ON SMARANDACHE TS CURVES OF BIHARMONIC S-CURVES
ACCORDING TO SABBAN FRAME IN HEISENBERG GROUP
HEIS\(^3\)

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Abstract. In this paper, we study Smarandache ts curves according to Sabb
can frame in the Heisenberg group Heis\(^3\). Finally, we find explicit parametric
equations of Smarandache ts curves according to Sabban Frame.

1. Introduction

A smooth map \( \phi : N \rightarrow M \) is said to be biharmonic if it is a critical point of
the bienergy functional:

\[
E_2(\phi) = \int_N \frac{1}{2} |T(\phi)|^2 \, dv_h,
\]

where \( T(\phi) := \text{tr} \nabla^\phi d\phi \) is the tension field of \( \phi \).

The Euler–Lagrange equation of the bienergy is given by \( T_2(\phi) = 0 \). Here the
section \( T_2(\phi) \) is defined by

\[
T_2(\phi) = -\Delta_\phi T(\phi) + \text{tr} R(T(\phi), d\phi) d\phi,
\]

and called the bitension field of \( \phi \). Non-harmonic biharmonic maps are called proper
biharmonic maps.

This study is organised as follows: Firstly, we study Smarandache ts curves
according to Sabban frame in the Heisenberg group Heis\(^3\). Finally, we find explicit
parametric equations of Smarandache ts curves according to Sabban Frame.

2. The Heisenberg Group Heis\(^3\)

Heisenberg group Heis\(^3\) can be seen as the space \( \mathbb{R}^3 \) endowed with the following
multiplication:

\[
(\bar{x}, \bar{y}, \bar{z})(x, y, z) = (\bar{x} + x, \bar{y} + y, \bar{z} + z - \frac{1}{2} \bar{x}y + \frac{1}{2} x\bar{y})
\]

Heis\(^3\) is a three-dimensional, connected, simply connected and 2-step nilpotent Lie
group.

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The Riemannian metric $g$ is given by
\[ g = dx^2 + dy^2 + (dz - xdy)^2. \]

The Lie algebra of $\text{Heis}^3$ has an orthonormal basis
\[ e_1 = \frac{\partial}{\partial x}, \quad e_2 = \frac{\partial}{\partial y} + x \frac{\partial}{\partial z}, \quad e_3 = \frac{\partial}{\partial z}, \]
for which we have the Lie products
\[ [e_1, e_2] = e_3, \quad [e_2, e_3] = [e_3, e_1] = 0 \]
with
\[ g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = 1. \]

We obtain
\[ \nabla_{e_1} e_1 = \nabla_{e_2} e_2 = \nabla_{e_3} e_3 = 0, \]
\[ \nabla_{e_1} e_2 = -\nabla_{e_2} e_1 = \frac{1}{2} e_3, \]
\[ \nabla_{e_1} e_3 = \nabla_{e_2} e_1 = -\frac{1}{2} e_2, \]
\[ \nabla_{e_2} e_3 = \nabla_{e_3} e_2 = \frac{1}{2} e_1. \]

The components $\{R_{ijkl}\}$ of $R$ relative to $\{e_1, e_2, e_3\}$ are defined by
\[ R_{ijk} = R(e_i, e_j)e_k, \quad R_{ijkl} = R(e_i, e_j, e_k, e_l) = g(R(e_i, e_j)e_l, e_k). \]

The non vanishing components of the above tensor fields are
\[ R_{121} = \frac{3}{4} e_2, \quad R_{131} = -\frac{1}{4} e_3, \quad R_{122} = -\frac{3}{4} e_1, \]
\[ R_{232} = -\frac{1}{4} e_3, \quad R_{133} = \frac{1}{4} e_1, \quad R_{233} = \frac{1}{4} e_2, \]
and
\[ R_{1212} = -\frac{3}{4}, \quad R_{1313} = R_{2323} = \frac{1}{4}. \]

3. Biharmonic $S$–Helices According To Sabban Frame In The Heisenberg Group $\text{Heis}^3$

Let $\gamma : I \rightarrow \text{Heis}^3$ be a non geodesic curve on the Heisenberg group $\text{Heis}^3$ parametrized by arc length. Let $\{T, N, B\}$ be the Frenet frame fields tangent to the Heisenberg group $\text{Heis}^3$ along $\gamma$ defined as follows:

$T$ is the unit vector field $\gamma'$ tangent to $\gamma$, $N$ is the unit vector field in the direction of $\nabla_T T$ (normal to $\gamma$), and $B$ is chosen so that $\{T, N, B\}$ is a positively oriented orthonormal basis. Then, we have the following Frenet formulas:
\[ \nabla_T T = \kappa N, \]
\[ \nabla_T N = -\kappa T + \tau B, \]
\[ \nabla_T B = -\tau N, \]
where $\kappa$ is the curvature of $\gamma$ and $\tau$ is its torsion,
\[ g(T, T) = 1, \quad g(N, N) = 1, \quad g(B, B) = 1, \]
\[ g(T, N) = g(T, B) = g(N, B) = 0. \]
In the rest of the paper, we suppose everywhere

$$\kappa \neq 0 \text{ and } \tau \neq 0.$$  

Now we give a new frame different from Frenet frame. Let \( \alpha : I \rightarrow \mathbb{S}^2_{\text{Heis}^3} \) be unit speed spherical curve. We denote \( \sigma \) as the arc-length parameter of \( \alpha \). Let us denote \( t(\sigma) = \alpha' (\sigma) \), and we call \( t(\sigma) \) a unit tangent vector of \( \alpha \). We now set a vector \( s(\sigma) = \alpha (\sigma) \times t(\sigma) \) along \( \alpha \). This frame is called the Sabban frame of \( \alpha \) on the Heisenberg group \( \text{Heis}^3 \). Then we have the following spherical Frenet-Serret formulae of \( \alpha \):

\[
\nabla_t \alpha = t, \\
\nabla_t t = -\alpha + \kappa_g s, \\
\nabla_t s = -\kappa_g t,
\]

where \( \kappa_g \) is the geodesic curvature of the curve \( \alpha \) on the \( \mathbb{S}^2_{\text{Heis}^3} \) and

\[
g(t,t) = 1, \quad g(\alpha,\alpha) = 1, \quad g(s,s) = 1, \quad g(t,\alpha) = g(t,s) = g(\alpha,s) = 0.
\]

With respect to the orthonormal basis \( \{e_1, e_2, e_3\} \), we can write

\[
\alpha = \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3, \\
t = t_1 e_1 + t_2 e_2 + t_3 e_3, \\
s = s_1 e_1 + s_2 e_2 + s_3 e_3.
\]

To separate a biharmonic curve according to Sabban frame from that of Frenet-Serret frame, in the rest of the paper, we shall use notation for the curve defined above as biharmonic \( S \)-curve.

**Lemma 3.1.** \( \alpha : I \rightarrow \mathbb{S}^2_{\text{Heis}^3} \) is a biharmonic \( S \)-curve if and only if

\[
\kappa_g = \text{constant } \neq 0, \\
1 + \kappa_g^2 = -\left[1 - s_3^2\right] + \kappa_g [-\alpha_3 s_3], \\
\kappa_g^3 = -\alpha_3 s_3 - \kappa_g \left[\frac{1}{4} - \alpha_3^2\right].
\]

Then the following result holds.

**Theorem 3.2.** ([9]) All of biharmonic \( S \)-curves in \( \mathbb{S}^2_{\text{Heis}^3} \) are helices.
Theorem 3.3. (9) Let $I : S^2_{Heis}^3$ be a unit speed non-geodesic biharmonic $S$-curve. Then, the position vector of $\alpha$ is

$$\alpha(\sigma) = -\frac{\sin^2 \mathcal{E}}{\mathcal{V}} \cos[M\sigma + M_1] + M_2]e_1 + \frac{\sin^2 \mathcal{E}}{2\mathcal{V}^2} \sin[M\sigma + M_1] + M_3]e_2$$

(3.5)

$$+ \left[ \frac{\mathcal{V}}{\mathcal{V}} \sin[M\sigma + M_1] + M_2] - \frac{\sin^2 \mathcal{E}}{\mathcal{V}} \cos[M\sigma + M_1] + M_2] + \frac{M_2}{\mathcal{V}} \sin^3 \mathcal{E} \sin[M\sigma + M_1] + M_3]e_3,$$

where $M_1, M_2, M_3, M_4$ are constants of integration and

$$\mathcal{M} = \left( \frac{1 + \kappa_g^2}{\sin \mathcal{E}} \right) \cos \mathcal{E} \right) \right) \frac{\mathcal{V}}{\sqrt{1 + 2\kappa_g^2 - \frac{\kappa_g^2}{2}}} \sin 2\mathcal{E}.$$

4. Smarandache $\gamma_\mathbf{ts}$ Curves Of Biharmonic $S$-Curves According To Sabban Frame In The Heisenberg Group $\text{Heis}^3$

Definition 4.1. Let $I : S^2_{Heis}^3$ be a unit speed regular curve in the Heisenberg group $\text{Heis}^3$ and $\{\alpha, t, s\}$ be its moving Bishop frame. Smarandache $\gamma_\mathbf{ts}$ curves are defined by

$$\gamma_\mathbf{ts} = \frac{1}{\sqrt{1 + 2\kappa_g^2}} (t + s).$$

Theorem 4.2. Let $I : S^2_{Heis}^3$ be a unit speed non-geodesic biharmonic $S$-curve $\gamma_\mathbf{ts}$ its Smarandache $\gamma_\mathbf{ts}$ curve. Then, the position vector of Smarandache $\gamma_\mathbf{ts}$ curve is

$$\gamma_\mathbf{ts}(\sigma) = \frac{1}{\sqrt{1 + 2\kappa_g^2}} \sin \mathcal{E} \sin[M\sigma + M_1] + \frac{1}{\kappa_g} \sin \mathcal{E} \cos[M\sigma + M_1](M + \cos \mathcal{E})$$

(4.2)

$$- \frac{\sin^2 \mathcal{E}}{\mathcal{V}} \cos[M\sigma + M_1] + M_2]e_1 + \frac{1}{\sqrt{1 + 2\kappa_g^2}} \sin \mathcal{E} \cos[M\sigma + M_1]$$

$$+ \frac{1}{\kappa_g} [- \sin \mathcal{E} \sin[M\sigma + M_1](M + \cos \mathcal{E}) + \frac{\sin^2 \mathcal{E}}{\mathcal{V}} \sin[M\sigma + M_1] + M_3]e_2$$

$$+ \frac{1}{\sqrt{1 + 2\kappa_g^2}} \cos \mathcal{E} + \frac{1}{\kappa_g} \cos \mathcal{E} \sigma - \frac{\mathcal{V}}{\mathcal{V}} \sin[M\sigma + M_1] + M_3]e_2$$

$$- \frac{\sin^2 \mathcal{E}}{\mathcal{V}} \sin[M\sigma + M_1] + M_3] - \frac{\sin^2 \mathcal{E}}{\mathcal{V}} \cos[M\sigma + M_1] + M_2]$$

$$+ \frac{M_2}{\mathcal{V}} \sin^3 \mathcal{E} \sin[M\sigma + M_1] + M_4]e_3,$$
where \( \mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3, \mathcal{M}_4 \) are constants of integration and

\[
(4.3) \quad \mathcal{M} = \left( \frac{\sqrt{1 + \kappa^2}}{\sin \mathcal{E}} - \cos \mathcal{E} \right) \quad \text{and} \quad \mathcal{V} = \sqrt{1 + \kappa^2} - \frac{1}{2} \sin 2\mathcal{E}.
\]

**Proof.** From definition of \( S \)-helix, we obviously obtain

\[
(4.4) \quad t = \sin \mathcal{E} \sin[\mathcal{M}\sigma + \mathcal{M}_1]e_1 + \sin \mathcal{E} \cos[\mathcal{M}\sigma + \mathcal{M}_1]e_2 + \cos \mathcal{E} e_3.
\]

We can easily verify that

\[
(4.5) \quad \nabla_t t = (t'_1 + t_2 t_3) e_1 + (t'_2 - t_1 t_3) e_2 + t'_3 e_3.
\]

Since, we immediately arrive at

\[
\nabla_t t = \sin \mathcal{E} \cos[\mathcal{M}\sigma + \mathcal{M}_1](\mathcal{M} + \cos \mathcal{E})e_1
\]

\[
- \sin \mathcal{E} \sin[\mathcal{M}\sigma + \mathcal{M}_1](\mathcal{M} + \cos \mathcal{E})e_2.
\]

Obviously, we also obtain

\[
s(\sigma) = \frac{1}{\kappa_g} \left[ \sin \mathcal{E} \cos[\mathcal{M}\sigma + \mathcal{M}_1](\mathcal{M} + \cos \mathcal{E}) - \frac{\sin^2 \mathcal{E}}{\mathcal{V}} \cos[\mathcal{M}\sigma + \mathcal{M}_1] + \mathcal{M}_2 \right] e_1
\]

\[
+ \frac{1}{\kappa_g} \left[ -\sin \mathcal{E} \sin[\mathcal{M}\sigma + \mathcal{M}_1](\mathcal{M} + \cos \mathcal{E}) + \frac{\sin^2 \mathcal{E}}{\mathcal{V}} \sin[\mathcal{M}\sigma + \mathcal{M}_1] + \mathcal{M}_3 \right] e_2
\]

\[
+ \frac{1}{\kappa_g} \left[ \cos \mathcal{E} \sigma - \frac{\mathcal{V}\sigma + \mathcal{M}_1}{2\mathcal{V}^2} \sin^4 \mathcal{E} - \frac{\sin 2[\mathcal{M}\sigma + \mathcal{M}_1]}{4\mathcal{V}^2} \sin^4 \mathcal{E}
\]

\[
- \frac{\sin^2 \mathcal{E}}{\mathcal{V}} \sin[\mathcal{M}\sigma + \mathcal{M}_1] + \mathcal{M}_3 \right] \right) e_3
\]

\[
+ \frac{\mathcal{M}_2}{\mathcal{V}} \sin^3 \mathcal{E} \sin[\mathcal{M}\sigma + \mathcal{M}_1] + \mathcal{M}_4 \right] e_3,
\]

where

\[
(4.6) \quad \mathcal{M} = \left( \frac{\sqrt{1 + \kappa^2}}{\sin \mathcal{E}} - \cos \mathcal{E} \right) \quad \text{and} \quad \mathcal{V} = \sqrt{1 + \kappa^2} - \frac{1}{2} \sin 2\mathcal{E}.
\]

Substituting (4.4) and (4.6) in (4.1) we have (4.3), which completes the proof.

**Corollary 4.3.** Let \( \alpha : I \rightarrow S^3_{Heis} \) be a unit speed non-geodesic biharmonic \( S \)-curve \( \gamma_{ts} \) its Smarandache \( ts \) curve. Then, the parametric equations of...
Smarandache ts curve are

\[ x_{ts}(\sigma) = \frac{1}{\sqrt{1 + 2\kappa_g^2}} \left[ \sin \mathcal{E} \sin[\mathcal{M}\sigma + \mathcal{M}_1] + \frac{1}{\kappa_g} [\sin \mathcal{E} \cos[\mathcal{M}\sigma + \mathcal{M}_1] \mathcal{M} + \sin \mathcal{E}] \right. \\
\left. + \frac{\sin^2 \mathcal{E}}{\mathcal{V}} \cos[\mathcal{M}\sigma + \mathcal{M}_1 + \mathcal{M}_2] \right], \]

(4.7)

\[ y_{ts}(\sigma) = \frac{1}{\sqrt{1 + 2\kappa_g^2}} \left[ \sin \mathcal{E} \cos[\mathcal{M}\sigma + \mathcal{M}_1] + \frac{1}{\kappa_g} [\sin \mathcal{E} \sin[\mathcal{M}\sigma + \mathcal{M}_1] \mathcal{M} + \cos \mathcal{E}] \right. \\
\left. + \frac{\sin^2 \mathcal{E}}{\mathcal{V}} \sin[\mathcal{M}\sigma + \mathcal{M}_1 + \mathcal{M}_3] \right], \]

\[ z_{ts}(\sigma) = \frac{1}{\sqrt{1 + 2\kappa_g^2}} \left[ \cos \mathcal{E} + \frac{1}{\kappa_g} \cos \mathcal{E} \sigma - \frac{\mathcal{V}\sigma + \mathcal{M}_1}{2\mathcal{V}^2} \sin^4 \mathcal{E} - \frac{\sin 2[\mathcal{M}\sigma + \mathcal{M}_1]}{4\mathcal{V}^2} \sin^4 \mathcal{E} \right. \\
\left. - \frac{\sin^2 \mathcal{E}}{\mathcal{V}} \sin[\mathcal{M}\sigma + \mathcal{M}_1 + \mathcal{M}_3] [- \frac{\sin^2 \mathcal{E}}{\mathcal{V}} \cos[\mathcal{M}\sigma + \mathcal{M}_1] + \mathcal{M}_2] \right. \\
\left. + \frac{\mathcal{M}_2}{\mathcal{V}} \sin^3 \mathcal{E} \sin[\mathcal{M}\sigma + \mathcal{M}_1 + \mathcal{M}_4] \right], \]

where \( \mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3, \mathcal{M}_4 \) are constants of integration and

\[ \mathcal{M} = \left( \frac{\sqrt{1 + 2\kappa_g^2}}{\sin \mathcal{E}} - \cos \mathcal{E} \right) \text{ and } \mathcal{V} = \sqrt{1 + \kappa_g^2} - \frac{1}{2} \sin 2\mathcal{E}. \]

**Proof.** Substituting (2.1) to (4.2), we have (4.7) as desired.

If we use Mathematica both unit speed non-geodesic biharmonic S-curve and its Smarandache ts curve, we have
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References


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