i)+\mu_B(x_i))}{2}\log\frac{(\mu_A(x_i)+\mu_B(x_i))}{2\sqrt{\mu_A(x_i)\mu_B(x_i)}}$$

$$+\frac{(2-\mu_A(x_i)-\mu_B(x_i))}{2}\log\frac{(2-\mu_A(x_i)-\mu_B(x_i))}{2\sqrt{(1-\mu_A(x_i))(1-\mu_B(x_i))}}]$$

$$= T(A,B)$$

that is , $T(A\cup B, A\cap B) = T(A,B)$
Thus proof of (a) follows.
Now,

$$T(A\cup B, A) = \sum_{W_2}[\frac{(\mu_A(x_i)+\mu_B(x_i))}{2}\log\frac{(\mu_A(x_i)+\mu_B(x_i))}{2\sqrt{\mu_A(x_i)\mu_B(x_i)}}$$

$$+\frac{(2-\mu_A(x_i)-\mu_B(x_i))}{2}\log\frac{(2-\mu_A(x_i)-\mu_B(x_i))}{2\sqrt{(1-\mu_A(x_i))(1-\mu_B(x_i))}}] \tag{19}$$

and

$$T(A\cap B, A) = \sum_{W_1}[\frac{(\mu_A(x_i)+\mu_B(x_i))}{2}\log\frac{(\mu_A(x_i)+\mu_B(x_i))}{2\sqrt{\mu_A(x_i)\mu_B(x_i)}}$$

$$+\frac{(2-\mu_A(x_i)-\mu_B(x_i))}{2}\log\frac{(2-\mu_A(x_i)-\mu_B(x_i))}{2\sqrt{(1-\mu_A(x_i))(1-\mu_B(x_i))}}] \tag{20}$$

Adding (19) and (20), we have

$$T(A\cup B, A) + T(A\cap B, A) = T(A,B)$$

Thus proof of (b) follows.

Theorem 3.3 $T(A,\overline{A})$= $n\log 2$ - $\frac{1}{2}\sum_{i=1}^{n}[\log\mu_A(x_i)+\log(1-\mu_A(x_i))]$.

Proof.$T(A,\overline{A}) = \sum_{i=1}^{n}\log\frac{1}{2\sqrt{\mu_A(x_i)(1-\mu_A(x_i))}}$
$= \sum_{i=1}^{n}[\log\frac{1}{2} - \frac{1}{2}\log\mu_A(x_i)(1-\mu_A(x_i))]$
$= n\log 2 - \frac{1}{2}\sum_{i=1}^{n}[\log\mu_A(x_i)+\log(1-\mu_A(x_i))]$

4 Generalized Triangular Discrimination between Fuzzy Sets

We propose generalized triangular discrimination between two arbitrary fuzzy sets A and B corresponding to (10) as follow:

$$\Delta_\alpha(A,B) = \sum_{i=1}^{n}(\mu_A(x_i)-\mu_B(x_i))^{2\alpha}[\frac{1}{(\mu_A(x_i)+\mu_B(x_i))^{2\alpha-1}} + \frac{1}{(2-\mu_A(x_i)-\mu_B(x_i))^{2\alpha-1}}] \tag{21}$$

Theorem 4.1 $\Delta_\alpha(A,B)$ *is a valid measure of fuzzy directed divergence.*

Proof. From (21),it is clear that
(a)$\Delta_\alpha(A,B) \geq 0$
(b)$\Delta_\alpha(A,B) = 0$ if $\mu_A(x_i) = \mu_B(x_i)$
(c)$\Delta_\alpha(A,B) = \Delta_\alpha(B,A)$
(d) Next, we examine the convexity of $\Delta_\alpha(A,B)$:
we have, $\frac{\partial\Delta_\alpha}{\partial\mu_A(x_i)} = 2\alpha(\mu_A(x_i)-\mu_B(x_i))^{2\alpha-1}[\frac{1}{(\mu_A(x_i)+\mu_B(x_i))^{2\alpha-1}}+\frac{1}{(2-\mu_A(x_i)-\mu_B(x_i))^{2\alpha-1}}]$
$+(2\alpha-1)(\mu_A(x_i)-\mu_B(x_i))^{2\alpha}[-\frac{1}{(\mu_A(x_i)+\mu_B(x_i))^{2\alpha}}+\frac{1}{(2-\mu_A(x_i)-\mu_B(x_i))^{2\alpha}}]$
$\frac{\partial^2\Delta_\alpha}{\partial\mu_A^2(x_i)} = 2\alpha(2\alpha-1)[\frac{\Delta_\alpha(A,B)}{(\mu_A(x_i)-\mu_B(x_i))^2}+\Delta_\alpha(A,B)+2\{-\frac{1}{(\mu_A(x_i)+\mu_B(x_i))^{2\alpha}}+\frac{1}{(2-\mu_A(x_i)-\mu_B(x_i))^{2\alpha}}\}] > 0$ for $\alpha > \frac{1}{2}$
Similarly, $\frac{\partial^2\Delta_\alpha}{\partial\mu_B^2(x_i)} > 0$ for $\alpha > \frac{1}{2}$
Thus $\Delta_\alpha(A,B)$ is convex function of A and B for $\alpha > \frac{1}{2}$and hence in view of the definition of fuzzy directed divergence in section 2, $\Delta_\alpha(A,B)$ is a valid measure of fuzzy directed divergence.

Theorem 4.2 *(a)*$\Delta_\alpha(A\cup B, A\cap B) = \Delta_\alpha(A,B)$
(b)$\Delta_\alpha(A\cup B, A) + \Delta_\alpha(A\cap B, A) = \Delta_\alpha(A,B)$

Proof:By the methodology followed in Theorem 2 proof is obvious.

Theorem 4.3 *Following results can be obtained from (21)*
(a)$\Delta_\alpha(A,\overline{A}) = 2n$
(b)$\Delta_\alpha(A,B) = n + \sum_{i=1}^{n}\frac{\mu_A(x_i)^{2\alpha}}{(2-\mu_A(x_i))^{2\alpha}}$ *when* $\mu_B(x_i) = 0$
(c)$\Delta_\alpha(A,B) = n + \sum_{i=1}^{n}\frac{(\mu_A(x_i)-1)^{2\alpha}}{(\mu_A(x_i)+1)^{2\alpha}}$ *when* $\mu_B(x_i) = 1$
(d)$\Delta_\alpha(A,B) = n + \sum_{i=1}^{n}\frac{(2\mu_A(x_i)-1)^{2\alpha}}{(2\mu_A(x_i)-3)^{2\alpha}}$ *when* $\mu_B(x_i) = \frac{1}{2}$

5 A New(α, β) Class of Measures of Fuzzy Directed Divergence

We propose a new (α, β) class of measures of fuzzy directed divergence for two arbitrary fuzzy sets A and B corresponding to (12) as follow:
Where

$$D_{\alpha}^{\beta}(A,B)=\frac{1}{\beta-1}\sum_{i=1}^{n}[(\mu_A(x_i)^{\alpha}\mu_B(x_i)^{1-\alpha}+(1-\mu_A(x_i))^{\alpha}(1-\mu_B(x_i))^{1-\alpha})^{\frac{\beta-1}{\alpha-1}}-1],$$

$$\alpha>0,\alpha\neq 1,\beta>0,\beta\neq 1 \tag{22}$$

Theorem 5.1 $D_{\alpha}^{\beta}(A,B)$ *is the valid measure of divergence.*

Proof. From (22),it is clear that
(a)$D_{\alpha}^{\beta}(A,B)=0$ if $\mu_A(x_i)=\mu_B(x_i)$
(b)$D_{\alpha}^{\beta}(A,B)\neq D_{\alpha}^{\beta}(B,A)$
but $J_{\alpha}^{\beta}(A,B)=D_{\alpha}^{\beta}(A,B)+D_{\alpha}^{\beta}(B,A)$ is symmetric.
(c) Next, we examine axiom d_3 of the definition of section 2.2 :
We divide X into following six subsets

$$W_1=\{x|x\in X,\ \mu_A(x)\le\mu_B(x)\le\mu_C(x)\},$$
$$W_2=\{x|x\in X,\ \mu_A(x)\le\mu_C(x)<\mu_B(x)\},$$
$$W_3=\{x|x\in X,\ \mu_B(x)<\mu_A(x)\le\mu_C(x)\},$$
$$W_4=\{x|x\in X,\ \mu_B(x)\le\mu_C(x)<\mu_A(x)\},$$
$$W_5=\{x|x\in X,\ \mu_C(x)<\mu_A(x)\le\mu_B(x)\},$$
$$W_6=\{x|x\in X,\ \mu_C(x)<\mu_B(x)<\mu_A(x)\}$$

In set W_1,
$A\cup C=$ Union of A and C $\Leftrightarrow \mu_{A\cup C}(x)=\max.\{\mu_A(x),\mu_C(x)\}=\mu_C(x)$
$B\cup C=$ Union of B and C $\Leftrightarrow \mu_{B\cup C}(x)=\max.\{\mu_B(x),\mu_C(x)\}=\mu_C(x)$
$A\cap C=$ Intersection of A and C $\Leftrightarrow \mu_{A\cap C}(x)=\min.\{\mu_A(x),\mu_C(x)\}=\mu_A(x)$
$B\cap C=$ Intersection of B and C $\Leftrightarrow \mu_{B\cap C}(x)=\min.\{\mu_B(x),\mu_C(x)\}=\mu_B(x)$
$D_{\alpha}^{\beta}(A\cup C,B\cup C)=D_{\alpha}^{\beta}(C,C)=0$,
$D_{\alpha}^{\beta}(A\cap C,B\cap C)=D_{\alpha}^{\beta}(A,B)$

Therefore, $max.\{D_\alpha^\beta(A\cup C, B\cup C), D_\alpha^\beta(A\cap C, B\cap C)\} = D_\alpha^\beta(A,B)$
Similarly, in the sets W_2, W_3, W_4, W_5, W_6 we have
$max.\{D_\alpha^\beta(A\cup C, B\cup C), D_\alpha^\beta(A\cap C, B\cap C)\} \le D_\alpha^\beta(A,B)$
Thus ,

$$max.\{D_\alpha^\beta(A\cup C, B\cup C), D_\alpha^\beta(A\cap C, B\cap C)\} \le D_\alpha^\beta(A,B) \tag{23}$$

for all $A, B, C \in F(X)$
(d) Now we show that $D_\alpha^\beta(A,B) > 0$ for two fuzzy sets empirically.
Consider two fuzzy sets A=(0.1, 0.3, 0.4, 0.2, 0.1)and B=(0.3, 0.5, 0.3, 0.1, 0.2). For sake of simplicity constructing Tables 1 , we denote

$$e = \sum_{i=1}^{n} e_i \tag{24}$$

where

$$e_i = [(\mu_A(x_i)^\alpha \mu_B(x_i)^{1-\alpha} + (1-\mu_A(x_i))^\alpha (1-\mu_B(x_i))^{1-\alpha})^{\frac{\beta-1}{\alpha-1}} - 1] \tag{25}$$

for $i = 1, 2, 3...n$.
Further taking all possible cases of α and β with two fuzzy sets considered here.Next we tabulate the values given in Table 1.
Case 1: $0 < \alpha < 1$ and $0 < \beta < 1$. In this case $\frac{1}{\beta-1} < 0$, $\frac{\beta-1}{\alpha-1} > 0$ and computed values are presented in the following table.

α	β	e_1	e_2	e_3	e_4	e_5	e	$D_\alpha^\beta(A,B)$
0.2	0.6	-0.0116	-0.0069	-0.0017	-0.0030	-0.0034	-0.0267	0.0333
0.4	0.8	-0.0110	-0.0068	-0.0017	-0.0031	-0.0032	-0.0200	0.0433
0.1	0.5	-0.0074	-0.0043	-0.0010	-0.0018	-0.0021	-0.0169	0.0187
0.3	0.9	-0.0042	-0.0025	-0.0006	-0.0011	-0.0012	-0.0099	0.0141
0.6	0.4	-0.0459	-0.0300	-0.0079	-0.0147	-0.0141	-0.1128	0.282
0.8	0.1	-0.0849	-0.0584	-0.016	-0.0303	-0.0270	-0.2168	1.084
0.7	0.2	-0.0686	-0.0460	-0.0124	-0.0232	-0.0275	-0.1719	0.573
0.5	0.7	-0.0199	-0.0127	-0.0033	-0.0060	-0.0060	-0.0480	0.096
0.9	0.3	-0.0727	-0.0509	-0.0140	-0.0270	-0.0232	-0.1880	1.88

From this table it is clear that $D_\alpha^\beta(A,B) > 0$
Similar, tables can be constructed for case 2, case 3, case 4 and it is observed that $D_\alpha^\beta(A,B) > 0$
Case 2: $0 < \alpha < 1$ and $\beta > 1$. In this case $\frac{1}{\beta-1} > 0$, $\frac{\beta-1}{\alpha-1} < 0$
Case 3: $\alpha > 1$ and $0 < \beta < 1$. In this case $\frac{1}{\beta-1} < 0$, $\frac{\beta-1}{\alpha-1} < 0$
Case 4: $\alpha > 1$ and $\beta > 1$. In this case $\frac{1}{\beta-1} < 0$, $\frac{\beta-1}{\alpha-1} > 0$
Note that , these tables can be constructed for any two standard fuzzy sets

and we shall have the same fact observed above. Thus, $D_\alpha^\beta(A, B)$ is a valid measure of fuzzy directed divergence.

Special cases: 1. When $\beta \to \alpha$ then $D_\alpha^\beta(A, B) = D_\alpha(A, B)$ as given in (6) i.e generalized measure of fuzzy directed divergence corresponding to Renyi's generalized divergence measure.
2. When $\beta \to \alpha \to 1$ then $D_\alpha^\beta(A, B) = D(A, B)$ as given in (3) i.e generalized measure of fuzzy directed divergence corresponding to Kullback-Leibler's generalized divergence measure.

6 Generalized Arithmetic-Geometric Measure of Fuzzy Directed Divergence

Now we introduce following (α, β) measure of fuzzy directed divergence corresponding to (16)

$$T_\alpha^\beta(A, B) = \frac{1}{2}[D_\alpha^\beta(\frac{A+B}{2}, A) + D_\alpha^\beta(\frac{A+B}{2}, B)] \tag{26}$$

This is Known as Generalized Arithmetic-Geometric measure of fuzzy directed divergence because $T_\alpha^\beta(A, B) = T(A, B)$ at $\alpha = \beta = 1$ which is Arithmetic-Geometric measure of divergence between fuzzy sets as discussed in section 3.

Theorem 6.1 *$T_\alpha^\beta(A, B)$ is a valid measure of divergence.*

Proof. In view of the properties of $D_\alpha^\beta(A, B)$ in theorem 7 and (24) we conclude that
$(T_1)T_\alpha^\beta(A, B) = 0$ if $\mu_A(x_i) = \mu_B(x_i)$
$(T_2)T_\alpha^\beta(A, B) = T_\alpha^\beta(B, A)$
$(T_3)max.\{T_\alpha^\beta(A \cup C, B \cup C), T_\alpha^\beta(A \cap C, B \cap C)\} \leq T_\alpha^\beta(A, B)$ for all $A, B, C \in F(X)$
$(T_4)T_\alpha^\beta(A, B) \geq 0$
Thus $T_\alpha^\beta(A, B)$ is a valid measure of divergence.

Theorem 6.2 *(a)$T_\alpha^\beta(A \cup B, A \cap B) = T_\alpha^\beta(A, B)$*
(b)$T_\alpha^\beta(A \cup B, A) + T_\alpha^\beta(A \cap B, A) = T_\alpha^\beta(A, B)$

Proof. By the methodology followed in Theorem 2 proof is obvious.

Theorem 6.3 *$T_\alpha^\beta(A, B)$ satisfies the following properties*
(a)$T_\alpha^\beta(A, B) = T_\alpha^\beta(\overline{A}, \overline{B})$
(b)$T_\alpha^\beta(A, \overline{A}) = T_\alpha^\beta(\overline{A}, A)$
(c)$T_\alpha^\beta(A, \overline{B}) = T_\alpha^\beta(\overline{A}, B)$

7 Normalized Measure of Fuzzy Directed Divergence

7.1 Need for Normalizing Measures of Directed Divergence

The value of normalized measure of directed divergence lies between 0 and 1. So, one of the main advantage of normalization is that we can compare two or more measures of the directed divergence.
Example. Consider the problem of comparing genomes. The E. coli genome is about 4.8 megabase long, whereas H. influenza, a sister species of E. coli, has genome length only 1.8 megabase. The information divergence E between the two genomes is dominated by their length difference rather than the amount of information they share. Such a measure will trivially classify H. influenza as being closer to a more remote species of similar genome length such as A. fulgidus (2.18 megabase), rather than with E. coli. In order to deal with such problems, we need to normalize.

7.2 Normalized Measures of Probabilistic Directed Divergence

Let$P = (p_1, p_2, p_3 ... p_n)$, $Q = (q_1, q_2, q_3 ... q_n)$ be two probability distributions. Then the measure of directed divergence $D(P, Q)$ can be normalized as:

$$\overline{D}(P, Q) = \frac{D(P, Q)}{\max_P D(P, Q)} \tag{27}$$

For more details [Kapur,1995] page 150 can be consulted.
The most important measure of directed divergence is due to [Csiszar,1963], viz.

$$D(P, Q) = \sum_{i=1}^{n} q_i f(\frac{p_i}{q_i}) \tag{28}$$

where $f(.)$ is any convex twice differentiable function which is such that $f(1) = 0$.
Corresponding to Csiszar,s type of measures of directed divergences [Kapur,1995] presented following normalized measure of directed divergence:

$$\overline{D}_{nor}(P, Q) = \frac{\sum_{i=1}^{n} q_i f(\frac{p_i}{q_i})}{q_k f(\frac{1}{q_k}) + (1 - q_k) f(0)} \tag{29}$$

where q_k= min.$\{q_1, q_2, q_3...q_n\}$

7.3 Normalized Measures of Fuzzy Directed Divergence

Many probabilistic measures of directed divergence have been normalized by [Kapur,1995]. Next, we explore the possibility of normalizing fuzzy measures of directed divergence.we propose the following normalized fuzzy directed divergences:

$$\overline{D}(A,B) = \frac{D(A,B)}{\max\limits_{A} D(A,B)} \tag{30}$$

$$\overline{D}_{nor}(A,B) = \frac{\sum_{i=1}^{n}[\mu_B(x_i)f(\frac{\mu_A(x_i)}{\mu_B(x_i)}) + (1-\mu_B(x_i))f(\frac{1-\mu_A(x_i)}{1-\mu_B(x_i)})]}{\mu_{B_k}(x_i)f(\frac{1}{\mu_{B_k}(x_i)}) + (1-\mu_{B_k}(x_i))f(0)} \tag{31}$$

where $\mu_{B_k}(x_i)$= min.$\{\mu_B(x_1), \mu_B(x_2), \mu_B(x_3)...\mu_B(x_n)\}$
In the present communication measure of fuzzy directed divergence in section 3 given by (18) and in section 4 given by (21) are csiszar's type with generating functions

$$f_1(x) = \frac{(x+1)}{2}\log\frac{x+1}{2\sqrt{x}} \tag{32}$$

and

$$f_2(x) = \frac{(x-1)^{2\alpha}}{(x+1)^{2\alpha-1}} \tag{33}$$

respectively.
Thus the normalized measures of fuzzy directed divergence for (18) and (21) can be obtained respectively by using (32) and (33) in (31).
Further,measure of fuzzy directed divergence in section 5 given by (22) and in section 6 given by (26) are not csiszar's type. Therefore normalized fuzzy directed divergence measure corresponding to these can be obtained using (30).

$$\max_{A} D_{\alpha}^{\beta}(A,B) = \frac{1}{\beta-1}[\mu_{B_k}(x_i)^{\beta-1}] \tag{34}$$

In view of (30), (22) and (34) gives

$$\overline{D}_{\alpha}^{\beta}(A,B) = \frac{\sum_{i=1}^{n}[(\mu_A(x_i)^{\alpha}\mu_B(x_i)^{1-\alpha} + (1-\mu_A(x_i))^{\alpha}(1-\mu_B(x_i))^{1-\alpha})^{\frac{\beta-1}{\alpha-1}} - 1]}{[\mu_{B_k}(x_i)^{\beta-1} - 1]} \tag{35}$$

where $\alpha \neq 1, \beta \neq 1$
Similarly , we can obtain the normalized measure of fuzzy directed divergence corresponding to (26).

8 Conclusion and Discission

In the proposed measures of fuzzy directed divergence, we observe that these have greater flexibility in applications because of the presence of the parameter α and (α, β). We think that our proposal is quite general and contains, as special cases,some classical measures of directed divergence due to Renyi's and Kullback-Leibler's. Some other important results are also obtained, which are useful in statistical and information sciences.In Section 7 we have introduced the concept of normalized measures of fuzzy directed divergence. It has been observed that there are redundancies and overlapping in similar situations which if removed can increase the efficiency of the process. The development of new generalized probabilistic and fuzzy measures of directed divergence is necessary to to get as much insight as possible into various physical situations. Many such measures of fuzzy directed divergence can be generated for different situations. The possible applications of the divergence measure could be useful in the detection processes when trying to identify an object (crisp or fuzzy, it does not matter) by means of partially reliable questions. In this case an unreliable answer (naturally represented by a fuzzy set) has to be compared with the set representing the object to be identified. The divergence seems to be the most natural index which measure how they agree. Other open problems are to find more possible applications of generalized measures of fuzzy directed divergence.

References

[1] Bajaj,R.K., Hooda,D.S, (2010), On Some New Generalized Measures of Fuzzy Information, *World Academy of Science,Engineering and Technology,* **Issue 62** , pp. 747- 753.

[2] Bhandari,D., and Pal,N. R.,(1993), Some new information measures for fuzzy sets, *Information Sciences,* **Vol 67**, pp. 204 - 228.

[3] Couso,Ines.,Janis, Vladimir., and Montes, Susana.,(2004), Fuzzy Divergence Measures, *Acta Univ. M. Belii Math.,* **Vol 8**,pp. 21 - 26.

[4] Csiszar,I., (1963),*Eine informationstheoretische ungleichung und ihre anwendung auf den beweis der ergodizitat von markoschen ketten, Magyar Tud. Akad. Mat. Kutato Int. Kozl,* **Vol 8** , pp.85 - 108.

[5] De Luca,A.,and Termini,S.,(1971), A definition of non-probabilistic entropy in the setting of fuzzy set theory,*Inform. and Control,* **Vol 20** ,pp. 301 - 312.

[6] Kapur,J.N.,(1995),Measures of Information and Their Applications,*Wiley Eastern Ltd. New Age International,New Delhi* .

[7] Kullback,S.,(1959),Information Theory and Statistics,*Wiley and Sons, New Delhi.*

[8] Kullback,S.,and Leibler,R. A.,(1952), On information and sufficiency,*Ann. Math. Stat.,* **Vol 22** , pp. 79 - 86.

[9] Renyi,A., (1961),On measures of entropy and information., *In: Proc. 4th Berkeley Symp. Math. Stat. Probab.,* **1** , pp. 547 - 561.

[10] Shannon,C E.,(1948),The mathematical theory of communication, *Bell Syst. Tech. Journal,* **Vol 27** , pp. 423 - 467.

[11] Sharma,B. D.,and Mittal,D. P.,(1975), New non-additive measures of entropy for discrete probability distributions, *J. Math. Sci (Calcutta),* **Vol 10** , pp. 28- 40.

[12] Taneja,I.J.,(2005),On mean divergence measures,*http://arXiv.org/abs/math/0501298v1,*, pp. 1 - 18.

[13] Taneja,I.J.,(2012),Seven means,Generelized triangular discrimination, and Generating divergence measures *http://arXiv.org/abs/math/1203.2778v1,* pp. 1 - 37.

[14] Taneja,I.J.,(2005),Generalized symmetric divergence measures and metric spaces *http://arXiv.org/abs/math/1105.2707v1,* pp. 1 - 11.

[15] Zadeh,L.A.,(1965), Fuzzy Sets, *Information and control,* **Vol 8**,pp. 338 - 353.