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TIMELIKE $\mathcal{B}^{\mathfrak{f}}$–HELICES ACCORDING TO FLAT METRIC AND BISHOP FRAME IN LORENTZIAN HEISENBERG GROUP HEIS^3

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ABSTRACT. In this paper, we study timelike $\mathcal{B}^{\mathfrak{f}}$-helices according to flat metric and Bishop frame in the Lorentzian Heisenberg group Heis^3. We characterize timelike $\mathcal{B}^{\mathfrak{f}}$-helices in terms of their curvature and torsion.

1. Introduction

A curve of constant slope or general helix in Euclidean 3-space $\mathbb{E}^3$, is defined by the property that the tangent makes a constant angle with a fixed straight line (the axis of the general helix). A classical result stated by Lancret in 1802 and first proved by de Saint Venant in 1845 (see [10]) is: A necessary and sufficient condition that a curve be a general helix is that the ratio of curvature to torsion be constant.

In this paper, we study timelike $\mathcal{B}^{\mathfrak{f}}$ helices according to flat metric and Bishop frame in the Lorentzian Heisenberg group Heis^3. We characterize timelike $\mathcal{B}^{\mathfrak{f}}$ helices in terms of their curvature and torsion.

2. The Lorentzian Heisenberg Group Heis^3

The Heisenberg group Heis^3 is a Lie group which is diffeomorphic to $\mathbb{R}^3$ and the group operation is defined as

$$(x, y, z) * (\overline{x}, \overline{y}, \overline{z}) = (x + \overline{x}, y + \overline{y}, z + \overline{z} - \overline{x}y + x\overline{y}).$$

The identity of the group is $(0, 0, 0)$ and the inverse of (x, y, z) is given by $(-x, -y, -z)$. The left-invariant Lorentz metric on Heis^3 is

$$g = dx^2 + (xdy + dz)^2 - ((1 - x)\, dy - dz)^2.$$

The following set of left-invariant vector fields forms an orthonormal basis for the corresponding Lie algebra:

$$\left\{ \mathbf{e}_1 = \frac{\partial}{\partial x},\ \mathbf{e}_2 = \frac{\partial}{\partial y} + (1 - x)\frac{\partial}{\partial z},\ \mathbf{e}_3 = \frac{\partial}{\partial y} - x\frac{\partial}{\partial z} \right\}. \tag{2.1}$$

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The characterising properties of this algebra are the following commutation relations:

$$[\mathbf{e}_2, \mathbf{e}_3] = 0, \quad [\mathbf{e}_3, \mathbf{e}_1] = \mathbf{e}_2 - \mathbf{e}_3, \quad [\mathbf{e}_2, \mathbf{e}_1] = \mathbf{e}_2 - \mathbf{e}_3,$$

with

$$g(\mathbf{e}_1, \mathbf{e}_1) = g(\mathbf{e}_2, \mathbf{e}_2) = 1, \quad g(\mathbf{e}_3, \mathbf{e}_3) = -1. \tag{2.2}$$

Proposition 2.1. *For the covariant derivatives of the Levi-Civita connection of the left-invariant metric g, defined above the following is true:*

$$\nabla = \begin{pmatrix} 0 & 0 & 0 \\ \mathbf{e}_2 - \mathbf{e}_3 & -\mathbf{e}_1 & -\mathbf{e}_1 \\ \mathbf{e}_2 - \mathbf{e}_3 & -\mathbf{e}_1 & -\mathbf{e}_1 \end{pmatrix}, \tag{2.3}$$

where the (i, j)-element in the table above equals $\nabla_{\mathbf{e}_i}\mathbf{e}_j$ for our basis

$$\{\mathbf{e}_k, k = 1, 2, 3\} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}.$$

So we obtain that

$$R(\mathbf{e}_1, \mathbf{e}_3) = R(\mathbf{e}_1, \mathbf{e}_2) = R(\mathbf{e}_2, \mathbf{e}_3) = 0. \tag{2.4}$$

Then, the Lorentz metric g is flat.

3. Timelike $\mathcal{B}^{\mathfrak{f}}$–Helices According to Flat Metric in the Lorentzian Heisenberg Group Heis^3

Let $\gamma : I \longrightarrow Heis^3$ be a non geodesic timelike curve on the Lorentzian Heisenberg group Heis^3 parametrized by arc length. Let $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ be the Frenet frame fields tangent to the Lorentzian Heisenberg group Heis^3 along γ defined as follows:

$\mathbf{T}$ is the unit vector field γ' tangent to γ, $\mathbf{N}$ is the unit vector field in the direction of $\nabla_{\mathbf{T}}\mathbf{T}$ (normal to γ), and $\mathbf{B}$ is chosen so that $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ is a positively oriented orthonormal basis. Then, we have the following Frenet formulas:

$$\begin{aligned} \nabla_{\mathbf{T}}\mathbf{T} &= \kappa\mathbf{N}, \\ \nabla_{\mathbf{T}}\mathbf{N} &= \kappa\mathbf{T} + \tau\mathbf{B}, \\ \nabla_{\mathbf{T}}\mathbf{B} &= -\tau\mathbf{N}, \end{aligned} \tag{3.1}$$

where κ is the curvature of γ and τ is its torsion and

$$\begin{aligned} g(\mathbf{T}, \mathbf{T}) = -1, \ g(\mathbf{N}, \mathbf{N}) = 1, \ g(\mathbf{B}, \mathbf{B}) = 1, \\ g(\mathbf{T}, \mathbf{N}) = g(\mathbf{T}, \mathbf{B}) = g(\mathbf{N}, \mathbf{B}) = 0. \end{aligned} \tag{3.2}$$

In the rest of the paper, we suppose everywhere $\kappa \neq 0$ and $\tau \neq 0$.

The Bishop frame or parallel transport frame is an alternative approach to defining a moving frame that is well defined even when the curve has vanishing second derivative. The Bishop frame is expressed as

$$\begin{aligned} \nabla_{\mathbf{T}}\mathbf{T} &= k_1\mathbf{M}_1 + k_2\mathbf{M}_2, \\ \nabla_{\mathbf{T}}\mathbf{M}_1 &= k_1\mathbf{T}, \\ \nabla_{\mathbf{T}}\mathbf{M}_2 &= k_2\mathbf{T}, \end{aligned} \tag{3.3}$$

where

$$g(\mathbf{T},\mathbf{T})=-1,\ g(\mathbf{M}_1,\mathbf{M}_1)=1,\ g(\mathbf{M}_2,\mathbf{M}_2)=1,$$
$$g(\mathbf{T},\mathbf{M}_1)=g(\mathbf{T},\mathbf{M}_2)=g(\mathbf{M}_1,\mathbf{M}_2)=0. \tag{3.4}$$

Here, we shall call the set $\{\mathbf{T},\mathbf{M}_1,\mathbf{M}_1\}$ as Bishop trihedra, k_1 and k_2 as Bishop curvatures. Thus, Bishop curvatures are defined by

$$k_1=\kappa(s)\cos\theta(s),$$
$$k_2=\kappa(s)\sin\theta(s), \tag{3.5}$$

where $\theta(s)=\arctan\frac{k_2}{k_1}$, $\tau(s)=\theta'(s)$ and $\kappa(s)=\sqrt{k_1^2+k_2^2}$.

With respect to the orthonormal basis $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ we can write

$$\mathbf{T}=T^1\mathbf{e}_1+T^2\mathbf{e}_2+T^3\mathbf{e}_3,$$
$$\mathbf{M}_1=M_1^1\mathbf{e}_1+M_1^2\mathbf{e}_2+M_1^3\mathbf{e}_3, \tag{3.6}$$
$$\mathbf{M}_2=M_2^1\mathbf{e}_1+M_2^2\mathbf{e}_2+M_2^3\mathbf{e}_3.$$

To separate a helix according to Bishop frame from that of Frenet- Serret frame, in the rest of the paper, we shall use notation for the helix defined above as $\mathcal{B}^{\mathfrak{f}}$-helix.

Theorem 3.1. *Let $\gamma:I\longrightarrow Heis^3$ is a unit speed timelike $\mathcal{B}^{\mathfrak{f}}-$helix according to flat metric. Then, the parametric equations of γ are*

$$x(s)=\sinh\mathfrak{A}s+\mathfrak{W}_1,$$

$$y(s)=\frac{1}{\mathfrak{W}}\cosh\mathfrak{A}\cosh[\mathfrak{W}s+\mathfrak{W}_0]+\frac{1}{\mathfrak{W}}\cosh\mathfrak{A}\sinh[\mathfrak{W}s+\mathfrak{W}_0]+\mathfrak{W}_2,$$

$$\begin{aligned} z(s)=&\frac{1}{\mathfrak{W}}\cosh\mathfrak{A}\cosh[\mathfrak{W}s+\mathfrak{W}_0]\\ &-\frac{1}{\mathfrak{W}}(\sinh\mathfrak{A}s+\mathfrak{W}_1)\cosh\mathfrak{A}\cosh[\mathfrak{W}s+\mathfrak{W}_0]\\ &+\frac{1}{\mathfrak{W}^2}\sinh\mathfrak{A}\cosh\mathfrak{A}\sinh[\mathfrak{W}s+\mathfrak{W}_0]\\ &-\frac{1}{\mathfrak{W}}(\sinh\mathfrak{A}s+\mathfrak{W}_1)\cosh\mathfrak{A}\sinh[\mathfrak{W}s+\mathfrak{W}_0]\\ &+\frac{1}{\mathfrak{W}^2}\sinh\mathfrak{A}\cosh\mathfrak{A}\cosh[\mathfrak{W}s+\mathfrak{W}_0]+\mathfrak{W}_3, \end{aligned} \tag{3.7}$$

where $\mathfrak{W},\mathfrak{W}_0,\mathfrak{W}_1,\mathfrak{W}_2,\mathfrak{W}_3$ are constants of integration.

Proof. Since γ is $\mathcal{B}^{\mathfrak{f}}-$helix according to flat metric without loss of generality, we take the axis of γ is parallel to the spacelike vector $\mathbf{e}_1$. Then,

$$g(\mathbf{T},\mathbf{e}_1)=T_1=\sinh\mathfrak{A},$$

where $\mathfrak{A}$ is constant angle.

The tangent vector can be written in the following form

$$\mathbf{T}=\sinh\mathfrak{A}\mathbf{e}_1+\cosh\mathfrak{A}\sinh[\mathfrak{W}s+\mathfrak{W}_0]\mathbf{e}_2+\cosh[\mathfrak{W}s+\mathfrak{W}_0]\cosh[\mathfrak{W}s+\mathfrak{W}_0]\mathbf{e}_3, \tag{3.8}$$

where $\mathfrak{W},\mathfrak{W}_0$ are constants of integration.

Using Eq.(2.1) in Eq.(3.8), we obtain

$$\begin{aligned}\mathbf{T} &= (\sinh \mathfrak{A}, \cosh \mathfrak{A} \sinh[\mathfrak{W}s + \mathfrak{W}_0] + \cosh \mathfrak{A} \cosh[\mathfrak{W}s + \mathfrak{W}_0], \\ &(1-x) \cosh \mathfrak{A} \sinh[\mathfrak{W}s + \mathfrak{W}_0] - x \cosh \mathfrak{A} \cosh[\mathfrak{W}s + \mathfrak{W}_0]).\end{aligned} \tag{3.9}$$

Also, from above Eq.(3.9), we get

$$\begin{aligned}\mathbf{T} &= (\sinh \mathfrak{A}, \cosh \mathfrak{A} \sinh[\mathfrak{W}s + \mathfrak{W}_0] + \cosh \mathfrak{A} \cosh[\mathfrak{W}s + \mathfrak{W}_0], \\ &(1-(\sinh \mathfrak{A}s + \mathfrak{W}_1)) \cosh \mathfrak{A} \sinh[\mathfrak{W}s + \mathfrak{W}_0] - (\sinh \mathfrak{A}s + \mathfrak{W}_1) \cosh \mathfrak{A} \cosh[\mathfrak{W}s + \mathfrak{W}_0]).\end{aligned} \tag{3.10}$$

If we take integrate above system we have Eq.(3.7). The proof is completed.

Theorem 3.2. *Let $\gamma : I \longrightarrow Heis^3$ be a unit speed timelike $\mathcal{B}^{\mathfrak{f}}-$helix according to flat metric. Then the position vector of γ is*

$$\begin{aligned}\gamma(s) &= (\sinh \mathfrak{A}s + \mathfrak{W}_1)\,\mathbf{e}_1 + [(\sinh \mathfrak{A}s + \mathfrak{W}_1)\,(\frac{1}{\mathfrak{W}} \cosh \mathfrak{A} \cosh[\mathfrak{W}s + \mathfrak{W}_0] \\ &+ \frac{1}{\mathfrak{W}} \cosh \mathfrak{A} \sinh[\mathfrak{W}s + \mathfrak{W}_0] + \mathfrak{W}_2) + \frac{1}{\mathfrak{W}} \cosh \mathfrak{A} \cosh[\mathfrak{W}s + \mathfrak{W}_0] \\ &- \frac{1}{\mathfrak{W}} (\sinh \mathfrak{A}s + \mathfrak{W}_1) \cosh \mathfrak{A} \cosh[\mathfrak{W}s + \mathfrak{W}_0] + \frac{1}{\mathfrak{W}^2} \sinh \mathfrak{A} \cosh \mathfrak{A} \sinh[\mathfrak{W}s + \mathfrak{W}_0] \\ &- \frac{1}{\mathfrak{W}} (\sinh \mathfrak{A}s + \mathfrak{W}_1) \cosh \mathfrak{A} \sinh[\mathfrak{W}s + \mathfrak{W}_0] + \frac{1}{\mathfrak{W}^2} \sinh \mathfrak{A} \cosh \mathfrak{A} \cosh[\mathfrak{W}s + \mathfrak{W}_0] + \mathfrak{W}_3]\mathbf{e}_2 \\ &[(1 - \sinh \mathfrak{A}s - \mathfrak{W}_1)\,(\frac{1}{\mathfrak{W}} \cosh \mathfrak{A} \cosh[\mathfrak{W}s + \mathfrak{W}_0] + \frac{1}{\mathfrak{W}} \cosh \mathfrak{A} \sinh[\mathfrak{W}s + \mathfrak{W}_0] + \mathfrak{W}_2) \\ &- \frac{1}{\mathfrak{W}} \cosh \mathfrak{A} \cosh[\mathfrak{W}s + \mathfrak{W}_0] + \frac{1}{\mathfrak{W}} (\sinh \mathfrak{A}s + \mathfrak{W}_1) \cosh \mathfrak{A} \cosh[\mathfrak{W}s + \mathfrak{W}_0] \\ &- \frac{1}{\mathfrak{W}^2} \sinh \mathfrak{A} \cosh \mathfrak{A} \sinh[\mathfrak{W}s + \mathfrak{W}_0] + \frac{1}{\mathfrak{W}} (\sinh \mathfrak{A}s + \mathfrak{W}_1) \cosh \mathfrak{A} \sinh[\mathfrak{W}s + \mathfrak{W}_0] \\ &- \frac{1}{\mathfrak{W}^2} \sinh \mathfrak{A} \cosh \mathfrak{A} \cosh[\mathfrak{W}s + \mathfrak{W}_0] - \mathfrak{W}_3]\mathbf{e}_3,\end{aligned} \tag{3.11}$$

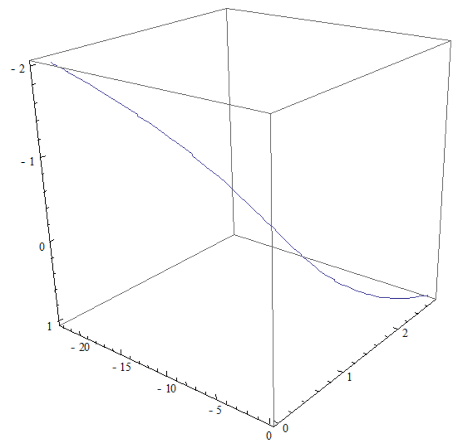
where $\mathfrak{W}, \mathfrak{W}_0, \mathfrak{W}_1, \mathfrak{W}_2, \mathfrak{W}_3$ are constants of integration.

Proof. Assume that γ be a unit speed timelike $\mathcal{B}^{\mathfrak{f}}-$helix according to flat metric. Using Eq.(2.1) we have

$$\begin{aligned}\frac{\partial}{\partial x} &= \mathbf{e}_1, \\ \frac{\partial}{\partial y} &= x\mathbf{e}_2 + (1-x)\,\mathbf{e}_3, \\ \frac{\partial}{\partial z} &= \mathbf{e}_2 - \mathbf{e}_3.\end{aligned} \tag{3.12}$$

Substituting Eq.(3.12) to Eq.(3.7), we have Eq.(3.11). This concludes the proof of theorem.

We can use Mathematica in Theorem 3.1, we obtain following figure:



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