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NEW APPROACH ON SPACELIKE BIHARMONIC CURVES WITH TIMELIKE BINORMAL IN TERMS OF ONE PARAMETER SUBGROUP ACCORDING TO FLAT METRIC IN LORENTZIAN HEISENBERG GROUP HEIS^3

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Abstract. In this paper, we obtain new approach on spacelike biharmonic curves with timelike binormal according to flat metric in terms of one parameter subgroup in the Lorentzian Heisenberg group Heis^3. We characterize spacelike biharmonic curves with timelike binormal in terms of one parameter subgroup in the Lorentzian Heisenberg group Heis^3.

1. Introduction

Lie groups represent the best-developed theory of continuous symmetry of mathematical objects and structures, which makes them indispensable tools for many parts of contemporary mathematics, as well as for modern theoretical physics. They provide a natural framework for analysing the continuous symmetries of differential equations (Differential Galois theory), in much the same way as permutation groups are used in Galois theory for analysing the discrete symmetries of algebraic equations.

Such one-parameter groups are of basic importance in the theory of Lie groups, for which every element of the associated Lie algebra defines such a homomorphism, the exponential map. In the case of matrix groups it is given by the matrix exponential.

Firstly, harmonic maps are given as follows:

Harmonic maps $f : (M, g) \longrightarrow (N, h)$ between Riemannian manifolds are the critical points of the energy

$$E(f) = \frac{1}{2} \int_M |df|^2 v_g, \tag{1.1}$$

and they are therefore the solutions of the corresponding Euler–Lagrange equation. This equation is given by the vanishing of the tension field

$$\tau(f) = \mathrm{trace} \nabla df. \tag{1.2}$$

Secondly, biharmonic maps are given as follows:

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The bienergy of a map f by

$$E_2(f) = \frac{1}{2}\int_M |\tau(f)|^2 v_g, \tag{1.3}$$

and say that is biharmonic if it is a critical point of the bienergy.

Jiang derived the first and the second variation formula for the bienergy, showing that the Euler–Lagrange equation associated to E_2 is

$$\begin{aligned}\tau_2(f) &= -\mathcal{J}^f(\tau(f)) = -\Delta\tau(f) - \operatorname{trace} R^N(df, \tau(f))\,df \\ &= 0,\end{aligned} \tag{1.4}$$

where $\mathcal{J}^f$ is the Jacobi operator of f . The equation $\tau_2(f) = 0$ is called the biharmonic equation. Since $\mathcal{J}^f$ is linear, any harmonic map is biharmonic.

In this paper, we obtain new approach on spacelike biharmonic curves with timelike binormal according to flat metric in terms of one parameter subgroup in the Lorentzian Heisenberg group Heis^3. We characterize spacelike biharmonic curves with timelike binormal in terms of one parameter subgroup in the Lorentzian Heisenberg group Heis^3.

2. The Lorentzian Heisenberg Group Heis^3

The Heisenberg group Heis^3 is a Lie group which is diffeomorphic to $\mathbb{R}^3$ and the group operation is defined as

$$(x, y, z) * (\overline{x}, \overline{y}, \overline{z}) = (x + \overline{x}, y + \overline{y}, z + \overline{z} - \overline{x}y + x\overline{y}).$$

The identity of the group is $(0, 0, 0)$ and the inverse of (x, y, z) is given by $(-x, -y, -z)$. The left-invariant Lorentz metric on Heis^3 is

$$g = dx^2 + (xdy + dz)^2 - ((1 - x)\,dy - dz)^2.$$

The following set of left-invariant vector fields forms an orthonormal basis for the corresponding Lie algebra:

$$\left\{\mathbf{e}_1 = \frac{\partial}{\partial x},\ \mathbf{e}_2 = \frac{\partial}{\partial y} + (1 - x)\frac{\partial}{\partial z},\ \mathbf{e}_3 = \frac{\partial}{\partial y} - x\frac{\partial}{\partial z}\right\}. \tag{2.1}$$

The characterising properties of this algebra are the following commutation relations:

$$[\mathbf{e}_2, \mathbf{e}_3] = 0,\quad [\mathbf{e}_3, \mathbf{e}_1] = \mathbf{e}_2 - \mathbf{e}_3,\quad [\mathbf{e}_2, \mathbf{e}_1] = \mathbf{e}_2 - \mathbf{e}_3,$$

with

$$g(\mathbf{e}_1, \mathbf{e}_1) = g(\mathbf{e}_2, \mathbf{e}_2) = 1,\quad g(\mathbf{e}_3, \mathbf{e}_3) = -1. \tag{2.2}$$

Proposition 2.1. *For the covariant derivatives of the Levi-Civita connection of the left-invariant metric g, defined above the following is true:*

$$\nabla = \begin{pmatrix} 0 & 0 & 0 \\ \mathbf{e}_2 - \mathbf{e}_3 & -\mathbf{e}_1 & -\mathbf{e}_1 \\ \mathbf{e}_2 - \mathbf{e}_3 & -\mathbf{e}_1 & -\mathbf{e}_1 \end{pmatrix}, \tag{2.3}$$

where the (i, j)-element in the table above equals $\nabla_{e_i} e_j$ for our basis

$$\{\mathbf{e}_k, k = 1, 2, 3\} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}.$$

So we obtain that

$$R(\mathbf{e}_1, \mathbf{e}_3) = R(\mathbf{e}_1, \mathbf{e}_2) = R(\mathbf{e}_2, \mathbf{e}_3) = 0. \tag{2.4}$$

Then, the Lorentz metric g is flat.

3. Spacelike Biharmonic Curves with Timelike Binormal According to Flat Metric in the Lorentzian Heisenberg Group Heis^3

Let $\gamma : I \longrightarrow Heis^3$ be a unit speed spacelike curve with timelike binormal and $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ are Frenet vector fields, then Frenet formulas are as follows

$$\begin{aligned} \nabla_{\mathbf{t}}\mathbf{t} &= \kappa_1 \mathbf{n}, \\ \nabla_{\mathbf{t}}\mathbf{n} &= -\kappa_1 \mathbf{t} + \kappa_2 \mathbf{b}, \\ \nabla_{\mathbf{t}}\mathbf{b} &= \kappa_2 \mathbf{n}, \end{aligned} \tag{3.1}$$

where κ_1, κ_2 are curvature function and torsion function, respectively and

$$\begin{aligned} &g(\mathbf{t}, \mathbf{t}) = 1, \ g(\mathbf{n}, \mathbf{n}) = 1, \ g(\mathbf{b}, \mathbf{b}) = -1, \\ &g(\mathbf{t}, \mathbf{n}) = g(\mathbf{t}, \mathbf{b}) = g(\mathbf{n}, \mathbf{b}) = 0. \end{aligned}$$

With respect to the orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ we can write

$$\begin{aligned} \mathbf{t} &= t_1 \mathbf{e}_1 + t_2 \mathbf{e}_2 + t_3 \mathbf{e}_3, \\ \mathbf{n} &= n_1 \mathbf{e}_1 + n_2 \mathbf{e}_2 + n_3 \mathbf{e}_3, \\ \mathbf{b} &= b_1 \mathbf{e}_1 + b_2 \mathbf{e}_2 + b_3 \mathbf{e}_3. \end{aligned}$$

Theorem 3.1. *If $\gamma : I \longrightarrow Heis^3$ is a unit speed spacelike biharmonic curve with timelike binormal according to flat metric, then*

$$\begin{aligned} \kappa_1 &= \text{constant} \neq 0, \\ \kappa_1^2 - \kappa_2^2 &= 0, \\ \kappa_2 &= \text{constant}. \end{aligned} \tag{3.2}$$

Corollary 3.2. *If $\gamma : I \longrightarrow Heis^3$ is a unit speed spacelike biharmonic curve with timelike binormal, then γ is a helix.*

4. Biharmonic Curves in terms of One-Parameter Subgroup of Lorentzian Heisenberg group

One-parameter groups describe dynamical systems. Furthermore, whenever a system of physical laws admits a one-parameter group of differentiable symmetries, then there is a conserved quantity, by Noether's theorem.

Definition 4.1. *For each $\mathbb{X} \in heis^3$, $\gamma : R \to Heis^3, t \to \gamma(t) = \exp t\mathbb{X}$ is analytic homomorphism then γ is called one-parameter subgroup of Lorentzian Heisenberg group.*

The action of a one-parameter group on a set is known as a flow.

Definition 4.2. *The mapping $\mathbb{X} \to \exp \mathbb{X}$ is called the exponential mapping.*
We have the formula

$$\exp(t+u)\mathbb{X} = \exp t\mathbb{X} \exp u\mathbb{X},$$

where $\forall s, u \in \mathbb{R}$ and $\forall \mathbb{X} \in heis^3$.

Firstly, let us calculate the arbitrary parameter t according to the arclength parameter s. It is well known that

$$s = \int_0^t g(\gamma'(t), \gamma'(t))^{\frac{1}{2}} dt, \tag{4.1}$$

where

$$\gamma'(t) = \mathbb{X}\gamma, \ \mathbb{X}\gamma = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3. \tag{4.2}$$

Substituting above equation in (4.1), we have

$$s = g(\mathbb{X}\gamma, \mathbb{X}\gamma)^{\frac{1}{2}} t$$

The first, second and third derivatives of γ are given as follows:

$$\begin{aligned} \gamma'(s) &= \frac{\mathbb{X}\gamma}{g(\mathbb{X}\gamma, \mathbb{X}\gamma)^{\frac{1}{2}}}, \\ \gamma''(s) &= \frac{\mathbb{X}^2\gamma}{g(\mathbb{X}\gamma, \mathbb{X}\gamma)^{2}}, \\ \gamma'''(s) &= \frac{\mathbb{X}^3\gamma}{g(\mathbb{X}\gamma, \mathbb{X}\gamma)^{\frac{3}{2}}}, \end{aligned} \tag{4.3}$$

where

$$\begin{aligned} \mathbb{X}^2\gamma &= [\mathbb{X} * \mathbb{X}]\gamma, \\ \mathbb{X}^3\gamma &= \left[\mathbb{X} * \mathbb{X}^2\right]\gamma. \end{aligned}$$

Using above sytems, we obtain following results.

Theorem 4.3. *Let $\gamma : I \longrightarrow Heis^3$ be a unit speed non-geodesic spacelike biharmonic curve in the $Heis^3$. Then,*

$$\mathbb{X}\gamma = g(\mathbb{X}\gamma, \mathbb{X}\gamma)^{\frac{1}{2}} [\cosh\varphi \mathbf{e}_1 + \sinh\varphi \sinh[\frac{\kappa_1 s}{\sinh\varphi} + \ell]\mathbf{e}_2 + \sinh\varphi\cosh[\frac{\kappa_1 s}{\sinh\varphi} + \ell]\mathbf{e}_3],$$

$$
\begin{aligned}
\mathbb{X}^2\gamma = & -\frac{g\left(\mathbb{X}^2\gamma,\mathbb{X}^2\gamma\right)^{\frac{1}{2}}}{\kappa_1}\sinh^2\varphi(\sinh^2[\frac{\kappa_1 s}{\cosh\varphi}+\ell]+\sinh[\frac{\kappa_1 s}{\cosh\varphi}+\ell]\cosh[\frac{\kappa_1 s}{\cosh\varphi}+\ell])\mathbf{e}_1 \\
& +\frac{g\left(\mathbb{X}^2\gamma,\mathbb{X}^2\gamma\right)^{\frac{1}{2}}}{\kappa_1}(\kappa_1\cosh[\frac{\kappa_1 s}{\cosh\varphi}+\ell]+\sinh\varphi\cosh\varphi\sinh[\frac{\kappa_1 s}{\cosh\varphi}+\ell] \\
& +\sinh\varphi\cosh\varphi\cosh[\frac{\kappa_1 s}{\cosh\varphi}+\ell])\mathbf{e}_2 \\
& +\frac{g\left(\mathbb{X}^2\gamma,\mathbb{X}^2\gamma\right)^{\frac{1}{2}}}{\kappa_1}(\kappa_1\sinh[\frac{\kappa_1 s}{\cosh\varphi}+\ell]-\sinh\varphi\cosh\varphi\sinh[\frac{\kappa_1 s}{\cosh\varphi}+\ell] \\
& -\sinh\varphi\cosh\varphi\cosh[\frac{\kappa_1 s}{\cosh\varphi}+\ell])\mathbf{e}_3,
\end{aligned}
\tag{4.4}
$$

$$
\begin{aligned}
\mathbb{X}^3\gamma = & \, g\left(\mathbb{X}\gamma,\mathbb{X}\gamma\right)^{\frac{3}{2}}\Bbbk[\frac{1}{\kappa_1}\sinh\varphi\sinh[\frac{\kappa_1 s}{\sinh\varphi}+C](\kappa_1\sinh[\frac{\kappa_1 s}{\cosh\varphi}+\ell] \\
& -\sinh\varphi\cosh\varphi\sinh[\frac{\kappa_1 s}{\cosh\varphi}+\ell]-\sinh\varphi\cosh\varphi\cosh[\frac{\kappa_1 s}{\cosh\varphi}+\ell]) \\
& -\frac{1}{\kappa_1}\sinh\varphi\cosh[\frac{\kappa_1 s}{\sinh\varphi}+C](\kappa_1\cosh[\frac{\kappa_1 s}{\cosh\varphi}+\ell] \\
& +\sinh\varphi\cosh\varphi\sinh[\frac{\kappa_1 s}{\cosh\varphi}+\ell]+\sinh\varphi\cosh\varphi\cosh[\frac{\kappa_1 s}{\cosh\varphi}+\ell])]\mathbf{e}_1 \\
& +g\left(\mathbb{X}\gamma,\mathbb{X}\gamma\right)^{\frac{3}{2}}\Bbbk[\frac{1}{\kappa_1}\cosh\varphi(\kappa_1\sinh[\frac{\kappa_1 s}{\cosh\varphi}+\ell]-\sinh\varphi\cosh\varphi\sinh[\frac{\kappa_1 s}{\cosh\varphi}+\ell] \\
& -\sinh\varphi\cosh\varphi\cosh[\frac{\kappa_1 s}{\cosh\varphi}+\ell]) \\
& +\frac{1}{\kappa_1}\sinh\varphi\cosh[\frac{\kappa_1 s}{\sinh\varphi}+C]\sinh^2\varphi(\sinh^2[\frac{\kappa_1 s}{\cosh\varphi}+\ell] \\
& +\sinh[\frac{\kappa_1 s}{\cosh\varphi}+\ell]\cosh[\frac{\kappa_1 s}{\cosh\varphi}+\ell])]\mathbf{e}_2 \\
& -g\left(\mathbb{X}\gamma,\mathbb{X}\gamma\right)^{\frac{3}{2}}\Bbbk[\frac{1}{\kappa_1}\cosh\varphi(\kappa_1\cosh[\frac{\kappa_1 s}{\cosh\varphi}+\ell]+\sinh\varphi\cosh\varphi\sinh[\frac{\kappa_1 s}{\cosh\varphi}+\ell] \\
& +\sinh\varphi\cosh\varphi\cosh[\frac{\kappa_1 s}{\cosh\varphi}+\ell])+\frac{1}{\kappa_1}\sinh^3\varphi(\sinh^2[\frac{\kappa_1 s}{\cosh\varphi}+\ell] \\
& +\sinh[\frac{\kappa_1 s}{\cosh\varphi}+\ell]\cosh[\frac{\kappa_1 s}{\cosh\varphi}+\ell])\sinh[\frac{\kappa_1 s}{\sinh\varphi}+C]]\mathbf{e}_3 \\
& +g\left(\mathbb{X}^3\gamma,\mathbb{X}\gamma\right)[\cosh\varphi\mathbf{e}_1+\sinh\varphi\sinh[\frac{\kappa_1 s}{\sinh\varphi}+\ell]\mathbf{e}_2+\sinh\varphi\cosh[\frac{\kappa_1 s}{\sinh\varphi}+\ell]\mathbf{e}_3],
\end{aligned}
$$

where ℓ is constant of integration and

$$
\Bbbk=[\frac{g\left(\mathbb{X}^3\gamma,\mathbb{X}^3\gamma\right)}{g\left(\mathbb{X}\gamma,\mathbb{X}\gamma\right)^3}-2\frac{g\left(\mathbb{X}^3\gamma,\mathbb{X}\gamma\right)^2}{g\left(\mathbb{X}\gamma,\mathbb{X}\gamma\right)^{\frac{7}{2}}}+\frac{g\left(\mathbb{X}^3\gamma,\mathbb{X}\gamma\right)}{g\left(\mathbb{X}\gamma,\mathbb{X}\gamma\right)^3}]^{\frac{1}{2}}.
$$

Proof. From (3.1) and Corollary 3.2, imply

$$
\mathbf{t}=\cosh\varphi\mathbf{e}_1+\sinh\varphi\sinh[\frac{\kappa_1 s}{\sinh\varphi}+\ell]\mathbf{e}_2+\sinh\varphi\cosh[\frac{\kappa_1 s}{\sinh\varphi}+\ell]\mathbf{e}_3,
$$

where ℓ is constant of integration.

On the other hand, first equation of (3.3) we have

(4.5)
$$\mathbb{X}\gamma = g\left(\mathbb{X}\gamma,\mathbb{X}\gamma\right)^{\frac{1}{2}}[\cosh\varphi\mathbf{e}_1+\sinh\varphi\sinh[\frac{\kappa_1 s}{\sinh\varphi}+\ell]\mathbf{e}_2+\sinh\varphi\cosh[\frac{\kappa_1 s}{\sinh\varphi}+\ell]\mathbf{e}_3].$$

So, we immediately arrive at

$$\mathbf{n}=\frac{1}{g\left(\mathbb{X}^2\gamma,\mathbb{X}^2\gamma\right)^{\frac{1}{2}}}\mathbb{X}^2\gamma. \tag{4.6}$$

Using first equation of the system (3.2) and (2.3), we have

$$\begin{aligned}\nabla_{\mathbf{t}}\mathbf{t}=&(t_1'-t_2^2-t_2t_3)\mathbf{e}_1+(t_2'+t_1t_2+t_1t_3)\mathbf{e}_2\\&+(t_3'-t_1t_2-t_1t_3)\mathbf{e}_3.\end{aligned}$$

By the use of Frenet formulas and above equation, we get

$$\begin{aligned}\mathbb{X}^2\gamma=&-\frac{g\left(\mathbb{X}^2\gamma,\mathbb{X}^2\gamma\right)^{\frac{1}{2}}}{\kappa_1}\sinh^2\varphi(\sinh^2[\frac{\kappa_1 s}{\cosh\varphi}+\ell]+\sinh[\frac{\kappa_1 s}{\cosh\varphi}+\ell]\cosh[\frac{\kappa_1 s}{\cosh\varphi}+\ell])\mathbf{e}_1\\&+\frac{g\left(\mathbb{X}^2\gamma,\mathbb{X}^2\gamma\right)^{\frac{1}{2}}}{\kappa_1}(\kappa_1\cosh[\frac{\kappa_1 s}{\cosh\varphi}+\ell]+\sinh\varphi\cosh\varphi\sinh[\frac{\kappa_1 s}{\cosh\varphi}+\ell]\\&+\sinh\varphi\cosh\varphi\cosh[\frac{\kappa_1 s}{\cosh\varphi}+\ell])\mathbf{e}_2\\&+\frac{g\left(\mathbb{X}^2\gamma,\mathbb{X}^2\gamma\right)^{\frac{1}{2}}}{\kappa_1}(\kappa_1\sinh[\frac{\kappa_1 s}{\cosh\varphi}+\ell]-\sinh\varphi\cosh\varphi\sinh[\frac{\kappa_1 s}{\cosh\varphi}+\ell]\\&-\sinh\varphi\cosh\varphi\cosh[\frac{\kappa_1 s}{\cosh\varphi}+\ell])\mathbf{e}_3.\end{aligned}$$

Using same calculations we get

$$\mathbf{b}=\frac{1}{\mathbb{k}}[\frac{\mathbb{X}^3\gamma}{g\left(\mathbb{X}\gamma,\mathbb{X}\gamma\right)^{\frac{3}{2}}}-\frac{g\left(\mathbb{X}^3\gamma,\mathbb{X}\gamma\right)}{g\left(\mathbb{X}\gamma,\mathbb{X}\gamma\right)^{2}}\mathbb{X}\gamma],$$

where

$$\mathbb{k}=[\frac{g\left(\mathbb{X}^3\gamma,\mathbb{X}^3\gamma\right)}{g\left(\mathbb{X}\gamma,\mathbb{X}\gamma\right)^{3}}-2\frac{g\left(\mathbb{X}^3\gamma,\mathbb{X}\gamma\right)^2}{g\left(\mathbb{X}\gamma,\mathbb{X}\gamma\right)^{\frac{7}{2}}}+\frac{g\left(\mathbb{X}^3\gamma,\mathbb{X}\gamma\right)}{g\left(\mathbb{X}\gamma,\mathbb{X}\gamma\right)^{3}}]^{\frac{1}{2}}.$$

From above equation, we have

$$\mathbb{X}^3\gamma=g\left(\mathbb{X}\gamma,\mathbb{X}\gamma\right)^{\frac{3}{2}}\mathbb{k}\mathbf{b}+\frac{g\left(\mathbb{X}^3\gamma,\mathbb{X}\gamma\right)}{g\left(\mathbb{X}\gamma,\mathbb{X}\gamma\right)^{\frac{1}{2}}}\mathbb{X}\gamma.$$

Cross product of $\mathbf{t}\times\mathbf{n}=\mathbf{b}$ gives us

$$\begin{aligned}
\mathbb{X}^3\gamma=&g\left(\mathbb{X}\gamma,\mathbb{X}\gamma\right)^{\frac{3}{2}}\mathbb{k}[\frac{1}{\kappa_1}\sinh\varphi\sinh[\frac{\kappa_1 s}{\sinh\varphi}+C](\kappa_1\sinh[\frac{\kappa_1 s}{\cosh\varphi}+\ell]\\
&-\sinh\varphi\cosh\varphi\sinh[\frac{\kappa_1 s}{\cosh\varphi}+\ell]-\sinh\varphi\cosh\varphi\cosh[\frac{\kappa_1 s}{\cosh\varphi}+\ell])\\
&-\frac{1}{\kappa_1}\sinh\varphi\cosh[\frac{\kappa_1 s}{\sinh\varphi}+C](\kappa_1\cosh[\frac{\kappa_1 s}{\cosh\varphi}+\ell]\\
&+\sinh\varphi\cosh\varphi\sinh[\frac{\kappa_1 s}{\cosh\varphi}+\ell]+\sinh\varphi\cosh\varphi\cosh[\frac{\kappa_1 s}{\cosh\varphi}+\ell])]\mathbf{e}_1\\
&+g\left(\mathbb{X}\gamma,\mathbb{X}\gamma\right)^{\frac{3}{2}}\mathbb{k}[\frac{1}{\kappa_1}\cosh\varphi(\kappa_1\sinh[\frac{\kappa_1 s}{\cosh\varphi}+\ell]-\sinh\varphi\cosh\varphi\sinh[\frac{\kappa_1 s}{\cosh\varphi}+\ell]\\
&-\sinh\varphi\cosh\varphi\cosh[\frac{\kappa_1 s}{\cosh\varphi}+\ell])\\
&+\frac{1}{\kappa_1}\sinh\varphi\cosh[\frac{\kappa_1 s}{\sinh\varphi}+C]\sinh^2\varphi(\sinh^2[\frac{\kappa_1 s}{\cosh\varphi}+\ell]\\
&+\sinh[\frac{\kappa_1 s}{\cosh\varphi}+\ell]\cosh[\frac{\kappa_1 s}{\cosh\varphi}+\ell])]\mathbf{e}_2\\
&-g\left(\mathbb{X}\gamma,\mathbb{X}\gamma\right)^{\frac{3}{2}}\mathbb{k}[\frac{1}{\kappa_1}\cosh\varphi(\kappa_1\cosh[\frac{\kappa_1 s}{\cosh\varphi}+\ell]+\sinh\varphi\cosh\varphi\sinh[\frac{\kappa_1 s}{\cosh\varphi}+\ell]\\
&+\sinh\varphi\cosh\varphi\cosh[\frac{\kappa_1 s}{\cosh\varphi}+\ell])+\frac{1}{\kappa_1}\sinh^3\varphi(\sinh^2[\frac{\kappa_1 s}{\cosh\varphi}+\ell]\\
&+\sinh[\frac{\kappa_1 s}{\cosh\varphi}+\ell]\cosh[\frac{\kappa_1 s}{\cosh\varphi}+\ell])\sinh[\frac{\kappa_1 s}{\sinh\varphi}+C]]\mathbf{e}_3\\
&+g\left(\mathbb{X}^3\gamma,\mathbb{X}\gamma\right)[\cosh\varphi\mathbf{e}_1+\sinh\varphi\sinh[\frac{\kappa_1 s}{\sinh\varphi}+\ell]\mathbf{e}_2+\sinh\varphi\cosh[\frac{\kappa_1 s}{\sinh\varphi}+\ell]\mathbf{e}_3],
\end{aligned}$$

where

$$\mathbb{k}=[\frac{g\left(\mathbb{X}^3\gamma,\mathbb{X}^3\gamma\right)}{g\left(\mathbb{X}\gamma,\mathbb{X}\gamma\right)^3}-2\frac{g\left(\mathbb{X}^3\gamma,\mathbb{X}\gamma\right)^2}{g\left(\mathbb{X}\gamma,\mathbb{X}\gamma\right)^{\frac{7}{2}}}+\frac{g\left(\mathbb{X}^3\gamma,\mathbb{X}\gamma\right)}{g\left(\mathbb{X}\gamma,\mathbb{X}\gamma\right)^3}]^{\frac{1}{2}}.$$

So, the proof is completed.

In the light of Theorem 4.3, we express the following corollary without proof:

Corollary 4.4.

$$\mathbb{X}^3\gamma=g\left(\mathbb{X}\gamma,\mathbb{X}\gamma\right)^{\frac{3}{2}}\mathbb{k}\mathbf{b}+g\left(\mathbb{X}^3\gamma,\mathbb{X}\gamma\right)\mathbf{t},$$

where

$$\mathbb{k}=[\frac{g\left(\mathbb{X}^3\gamma,\mathbb{X}^3\gamma\right)}{g\left(\mathbb{X}\gamma,\mathbb{X}\gamma\right)^3}-2\frac{g\left(\mathbb{X}^3\gamma,\mathbb{X}\gamma\right)^2}{g\left(\mathbb{X}\gamma,\mathbb{X}\gamma\right)^{\frac{7}{2}}}+\frac{g\left(\mathbb{X}^3\gamma,\mathbb{X}\gamma\right)}{g\left(\mathbb{X}\gamma,\mathbb{X}\gamma\right)^3}]^{\frac{1}{2}}.$$

References

[1] D. E. Blair: *Contact Manifolds in Riemannian Geometry*, Lecture Notes in Mathematics, Springer-Verlag 509, Berlin-New York, 1976.
[2] M.do Carmo: *Differential Geometry of Curves and Surfaces,* Prentice Hall, New Jersey 1976.
[3] J. Eells and J. H. Sampson: *Harmonic mappings of Riemannian manifolds,* Amer. J. Math. 86 (1964), 109–160.
[4] R.T. Farouki, C.A. Neff: *Algebraic properties of plane offset curves*, Comput. Aided Geom. Design 7 (1990), 101–127.
[5] S. Helgason: *Differential Geometry, Lie Groups, and Symmetric Spaces.* Pure and Applied Mathematics. Academic Press, 1978.
[6] G. Y.Jiang: *2-harmonic isometric immersions between Riemannian manifolds,* Chinese Ann. Math. Ser. A 7(2) (1986), 130–144.
[7] G. Y.Jiang: *2-harmonic isometric immersions between Riemannian manifolds,* Chinese Ann. Math. Ser. A 7(2) (1986), 130–144.
[8] T. Körpınar, E. Turhan: *On characterization of B-canal surfaces in terms of biharmonic B-slant helices according to Bishop frame in Heisenberg group* $Heis^3$, J. Math. Anal. Appl. 382 (2011), 57–65.
[9] E. Kruppa: *Analytische und konstruktive Differentialgeometrie,* Springer Verlag, Wien, 1957.
[10] J. Milnor: *Curvatures of left invariant metrics on Lie groups.* Adv. Math. 21(3) (1976), 293–329.
[11] Y. Ou and Z. Wang: *Linear Biharmonic Maps into Sol, Nil and Heisenberg Spaces,* Mediterr. j. math. 5 (2008), 379–394.
[12] S. Rahmani: *Metriqus de Lorentz sur les groupes de Lie unimodulaires, de dimension trois*, Journal of Geometry and Physics 9 (1992), 295-302.
[13] E. Turhan, T. Körpınar, *On spacelike biharmonic new type b-slant helices with timelike* m_2 *according to Bishop frame in Lorentzian Heisenberg group* H^3, Advanced Modeling and Optimization, 14 (1) (2012), 297-302.
[14] E. Turhan and T. Körpınar: *On Characterization Of Timelike Horizontal Biharmonic Curves In The Lorentzian Heisenberg Group* $Heis^3$, Zeitschrift für Naturforschung A- A Journal of Physical Sciences 65a (2010), 641-648.
[15] E. Turhan and T. Körpınar: *Parametric equations of general helices in the sol space* $\mathfrak{Sol}^3$, Bol. Soc. Paran. Mat. 31 (1) (2013), 99–104.

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