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HORIZONTAL GEODESICS IN LORENTZIAN HEISENBERG GROUP HEIS^3

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Abstract. In this paper, we study different geodesic lines on the Lorentzian Heis^3. We consider the Lorentzian left invariant metric and use some results of Levi-Civita connection and curvature tensor to present solution of equations for geodesic lines in Lorentzian Heis^3. Furthermore it is obtained equations of geodesic lines with respect to the left invariant Lorentzian metric of the Heis^3. We characterize horizontal geodesic curves in Lorentzian Heis^3 and prove that there exist no null horizontal geodesic curves in Lorentzian Heis^3..

1. Introduction

Rahmani (see [4]) described one motivation for our study that there are three classes of left invariant Lorentzian metrics on the Heis^3.

In (see [2]) V. Marenich used Heisenberg left invariant metric and obtained geodesic lines in Heis^3.

Returning to geometry, we note that the Heis^3 serves also as a contact manifold. This is the odd dimension analog for a symplectic structure. The 1-form $\omega^1 = dt + xdy$, which anihilates e_1 and e_2 is a contact form, i.e. $\omega^1 \wedge d\omega^1$ never vanishes. The 2-form $\Omega = d\omega^1 = dx \wedge dy$ is a symplectic form on the distribution generated by e_1 and e_2. It can be considered also as a magnetic field. Then the Maxwell's equation can be written as $d\Omega = 0$.

In this paper, we study different geodesic lines on the Lorentzian Heis^3. We consider the Lorentzian left invariant metric and use some results of Levi-Civita connection and curvature tensor to present solution of equations for geodesic lines in Lorentzian Heis^3. Furthermore it is obtained equations of geodesic lines with respect to the left invariant Lorentzian metric of the Heis^3. We characterize horizontal geodesic curves in Lorentzian Heis^3 and prove that there exist no null horizontal geodesic curves in Lorentzian Heis^3 .

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2. Lorentzian Heisenberg Group Heis^3

This group is formed by all matrices of the form

$$Heis^3 = \left\{ \begin{pmatrix} 1 & x & t \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, t \in \mathbb{R} \right\}$$

with the group multiplication induced by the standard matrix product.

The Lorentzian Heis^3 can be seen as the space $\mathbb{R}^3$ endowed with multiplication

$$(\overline{x}, \overline{y}, \overline{t})(x, y, t) = (\overline{x} + x, \overline{y} + y, \overline{t} + t - \overline{x}y + x\overline{y}).$$

The orthonormal basis

$$E_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

of tangent space at the identity, determines on Heis^3 a left invariant Lorentzian metric

$$ds^2 = (\omega^1)^2 + (\omega^2)^2 - (\omega^3)^2 \tag{2.1}$$

where

$$\omega^1 = dt + xdy, \qquad \omega^2 = dy, \qquad \omega^3 = dx$$

is the left invariant orthonormal coframe associated with the orthonormal left invariant frame,

$$e_1 = \frac{\partial}{\partial t}, \quad e_2 = \frac{\partial}{\partial y} - x\frac{\partial}{\partial t}, \quad e_3 = \frac{\partial}{\partial x}. \tag{2.2}$$

Now let C_{ij}^k be the structure's constants of the Lie algebra g of G (see [5]), that is

$$[e_i, e_j] = C_{ij}^k e_k.$$

The corresponding Lie brackets are

$$[e_2, e_3] = 2e_1, \qquad [e_1, e_3] = [e_1, e_2] = 0.$$

The Coshul formula for the Levi–Civita connection is:

$$2g(\nabla_{e_i} e_j, e_k) = C_{ij}^k - C_{jk}^i + C_{ki}^j := L_{ij}^k$$

where the non zero L_{ij}^k 's are

$$L_{12}^3 = 2, \;\; L_{21}^3 = 2, \;\; L_{13}^2 = 2, \;\; L_{31}^2 = 2, \;\; L_{23}^1 = 2, \;\; L_{32}^1 = -2. \tag{2.3}$$

The element zero $0 = (0, 0, 0)$ is the unit of this group strucure and the inverse element for $(z,\ t)$ is $(z,\ t)^{-1} = (-z,\ -t)$. Let $a = (z, t)$, $b = (w, s)$. The commutator of the elements a, $b \in Heis^3$ is equal to

$$\begin{aligned} [a, b] &= aba^{-1}b^{-1} \\ &= (z, t)\,(w, s)\,(-z, -t)\,(-w, -s) \\ &= (z + w - z + w, t + s - t - s + \alpha) = (0, \alpha) \end{aligned}$$

where $\alpha \neq 0$ in general. Which shows that Heis^3 is not abelian. On the other hand for any a, b, $c \in Heis^3$, their double commutator is

$$[[a, b], c] = (0, 0).$$

Heisenberg group plays an important role in many branches of mathematics such as representation theory, harmonic analysis, PDEs or even quantum mechanic. Heis^3 has a rich geometric structure. In fact its group of isometries is of dimension 4, which is the maximal possible dimension for a non-constant curvature metric on a 3-manifold. Also, from the algebraic point of view, Heis^3 is a 2-step nilpotent Lie group, i.e. "almost Abelian".

3. Horizontal Geodesics in the Lorentzian $\mathrm{H{\small EIS}}^3$

Consider a nonintegrable 2-dimensional distribution $x \longrightarrow \mathcal{H}_x$ defined as $\mathcal{H} = \ker\omega$, where ω is a 1-form on Heis^3. The distribution $\mathcal{H}$ is called the horizontal distribution. A curve $s \longrightarrow c(s) = (x(s), y(s), t(s))$ is called horizontal curve if $c'(s) \in \mathcal{H}_{c(s)}$, for every s (see [6]).

Theorem 3.1. *Spacelike geodesic lines issuing from* 0 *in the Lorentzian* $Heis^3$ *is solution following differential equation system:*

$$\begin{aligned} x'(s) &= \sqrt{1-\xi^2}\sinh(2\xi s+\phi), \\ y'(s) &= \sqrt{1-\xi^2}\cosh(2\xi s+\phi), \\ t'(s) &= \xi - x(s)\sqrt{1-\xi^2}\cosh(2\xi s+\phi). \end{aligned} \tag{3.1}$$

Proof. Let $c(s)$ be such a geodesics with a natural parameters s and its vector of velocity given by

$$c'(s) = \xi(s)\, e_1 + \eta(s)\, e_2 + \rho(s)\, e_3. \tag{3.2}$$

Then the equation of a geodesic $\nabla_{c'(s)}c'(s) \equiv 0$ and our table of covariant derivatives (3.2) give

$$\xi'(s)\, e_1 + (\eta'(s) + 2\xi(s)\rho(s))\, e_2 + (\rho'(s) - 2\xi(s)\eta(s))\, e_3 = 0. \tag{3.3}$$

Thus we easily obtain the following equations for coordinates of the velocity of the geodesic $c(s)$ in our left-invariant moving frame

$$\begin{aligned} &\xi'(s) = 0, \\ &\eta'(s) + 2\xi(s)\rho(s) = 0, \\ &\rho'(s) - 2\xi(s)\eta(s) = 0 \end{aligned} \tag{3.4}$$

or

$$\begin{aligned} &(\eta(s)+\rho(s))' + 2\xi(s)(\rho(s)-\eta(s)) = 0, \\ &(\eta(s)-\rho(s))' + 2\xi(\rho(s)+\eta(s)) = 0. \end{aligned} \tag{3.5}$$

Since $c : I \longrightarrow Heis^3$ is spacelike curve, we have

$$\xi^2(s) + \eta^2(s) - \rho^2(s) = 1 \tag{3.6}$$

and from (3.4) we could take

$$\xi = const.$$

where $|\xi| \leq 1$. From (3.6) we could find that,

$$\begin{aligned} &\xi = const. \\ &\eta(s) = r\cosh(2\xi s + \phi), \\ &\rho(s) = r\sinh(2\xi s + \phi) \end{aligned} \tag{3.7}$$

where $r = \sqrt{\eta^2(s) - \rho^2(s).}$

To find equations for geodesic $c(s) = (x(s), y(s), z(s))$ issuing from 0, we note that if

$$c'(s) = \xi(s)e_1 + \eta(s)e_2 + \rho(s)e_3 \tag{3.8}$$

and our left-invariant vector fields are

$$e_1 = \frac{\partial}{\partial z}, \quad e_2 = \frac{\partial}{\partial y} - x\frac{\partial}{\partial z}, \quad e_3 = \frac{\partial}{\partial x}.$$

Then we easily have (3.1).

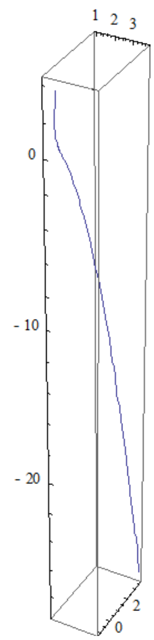
Corollary 3.2. *Let $\gamma : I \longrightarrow Heis^3$ be a unit speed spacelike geodesic curve. Then, the parametric equation of spacelike geodesic are*

$$\begin{aligned} x(s) &= \frac{\sqrt{1-\xi^2}}{2\xi}\cosh(2\xi s + \phi) + c_1, \\ y(s) &= \frac{\sqrt{1-\xi^2}}{2\xi}\sinh(2\xi s + \phi) + c_2, \\ t(s) &= \left(\xi - \frac{1-\xi^2}{4\xi}\right)s - \frac{1-\xi^2}{8\xi^2}\sinh 2(2\xi s + \phi) \\ &- \frac{c_1\sqrt{1-\xi^2}}{2\xi}\sinh(2\xi s + \phi), \end{aligned} \tag{3.9}$$

where c_1, c_2, c_3 are constants of integration.

Next, we apply Corollary 3.2.

Example 3.3. *Let us consider parametric equation of spacelike geodesic lines with $\xi = \frac{1}{2}$, $\phi = 0$ and $c_1 = c_2 = c_3 = 1$. Then, we can draw spacelike geodesic* *lines with helping the programme of Mathematica as follow*



Theorem 3.4. *If* $c : I \longrightarrow Heis^3$ *spacelike horizontal geodesic lines issuing from zero 0, then satisfy to the following equations:*

$$\begin{aligned} x(s) &= s\sinh\phi + a, \\ y(s) &= s\cosh\phi + b, \\ t(s) &= \frac{s^2}{2}\sinh\phi\cosh\phi + a\cosh\phi s + c, \end{aligned} \tag{3.10}$$

where a, b, c *are constants of integration.*

Proof. Let $c(s)$ be a horizontal geodesics. From horizontal curve equation, we get

$$w(c'(s)) = \xi = 0 \tag{3.11}$$

If we use (3.11) in (3.7) and some calculations,we have

$$\begin{aligned} &\xi = 0 \\ &\eta(s) = \cosh\phi, \\ &\rho(s) = \sinh\phi \end{aligned} \tag{3.12}$$

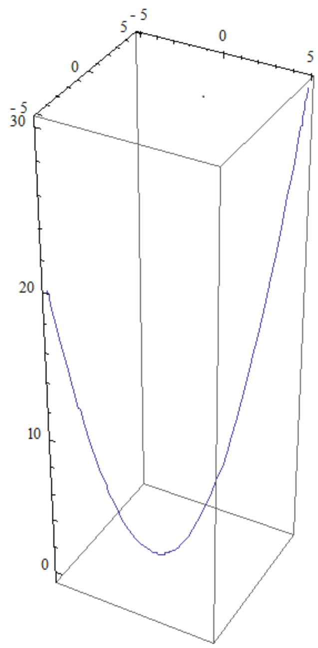
and (3.12) in (3.8) we obtain

$$\begin{aligned} x'(s) &= \sinh\phi \\ y'(s) &= \cosh\phi \\ t'(s) &= x(s)\cosh\phi. \end{aligned}$$

The integration is immediate and yelds (3.10).

Next, we apply Theorem 3.4.

Example 3.5. *Let us consider parametric equation of spacelike horizontal geodesic lines with $a = b = c = 1$. Then, we can draw spacelike horizontal geodesic with helping the programme of Mathematica as follow*



Lemma 3.6. *Timelike geodesic lines issuing from 0 in the Lorentzian $Heis^3$ is solution following differential equation system;*

$$\begin{aligned} x'(s) &= \sqrt{1+\xi^2}\cosh(2\xi s+\phi), \\ y'(s) &= \sqrt{1+\xi^2}\sinh(2\xi s+\phi), \\ t'(s) &= \xi - x(s)\sqrt{1+\xi^2}\sinh(2\xi s+\phi). \end{aligned} \tag{3.13}$$

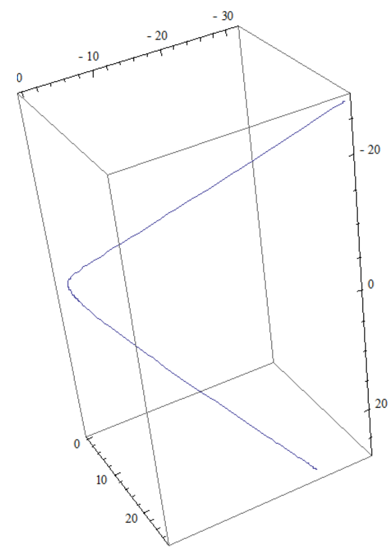
Corollary 3.7. *Let $\gamma : I \longrightarrow Heis^3$ be a unit speed geodesic curve. Then, the parametric equation of spacelike geodesic are*

$$\begin{aligned} x(s) &= \frac{\sqrt{1+\xi^2}}{2\xi}\sinh(2\xi s+\phi)+c_1, \\ y(s) &= \frac{\sqrt{1+\xi^2}}{2\xi}\cosh(2\xi s+\phi)+c_2, \\ t(s) &= \left(\xi - \frac{1+\xi^2}{4\xi}\right)s - \frac{1+\xi^2}{8\xi^2}\sinh 2(2\xi s+\phi) \\ &- \frac{c_1\sqrt{1+\xi^2}}{2\xi}\cosh(2\xi s+\phi), \end{aligned} \tag{3.14}$$

where c_1, c_2, c_3 are constants of integration.

Next, we apply Corollary 3.7.

Example 3.8. *Let us consider parametric equation of spacelike geodesic lines with $\xi = 1$, $\phi = 0$ and $c_1 = c_2 = c_3 = 1$. Then, we can draw spacelike geodesic lines with helping the programme of Mathematica as follow*



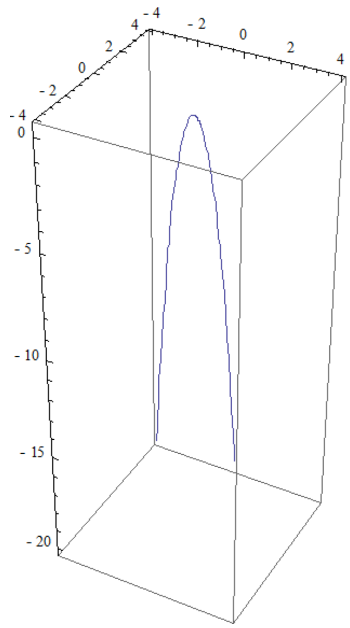
Corollary 3.9. *If $c : I \longrightarrow Heis^3$ timelike horizontal geodesic lines issuing from zero 0, then satisfy to the following equations:*

$$\begin{aligned} x(s) &= s\cosh\phi + a, \\ y(s) &= s\sinh\phi + b, \\ t(s) &= \frac{s^2}{2}\sinh\phi\cosh\phi + a\sinh\phi s + c, \end{aligned} \tag{3.13}$$

where a, b, c are constants of integration.

Next, we apply Corollary 3.9.

Example 3.10. *Let us consider parametric equation of timelike geodesic lines with $a = b = c = 1$. Then, we can draw timelike horizontal geodesic lines with* *helping the programme of Mathematica as follow*



Theorem 3.11. *There exists no null horizontal geodesic in Lorentzian* $Heis^3$.

Proof. We prove that by contradiction. Let us assume that $c : I \longrightarrow Heis^3$ is null horizontal geodesic curve parametrized by arc length. Then, null geodesic lines issuing from 0 in the Lorentzian $Heis^3$ is solution following differential equation system;

$$x'(s) = |\xi| \sinh(2\xi s + \phi)$$
$$y'(s) = |\xi| \cosh(2\xi s + \phi) \tag{3.14}$$
$$t'(s) = \xi - x(s) |\xi| \cosh(2\xi s + \phi).$$

Substitute equations (3.11) into (3.16) we have

$$\eta(s) = 0, \ \rho(s) = 0. \tag{3.15}$$

From (3.8) we get $c'(s) = 0$. This contradiction proves the claim.

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