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# CHARACTERIZATION OF SMARANDACHE $\mathbf{M_1}\mathrm{M}_2$ CURVES OF SPACELIKE BIHARMONIC $\mathcal{B}-$SLANT HELICES ACCORDING TO BISHOP FRAME IN $\mathbb{E}(1,1)$

TALAT KÖRPINAR AND ESSIN TURHAN

ABSTRACT. In this paper, we find parametric equations of Smarandache $\mathbf{M_1M_2}$ curves of spacelike biharmonic $\mathcal{B}-$slant helices according to Bishop frame in terms of Bishop curvatures in the Lorentzian group of rigid motions $\mathbb{E}(1,1)$. Finally, we construct Bishop equations of Smarandache $\mathbf{M_1M_2}$ curves of spacelike biharmonic $\mathcal{B}-$slant helices in $\mathbb{E}(1,1)$.

## 1. Introduction

A curve of constant slope or general helix is defined by the property that the tangent lines make a constant angle with a fixed direction. A necessary and sufficient condition that a curve to be general helix is that ratio of curvature to torsion be constant. Indeed, a helix is a special case of the gerenal helix. If both curvature and torsion are non-zero constants, it is called a helix or only a W-curve.

On the other hand, the Bishop frame or parallel transport frame is an alternative approach to defining a moving frame that is well defined even when the curve has vanishing second derivative. We can parallel transport an orthonormal frame along a curve simply by parallel transporting each component of the frame.

In this paper, we find parametric equations of Smarandache $\mathbf{M_1M_2}$ curves of spacelike biharmonic $\mathcal{B}-$slant helices according to Bishop frame in terms of Bishop curvatures in the Lorentzian group of rigid motions $\mathbb{E}(1,1)$. Finally, we construct Bishop equations of Smarandache $\mathbf{M_1M_2}$ curves of spacelike biharmonic $\mathcal{B}-$slant helices in $\mathbb{E}(1,1)$.

## 2. Preliminaries

Let $\mathbb{E}(1,1)$ be the group of rigid motions of Euclidean 2-space. This consists of all matrices of the form

$$\begin{pmatrix} \cosh x & \sinh x & y \\ \sinh x & \cosh x & z \\ 0 & 0 & 1 \end{pmatrix}.$$

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Topologically, $\mathbb{E}(1,1)$ is diffeomorphic to $\mathbb{R}^3$ under the map

$$\mathbb{E}(1,1) \longrightarrow \mathbb{R}^3 : \begin{pmatrix} \cosh x & \sinh x & y \\ \sinh x & \cosh x & z \\ 0 & 0 & 1 \end{pmatrix} \longrightarrow (x,y,z),$$

It's Lie algebra has a basis consisting of

$$\mathbf{X}_1 = \frac{\partial}{\partial x},\ \mathbf{X}_2 = \cosh x \frac{\partial}{\partial y} + \sinh x \frac{\partial}{\partial z},\ \mathbf{X}_3 = \sinh x \frac{\partial}{\partial y} + \cosh x \frac{\partial}{\partial z},$$

for which

$$[\mathbf{X}_1, \mathbf{X}_2] = \mathbf{X}_3,\ [\mathbf{X}_2, \mathbf{X}_3] = 0,\ [\mathbf{X}_1, \mathbf{X}_3] = \mathbf{X}_2.$$

Put

$$x^1 = x,\ x^2 = \frac{1}{2}(y+z),\ x^3 = \frac{1}{2}(y-z).$$

Then, we get

(2.1)
$$\mathbf{X}_1 = \frac{\partial}{\partial x^1},\ \mathbf{X}_2 = \frac{1}{2}\left(e^{x^1}\frac{\partial}{\partial x^2} + e^{-x^1}\frac{\partial}{\partial x^3}\right),\ \mathbf{X}_3 = \frac{1}{2}\left(e^{x^1}\frac{\partial}{\partial x^2} - e^{-x^1}\frac{\partial}{\partial x^3}\right).$$

The bracket relations are

$$[\mathbf{X}_1, \mathbf{X}_2] = \mathbf{X}_3,\ [\mathbf{X}_2, \mathbf{X}_3] = 0,\ [\mathbf{X}_1, \mathbf{X}_3] = \mathbf{X}_2.$$

We consider left-invariant Lorentzian metrics which has a pseudo-orthonormal basis $\{\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3\}$. We consider left-invariant Lorentzian metric [12], given by

$$g = -\left(dx^1\right)^2 + \left(e^{-x^1}dx^2 + e^{x^1}dx^3\right)^2 + \left(e^{-x^1}dx^2 - e^{x^1}dx^3\right)^2,$$

where

$$g(\mathbf{X}_1, \mathbf{X}_1) = -1,\ g(\mathbf{X}_2, \mathbf{X}_2) = g(\mathbf{X}_3, \mathbf{X}_3) = 1.$$

Let coframe of our frame be defined by

$$\theta^1 = dx^1,\ \theta^2 = e^{-x^1}dx^2 + e^{x^1}dx^3,\ \theta^3 = e^{-x^1}dx^2 - e^{x^1}dx^3.$$

## 3. Smarandache $\mathbf{M_1M_2}$ Curves of Spacelike Biharmonic B-Slant Helices in the Lorentzian Group of Rigid Motions $\mathbb{E}(1,1)$

Let $\gamma : I \longrightarrow \mathbb{E}(1,1)$ be a non geodesic spacelike curve on the $\mathbb{E}(1,1)$ parametrized by arc length. Let $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ be the Frenet frame fields tangent to the $\mathbb{E}(1,1)$ along $\gamma$ defined as follows:

$\mathbf{T}$ is the unit vector field $\gamma'$ tangent to $\gamma$, $\mathbf{N}$ is the unit vector field in the direction of $\nabla_{\mathbf{T}}\mathbf{T}$ (normal to $\gamma$), and $\mathbf{B}$ is chosen so that $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ is a positively oriented orthonormal basis. Then, we have the following Frenet formulas:

(3.1)
$$\begin{aligned} \nabla_{\mathbf{T}}\mathbf{T} &= \kappa\mathbf{N}, \\ \nabla_{\mathbf{T}}\mathbf{N} &= \kappa\mathbf{T} + \tau\mathbf{B}, \\ \nabla_{\mathbf{T}}\mathbf{B} &= \tau\mathbf{N}, \end{aligned}$$

where $\kappa$ is the curvature of $\gamma$ and $\tau$ is its torsion and

$$\begin{aligned} g(\mathbf{T}, \mathbf{T}) = 1,\ g(\mathbf{N}, \mathbf{N}) = -1,\ g(\mathbf{B}, \mathbf{B}) = 1, \\ g(\mathbf{T}, \mathbf{N}) = g(\mathbf{T}, \mathbf{B}) = g(\mathbf{N}, \mathbf{B}) = 0. \end{aligned}$$

The Bishop frame or parallel transport frame is an alternative approach to defining a moving frame that is well defined even when the curve has vanishing second derivative. The Bishop frame is expressed as

$$\begin{aligned} \nabla_{\mathbf{T}}\mathbf{T} &= k_1\mathbf{M}_1 - k_2\mathbf{M}_2, \\ \nabla_{\mathbf{T}}\mathbf{M}_1 &= k_1\mathbf{T}, \\ \nabla_{\mathbf{T}}\mathbf{M}_2 &= k_2\mathbf{T}, \end{aligned} \tag{3.2}$$

where

$$g(\mathbf{T},\mathbf{T}) = 1,\ g(\mathbf{M}_1,\mathbf{M}_1) = -1,\ g(\mathbf{M}_2,\mathbf{M}_2) = 1,$$
$$g(\mathbf{T},\mathbf{M}_1) = g(\mathbf{T},\mathbf{M}_2) = g(\mathbf{M}_1,\mathbf{M}_2) = 0.$$

Here, we shall call the set $\{\mathbf{T},\mathbf{M}_1,\mathbf{M}_2\}$ as Bishop trihedra, $k_1$ and $k_2$ as Bishop curvatures and $\tau(s) = \psi'(s)$, $\kappa(s) = \sqrt{|k_2^2 - k_1^2|}$. Thus, Bishop curvatures are defined by

$$\begin{aligned} k_1 &= \kappa(s)\sinh\psi(s), \\ k_2 &= \kappa(s)\cosh\psi(s). \end{aligned}$$

With respect to the orthonormal basis $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ we can write

$$\begin{aligned} \mathbf{T} &= T^1\mathbf{e}_1 + T^2\mathbf{e}_2 + T^3\mathbf{e}_3, \\ \mathbf{M}_1 &= M_1^1\mathbf{e}_1 + M_1^2\mathbf{e}_2 + M_1^3\mathbf{e}_3, \\ \mathbf{M}_2 &= M_2^1\mathbf{e}_1 + M_2^2\mathbf{e}_2 + M_2^3\mathbf{e}_3. \end{aligned} \tag{3.3}$$

**Definition 3.1.** *Let $\gamma : I \longrightarrow \mathbb{E}(1,1)$ be a unit speed regular curve in the Lorentzian group of rigid motions $\mathbb{E}(1,1)$. and $\{\mathbf{T},\mathbf{M_1},\mathbf{M}_2\}$ be its moving Bishop frame. Smarandache $\mathbf{M_1M_2}$ curves are defined by*

$$\gamma_{\mathbf{M_1M_2}} = \frac{1}{|k_1+k_2|}\left(\mathbf{M_1}+\mathbf{M}_2\right). \tag{3.4}$$

**Definition 3.2**. *[7], A regular spacelike curve $\gamma : I \longrightarrow \mathbb{E}(1,1)$ is called a $\mathcal{B}-$slant helix provided the timelike unit vector $\mathbf{M}_1$ of the curve $\gamma$ has constant angle $\theta$ with some fixed timelike unit vector $u$, that is*

$$g(\mathbf{M}_1(s),u) = \cosh\wp \text{ for all } s \in I.$$

**Lemma 3.3.** *[7], Let $\gamma : I \longrightarrow \mathbb{E}(1,1)$ be a unit speed spacelike curve with non-zero natural curvatures. Then $\gamma$ is a $\mathcal{B}-$slant helix if and only if*

$$\frac{k_1}{k_2} = \tanh\wp. \tag{3.5}$$

**Theorem 3.4.** *Let $\gamma : I \longrightarrow \mathbb{E}(1,1)$ is a non geodesic spacelike biharmonic $\mathcal{B}-$slant helix in the Lorentzian group of rigid motions $\mathbb{E}(1,1)$. Then, the parametric* *equations of Smarandache* $\mathbf{M_1M_2}$ *curves of spacelike biharmonic slant helix are*

$$
\begin{aligned}
\gamma_{\mathbf{M_1M_2}}(s) &= \mathfrak{C}(\cosh\wp)\mathbf{X}_1 \\
&+\mathfrak{C}(\sinh\wp\cos[D_1 s + D_2] - \sin[D_1 s + D_2])\mathbf{X}_2 \\
&+\mathfrak{C}(\sinh\wp\sin[D_1 s + D_2] + \cos[D_1 s + D_2])\mathbf{X}_3,
\end{aligned} \tag{3.6}
$$

*where* $D_1, D_2$ *are constants of integration and*

$$\mathfrak{C} = \frac{1}{|k_1 + k_2|}.$$

**Proof.** Assume that $\gamma$ is a non geodesic spacelike biharmonic $\mathcal{B}-$slant helix according to Bishop frame.

Hence, from Definition 3.2, we obtain

$$\mathbf{M}_1 = \cosh\wp\mathbf{X}_1 + \sinh\wp\cos[D_1 s + D_2]\,\mathbf{X}_2 + \sinh\wp\sin[D_1 s + D_2]\,\mathbf{X}_3. \tag{3.7}$$

Using (2.1) in (3.7), we may be written as

$$\mathbf{M}_2 = -\sin[D_1 s + D_2]\,\mathbf{X}_2 + \cos[D_1 s + D_2]\,\mathbf{X}_3. \tag{3.8}$$

Substituting (3.7) and (3.8) in (3.4) we have (3.6), which completes the proof.

Then, we obtain the following corollary.

**Corollary 3.5.** *Let* $\gamma : I \longrightarrow \mathbb{E}(1,1)$ *is a non geodesic spacelike biharmonic* $\mathcal{B}-$*slant helix in the Lorentzian group of rigid motions* $\mathbb{E}(1,1)$. *Then, the parametric equations of Smarandache* $\mathbf{M_1M_2}$ *curves of spacelike biharmonic slant helix are*

$$
\begin{aligned}
x^1_{\mathbf{M_1M_2}}(s) &= \mathfrak{C}(\cosh\wp), \\
x^2_{\mathbf{M_1M_2}}(s) &= \frac{\mathfrak{C}}{2} e^{\frac{\cosh\wp}{|k_1+k_2|}}((\sinh\wp)\cos[D_1 s + D_2] - \sin[D_1 s + D_2]) \\
&+ \frac{\mathfrak{C}}{2} e^{\frac{\cosh\wp}{|k_1+k_2|}}((\sinh\wp)\sin[D_1 s + D_2] + \cos[D_1 s + D_2]), \\
x^3_{\mathbf{M_1M_2}}(s) &= \frac{\mathfrak{C}}{2} e^{-\frac{\cosh\wp}{|k_1+k_2|}}((\sinh\wp)\cos[D_1 s + D_2] - \sin[D_1 s + D_2]) \\
&- \frac{\mathfrak{C}}{2} e^{-\frac{\cosh\wp}{|k_1+k_2|}}((\sinh\wp)\sin[D_1 s + D_2] + \cos[D_1 s + D_2]),
\end{aligned} \tag{3.9}
$$

*where* $D_1, D_2$ *are constants of integration and*

$$\mathfrak{C} = \frac{1}{|k_1 + k_2|}.$$

**Proof.** Substituting (2.1) to (3.6), we have (3.9) as desired.

We may use Mathematica in Corollary 3.5, yields

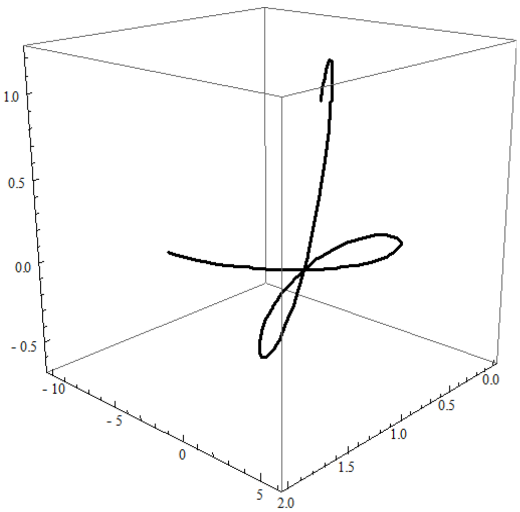


Figure 1.

## 4. Bishop Equations of Smarandache $\mathbf{M_1M_2}$ Curves of Spacelike Biharmonic $\mathcal{B}-$Slant Helices in the $\mathbb{E}(1,1)$

In this section, we shall call the set $\{\mathbf{T}^{\mathfrak{P}},\mathbf{M}_1^{\mathfrak{P}},\mathbf{M}_2^{\mathfrak{P}}\}$ as Bishop trihedra, $k_1^{\mathfrak{P}}$ and $k_2^{\mathfrak{P}}$ as Bishop curvatures of Smarandache $\mathbf{M_1M_2}$ curve.

We can now state the main result of the paper.

**Theorem 4.1.** *Let $\gamma : I \longrightarrow \mathbb{E}(1,1)$ is a non geodesic spacelike biharmonic $\mathcal{B}-$slant helix in the Lorentzian group of rigid motions $\mathbb{E}(1,1)$. Then, the Bishop equations of Smarandache $\mathbf{M_1M_2}$ curves of spacelike biharmonic $\mathcal{B}-$slant helix are*

$$\begin{aligned}\nabla_{\mathbf{T}^{\mathfrak{P}}}\mathbf{T}^{\mathfrak{P}} &= \mathfrak{C}([(k_1+k_2)\,k_1\cosh\wp]\mathbf{X}_1\\&+\mathfrak{C}[(k_1+k_2)\,k_1\sinh\wp\cos[D_1s+D_2]-(k_1+k_2)\,k_2\sin[D_1s+D_2]]\mathbf{X}_2\\&+\mathfrak{C}[(k_1+k_2)\,k_1\sinh\wp\sin[D_1s+D_2]+(k_1+k_2)\,k_2\cos[D_1s+D_2]]\mathbf{X}_3,\end{aligned}$$

$$\begin{aligned}\nabla_{\mathbf{T}^{\mathfrak{P}}}\mathbf{M}_1^{\mathfrak{P}} &= k_1^{\mathfrak{P}}\mathfrak{C}([-(k_1+k_2)\sinh\wp]\mathbf{X}_1\\&+k_1^{\mathfrak{P}}\mathfrak{C}[-(k_1+k_2)\cosh\wp\cos[D_1s+D_2]]\mathbf{X}_2\\&+k_1^{\mathfrak{P}}\mathfrak{C}[-(k_1+k_2)\cosh\wp\sin[D_1s+D_2]]\mathbf{X}_3,\end{aligned}\tag{4.1}$$

$$\begin{aligned}\nabla_{\mathbf{T}^{\mathfrak{P}}}\mathbf{M}_2^{\mathfrak{P}} &= k_2^{\mathfrak{P}}\mathfrak{C}([-(k_1+k_2)\sinh\wp]\mathbf{X}_1\\&+k_2^{\mathfrak{P}}\mathfrak{C}[-(k_1+k_2)\cosh\wp\cos[D_1s+D_2]]\mathbf{X}_2\\&+k_2^{\mathfrak{P}}\mathfrak{C}[-(k_1+k_2)\cosh\wp\sin[D_1s+D_2]]\mathbf{X}_3,\end{aligned}$$

*where* $k_1^{\mathfrak{P}}, k_2^{\mathfrak{P}}$ *are Bishop curvatures of* $\gamma_{\mathbf{M_1 M_2}}$, $D_1, D_2$ *are constants of integration and*

$$\mathfrak{C} = \frac{1}{|k_1 + k_2|}.$$

**Proof.** Assume that $\gamma$ is a non geodesic spacelike biharmonic $\mathcal{B}-$slant helix and its Smarandache $\mathbf{M_1 M_2}$ curve is $\gamma_{\mathbf{M_1 M_2}}$.

From (3.4), we have

$$\mathbf{T}^{\mathfrak{P}} = \mathfrak{C}\left(\left(k_1 + k_2\right)\mathbf{T}\right), \tag{4.2}$$

where $\mathfrak{C} = \frac{1}{|k_1+k_2|}$.

On the other hand, a straightforward computation gives

$$\begin{aligned} \mathbf{T}^{\mathfrak{P}} &= \mathfrak{C}([-\left(k_1 + k_2\right)\sinh\wp]\mathbf{X}_1 \\ &+ \mathfrak{C}[-\left(k_1 + k_2\right)\cosh\wp\cos\left[D_1 s + D_2\right]]\mathbf{X}_2 \\ &+ \mathfrak{C}[-\left(k_1 + k_2\right)\cosh\wp\sin\left[D_1 s + D_2\right]]\mathbf{X}_3. \end{aligned} \tag{4.3}$$

From (2.1), we have

$$\begin{aligned} \nabla_{\mathbf{T}^{\mathfrak{P}}}\mathbf{T}^{\mathfrak{P}} &= \mathfrak{C}([\left(k_1 + k_2\right)k_1\cosh\wp]\mathbf{X}_1 \\ &+ \mathfrak{C}[\left(k_1 + k_2\right)k_1\sinh\wp\cos\left[D_1 s + D_2\right] - \left(k_1 + k_2\right)k_2\sin\left[D_1 s + D_2\right]]\mathbf{X}_2 \\ &+ \mathfrak{C}[\left(k_1 + k_2\right)k_1\sinh\wp\sin\left[D_1 s + D_2\right] + \left(k_1 + k_2\right)k_2\cos\left[D_1 s + D_2\right]]\mathbf{X}_3. \end{aligned}$$

Considering Eqs.(3.2) and (3.3), we obtain the theorem. This concludes the proof of theorem.

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Firat University, Department of Mathematics,23119, Elaziğ, Turkey
*E-mail address*: talatkorpinar@gmail.com, essin.turhan@gmail.com