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NEW INEXTENSIBLE FLOWS OF TIMELIKE CURVES ON THE ORIENTED TIMELIKE SURFACES ACCORDING TO DARBOUX FRAME IN $\mathbb{M}_1^3$

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ABSTRACT. In this paper, we study inextensible flows of timelike curves on oriented time-like surface in $\mathbb{M}_1^3$. We research inextensible flows of timelike curves according to Darboux frame in $\mathbb{M}_1^3$. Finally, we obtain partial differential equations about curvatures of timelike curves.

1. Introduction

Fluid flow researchers have been studying fluid flows in various ways, and today fluid flow is still an important field of research. The areas in which fluid flow plays a role are numerous. Gaseous flows are studied for the development of cars, aircraft and spacecrafts, and also for the design of machines such as turbines and combustion engines. Liquid flow research is necessary for naval applications, such as ship design, and is widely used in civil engineering projects such as harbour design and coastal protection.

The visualization of fluid flow simulation data may have several different purposes. One purpose is the verification of theoretical models in fundamental research. When a flow phenomenon is described by a model, this flow model should be compared with the 'real' fluid flow. The accuracy of the model can be verified by calculation and visualization of a flow with the model, and comparison of the results with experimental results.

This study is organised as follows: Firstly, we study inextensible flows of timelike curves on oriented time-like surface in $\mathbb{M}_1^3$. Secondly, we research inextensible flows of timelike curves according to Darboux frame in $\mathbb{M}_1^3$. Finally, we obtain partial differential equations about curvatures of timelike curves.

2. Preliminaries

The Minkowski 3-space $\mathbb{M}_1^3$ provided with the standard flat metric given by

$$\langle , \rangle = -dx_1^2 + dx_2^2 + dx_3^2,$$

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where (x_1, x_2, x_3) is a rectangular coordinate system of $\mathbb{M}_1^3$. Recall that, the norm of an arbitrary vector $a \in \mathbb{M}_1^3$ is given by $\|a\| = \sqrt{\langle a, a\rangle}$. γ is called a unit speed curve if velocity vector v of γ satisfies $\|a\| = 1$.

Denote by $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ the moving Frenet–Serret frame along the timelike curve γ in the space $\mathbb{M}_1^3$. For an arbitrary timelike curve γ with first and second curvature, κ and τ in the space $\mathbb{M}_1^3$, the following Frenet–Serret formulae is given

$$\begin{aligned} \mathbf{T}' &= \kappa \mathbf{N} \\ \mathbf{N}' &= \kappa \mathbf{T} + \tau \mathbf{B} \\ \mathbf{B}' &= -\tau \mathbf{N}, \end{aligned} \tag{2.1}$$

where

$$\begin{aligned} &\langle \mathbf{T}, \mathbf{T}\rangle = -1, \langle \mathbf{N}, \mathbf{N}\rangle = 1, \langle \mathbf{B}, \mathbf{B}\rangle = 1, \\ &\langle \mathbf{T}, \mathbf{N}\rangle = \langle \mathbf{T}, \mathbf{B}\rangle = \langle \mathbf{N}, \mathbf{B}\rangle = 0. \end{aligned}$$

Here, curvature functions are defined by $\kappa = \kappa(s) = \|\mathbf{T}'(s)\|$ and $\tau(s) = -\langle \mathbf{N}, \mathbf{B}'\rangle$.

Torsion of the curve γ is given by the aid of the mixed product

$$\tau = \frac{[\gamma', \gamma'', \gamma''']}{\kappa^2}.$$

A surface M in the Minkowski 3-space $\mathbb{M}_1^3$ is said to be space-like, time-like surface if, respectively the induced metric on the surface is a positive definite Riemannian metric, Lorentz metric. In other words, the normal vector on the space-like (time-like) surface is a time-like (space-like) vector [9].

If the surface $\mathcal{M}$ is an oriented time-like surface, then the curve $\alpha(s)$ lying on $\mathcal{M}$ is a time-like curve. Thus, the equations which describe the Darboux frame of $\alpha(s)$ is given by :

$$\begin{aligned} \mathbf{T}' &= \kappa_g \mathbf{P} + \kappa_n \mathbf{n}, \\ \mathbf{P}' &= \kappa_g \mathbf{T} - \tau_g \mathbf{n}, \\ \mathbf{n}' &= \kappa_n \mathbf{T} + \tau_g \mathbf{P}, \end{aligned} \tag{2.2}$$

where $\mathbf{T}, \mathbf{P}, \mathbf{n}$ satisfy the following properties:

$$\begin{aligned} &< \mathbf{T}, \mathbf{T} > = -1, \ \ < \mathbf{n}, \mathbf{n} > = 1, \ \ < \mathbf{P}, \mathbf{P} > = 1, \\ &< \mathbf{T}, \mathbf{n} > = < \mathbf{T}, \mathbf{P} > = < \mathbf{n}, \mathbf{P} > = 0. \end{aligned}$$

In this frame $\mathbf{T}$ is the unit tangent of the curve, $\mathbf{n}$ is the unit normal of the surface $\mathcal{M}$ and $\mathbf{P}$ is a unit vector given by $\mathbf{T} \times \mathbf{n} = -\mathbf{P}$, $\mathbf{n} \times \mathbf{P} = \mathbf{T}$, $\mathbf{P} \times \mathbf{T} = -\mathbf{n}$.

3. Inextensible Flows of Timelike Curves on Oriented Time-like Surfaces According to Darboux Frame in $\mathbb{M}_1^3$

Let $\alpha(u, t)$ is a one parameter family of smooth timelike curves on oriented time-like surface in $\mathbb{M}_1^3$.

The arclength of α is given by

$$s(u)=\int_0^u\left|\frac{\partial\alpha}{\partial u}\right|du, \tag{3.1}$$

where

$$\left|\frac{\partial\alpha}{\partial u}\right|=\left|\left\langle\frac{\partial\alpha}{\partial u},\frac{\partial\alpha}{\partial u}\right\rangle\right|^{\frac{1}{2}}. \tag{3.2}$$

The operator $\dfrac{\partial}{\partial s}$ is given in terms of u by

$$\frac{\partial}{\partial s}=\frac{1}{\nu}\frac{\partial}{\partial u},$$

where $v=\left|\dfrac{\partial\alpha}{\partial u}\right|$ and the arclength parameter is $ds=vdu$.

Any flow of α can be represented as

$$\frac{\partial\alpha}{\partial t}=\mathcal{A}_1^{\mathcal{D}}\mathbf{T}+\mathcal{A}_2^{\mathcal{D}}\mathbf{P}+\mathcal{A}_3^{\mathcal{D}}\mathbf{n}. \tag{3.3}$$

Letting the arclength variation be

$$s(u,t)=\int_0^u vdu.$$

In the $\mathbb{E}_1^3$ the requirement that the curve not be subject to any elongation or compression can be expressed by the condition

$$\frac{\partial}{\partial t}s(u,t)=\int_0^u\frac{\partial v}{\partial t}du=0, \tag{3.4}$$

for all $u\in[0,l]$.

Definition 3.1. *Let $\mathcal{M}$ be an oriented time-like surface and α lying on $\mathcal{M}$ in Minkowski 3-space $\mathbb{M}_1^3$. The flow $\dfrac{\partial\alpha}{\partial t}$ on $\mathcal{M}$ are said to be inextensible if*

$$\frac{\partial}{\partial t}\left|\frac{\partial\alpha}{\partial u}\right|=0.$$

Lemma 3.2. *Let $\mathcal{M}$ be an oriented time-like surface and α lying on $\mathcal{M}$ in Minkowski 3-space $\mathbb{M}_1^3$. The flow $\dfrac{\partial\alpha}{\partial t}=\mathcal{A}_1^{\mathcal{D}}\mathbf{T}+\mathcal{A}_2^{\mathcal{D}}\mathbf{P}+\mathcal{A}_3^{\mathcal{D}}$ is inextensible if and only if*

$$\frac{\partial v}{\partial t}+\frac{\partial\mathcal{A}_1^{\mathcal{D}}}{\partial u}=-\mathcal{A}_2^{\mathcal{D}}v\kappa_g-\mathcal{A}_3^{\mathcal{D}}v\kappa_n. \tag{3.5}$$

Proof. Suppose that $\dfrac{\partial\alpha}{\partial u}$ be a smooth flow of the timelike curve α. Using definition of α, we have

$$v^2=\left\langle\frac{\partial\alpha}{\partial u},\frac{\partial\alpha}{\partial u}\right\rangle. \tag{3.6}$$

$\frac{\partial}{\partial u}$ and $\frac{\partial}{\partial t}$ commute since and are independent coordinates. So, by differentiating of the formula (3.6), we get

$$2v\frac{\partial v}{\partial t}=\frac{\partial}{\partial t}\left\langle \frac{\partial \alpha}{\partial u},\frac{\partial \alpha}{\partial u}\right\rangle .$$

On the other hand, changing $\frac{\partial}{\partial u}$ and $\frac{\partial}{\partial t}$, we have

$$v\frac{\partial v}{\partial t}=\left\langle \frac{\partial \alpha}{\partial u},\frac{\partial}{\partial u}(\frac{\partial \alpha}{\partial t})\right\rangle .$$

From (3.3), we obtain

$$v\frac{\partial v}{\partial t}=\left\langle \frac{\partial \alpha}{\partial u},\frac{\partial}{\partial u}\left(\mathcal{A}_1^{\mathcal{D}}\mathbf{T}+\mathcal{A}_2^{\mathcal{D}}\mathbf{P}+\mathcal{A}_3^{\mathcal{D}}\mathbf{n}\right)\right\rangle .$$

By the formula of the Darboux, we have

$$\frac{\partial v}{\partial t}=<\mathbf{T},\left(\frac{\partial \mathcal{A}_1^{\mathcal{D}}}{\partial u}+\mathcal{A}_2^{\mathcal{D}}v\kappa_g+\mathcal{A}_3^{\mathcal{D}}v\kappa_n\right)\mathbf{T}+\left(\mathcal{A}_1^{\mathcal{D}}v\kappa_g+\frac{\partial \mathcal{A}_2^{\mathcal{D}}}{\partial u}+\mathcal{A}_3^{\mathcal{D}}v\tau_g\right)\mathbf{P}$$
$$+\left(\mathcal{A}_1^{\mathcal{D}}v\kappa_n-\mathcal{A}_2^{\mathcal{D}}v\tau_g+\frac{\partial \mathcal{A}_3^{\mathcal{D}}}{\partial u}\right)\mathbf{n}>.$$

Making necessary calculations from above equation, we have (3.5), which proves the lemma.

Theorem 3.3. *Let $\mathcal{M}$ be an oriented time-like surface and α lying on $\mathcal{M}$ in Minkowski 3-space $\mathbb{M}_1^3$. The flow $\frac{\partial \alpha}{\partial t}$ is inextensible if and only if*

$$\frac{\partial \mathcal{A}_1^{\mathcal{D}}}{\partial u}=-\mathcal{A}_2^{\mathcal{D}}v\kappa_g-\mathcal{A}_3^{\mathcal{D}}v\kappa_n. \tag{3.7}$$

Proof. Assume that $\frac{\partial \alpha}{\partial t}$ be inextensible. From (3.4), we have

$$\frac{\partial}{\partial t}s(u,t)=\int_0^u\frac{\partial v}{\partial t}du=\int_0^u\left(\frac{\partial \mathcal{A}_1^{\mathcal{D}}}{\partial u}+\mathcal{A}_2^{\mathcal{D}}v\kappa_g+\mathcal{A}_3^{\mathcal{D}}v\kappa_n\right)du=0, \tag{3.8}$$

$\forall u\in[0,l]$. Substituting (3.5) in (3.8) complete the proof of the theorem.

We now restrict ourselves to arc length parametrized curves. That is, $v=1$ and the local coordinate u corresponds to the curve arc length s. We require the following lemma.

Lemma 3.4.

$$\frac{\partial \mathbf{T}}{\partial t}=\left(\mathcal{A}_1^{\mathcal{D}}\kappa_g+\frac{\partial \mathcal{A}_2^{\mathcal{D}}}{\partial s}+\mathcal{A}_3^{\mathcal{D}}\tau_g\right)\mathbf{P}+\left(\mathcal{A}_1^{\mathcal{D}}\kappa_n-\mathcal{A}_2^{\mathcal{D}}\tau_g+\frac{\partial \mathcal{A}_3^{\mathcal{D}}}{\partial s}\right)\mathbf{n}, \tag{3.9}$$

$$\frac{\partial \mathbf{P}}{\partial t}=\left(\mathcal{A}_1^{\mathcal{D}}\kappa_g+\frac{\partial \mathcal{A}_2^{\mathcal{D}}}{\partial s}+\mathcal{A}_3^{\mathcal{D}}\tau_g\right)\mathbf{T}+\psi\mathbf{n}, \tag{3.10}$$

$$\frac{\partial \mathbf{n}}{\partial t}=\left(\mathcal{A}_1^{\mathcal{D}}\kappa_n-\mathcal{A}_2^{\mathcal{D}}\tau_g+\frac{\partial \mathcal{A}_3^{\mathcal{D}}}{\partial s}\right)\mathbf{T}-\psi\mathbf{P}, \tag{3.11}$$

where $\psi = \left\langle \dfrac{\partial \mathbf{P}}{\partial t}, \mathbf{n} \right\rangle$.

Proof. Using definition of α, we have

$$\frac{\partial \mathbf{T}}{\partial t} = \frac{\partial}{\partial t}\frac{\partial \alpha}{\partial s} = \frac{\partial}{\partial s}(\mathcal{A}_1^{\mathcal{S}}\mathbf{T} + \mathcal{A}_2^{\mathcal{S}}\mathbf{P} + \mathcal{A}_3^{\mathcal{S}}\mathbf{n}).$$

Using the Darboux equations, we have

$$\begin{aligned}\frac{\partial \mathbf{T}}{\partial t} &= \left(\frac{\partial \mathcal{A}_1^{\mathcal{D}}}{\partial s} + \mathcal{A}_2^{\mathcal{D}} \upsilon \kappa_g + \mathcal{A}_3^{\mathcal{D}} \kappa_n\right)\mathbf{T} + \left(\mathcal{A}_1^{\mathcal{D}} \upsilon \kappa_g + \frac{\partial \mathcal{A}_2^{\mathcal{D}}}{\partial s} + \mathcal{A}_3^{\mathcal{D}} \tau_g\right)\mathbf{P} \\ &\quad + \left(\mathcal{A}_1^{\mathcal{D}} \kappa_n - \mathcal{A}_2^{\mathcal{D}} \tau_g + \frac{\partial \mathcal{A}_3^{\mathcal{D}}}{\partial s}\right)\mathbf{n}.\end{aligned} \tag{3.12}$$

Substituting (3.7) in (3.12), we get

$$\frac{\partial \mathbf{T}}{\partial t} = \left(\mathcal{A}_1^{\mathcal{D}} \upsilon \kappa_g + \frac{\partial \mathcal{A}_2^{\mathcal{D}}}{\partial s} + \mathcal{A}_3^{\mathcal{D}} \upsilon \tau_g\right)\mathbf{P} + \left(\mathcal{A}_1^{\mathcal{D}} \kappa_n - \mathcal{A}_2^{\mathcal{D}} \tau_g + \frac{\partial \mathcal{A}_3^{\mathcal{D}}}{\partial s}\right)\mathbf{n}.$$

Also, we have

$$\begin{aligned}\mathcal{A}_1^{\mathcal{D}} \kappa_g + \frac{\partial \mathcal{A}_2^{\mathcal{D}}}{\partial s} + \mathcal{A}_3^{\mathcal{D}} \tau_g + \left\langle \mathbf{T}, \frac{\partial \mathbf{P}}{\partial t}\right\rangle &= 0, \\ \mathcal{A}_1^{\mathcal{D}} \kappa_n - \mathcal{A}_2^{\mathcal{D}} \tau_g + \frac{\partial \mathcal{A}_3^{\mathcal{D}}}{\partial s} + \left\langle \mathbf{T}, \frac{\partial \mathbf{n}}{\partial t}\right\rangle &= 0, \\ \psi + \left\langle \mathbf{P}, \frac{\partial \mathbf{n}}{\partial t}\right\rangle &= 0.\end{aligned}$$

Then, a straightforward computation using above system gives

$$\begin{aligned}\frac{\partial \mathbf{P}}{\partial t} &= \left(\mathcal{A}_1^{\mathcal{D}} \kappa_g + \frac{\partial \mathcal{A}_2^{\mathcal{D}}}{\partial s} + \mathcal{A}_3^{\mathcal{D}} \tau_g\right)\mathbf{T} + \psi \mathbf{n}, \\ \frac{\partial \mathbf{n}}{\partial t} &= \left(\mathcal{A}_1^{\mathcal{D}} \kappa_n - \mathcal{A}_2^{\mathcal{D}} \tau_g + \frac{\partial \mathcal{A}_3^{\mathcal{D}}}{\partial s}\right)\mathbf{T} - \psi \mathbf{P},\end{aligned}$$

where $\psi = \left\langle \dfrac{\partial \mathbf{P}}{\partial t}, \mathbf{n} \right\rangle$.

Thus, we obtain the theorem.

The following theorem states the conditions on the curvature and torsion for the flow to be inextensible.

Theorem 3.5. *Let $\mathcal{M}$ be an oriented time-like surface and α lying on $\mathcal{M}$ in Minkowski 3-space $\mathbb{M}_1^3$. If $\dfrac{\partial \alpha}{\partial t}$ is inextensible, then the following system of partial differential equations holds:*

$$\begin{aligned}\frac{\partial \kappa_g}{\partial t} - \psi \kappa_n &= \frac{\partial}{\partial s}(\mathcal{A}_1^{\mathcal{D}} \kappa_g) + \frac{\partial^2 \mathcal{A}_2^{\mathcal{D}}}{\partial s^2} + \frac{\partial}{\partial s}(\mathcal{A}_3^{\mathcal{D}} \tau_g) + \mathcal{A}_1^{\mathcal{D}} \kappa_n \tau_g - \mathcal{A}_2^{\mathcal{D}} \tau_g^2 + \frac{\partial \mathcal{A}_3^{\mathcal{D}}}{\partial s}\tau_g, \\ \frac{\partial \kappa_n}{\partial t} + \psi \kappa_g &= \left(\frac{\partial}{\partial s}(\mathcal{A}_1^{\mathcal{D}} \kappa_n) - \frac{\partial}{\partial s}(\mathcal{A}_2^{\mathcal{D}} \tau_g) + \frac{\partial^2 \mathcal{A}_3^{\mathcal{D}}}{\partial s^2}\right) - \left(\mathcal{A}_1^{\mathcal{D}} \kappa_g + \frac{\partial \mathcal{A}_2^{\mathcal{D}}}{\partial s} + \mathcal{A}_3^{\mathcal{D}} \tau_g\right)\tau_g.\end{aligned} \tag{3.13}$$

Proof. Using (3.9), we have

$$
\begin{aligned}
\frac{\partial}{\partial s}\frac{\partial \mathbf{T}}{\partial t} &= \frac{\partial}{\partial s}\left[\left(\mathcal{A}_1^{\mathcal{D}}\kappa_g + \frac{\partial \mathcal{A}_2^{\mathcal{D}}}{\partial s} + \mathcal{A}_3^{\mathcal{D}}\tau_g\right)\mathbf{P} + \left(\mathcal{A}_1^{\mathcal{D}}\kappa_n - \mathcal{A}_2^{\mathcal{D}}\tau_g + \frac{\partial \mathcal{A}_3^{\mathcal{D}}}{\partial s}\right)\mathbf{n}\right] \\
&= [\left(\mathcal{A}_1^{\mathcal{D}}\kappa_g + \frac{\partial \mathcal{A}_2^{\mathcal{D}}}{\partial s} + \mathcal{A}_3^{\mathcal{D}}\tau_g\right)\kappa_g + \left(\mathcal{A}_1^{\mathcal{D}}\kappa_n - \mathcal{A}_2^{\mathcal{D}}\tau_g + \frac{\partial \mathcal{A}_3^{\mathcal{D}}}{\partial s}\right)\kappa_n]\mathbf{T} \\
&+ \left(\frac{\partial}{\partial s}(\mathcal{A}_1^{\mathcal{D}}\kappa_g) + \frac{\partial^2 \mathcal{A}_2^{\mathcal{D}}}{\partial s^2} + \frac{\partial}{\partial s}(\mathcal{A}_3^{\mathcal{D}}\tau_g) + \mathcal{A}_1^{\mathcal{D}}\kappa_n\tau_g - \mathcal{A}_2^{\mathcal{D}}\tau_g^2 + \frac{\partial \mathcal{A}_3^{\mathcal{D}}}{\partial s}\tau_g\right)\mathbf{P} \\
&[\left(\frac{\partial}{\partial s}(\mathcal{A}_1^{\mathcal{D}}\kappa_n) - \frac{\partial}{\partial s}(\mathcal{A}_2^{\mathcal{D}}\tau_g) + \frac{\partial^2 \mathcal{A}_3^{\mathcal{D}}}{\partial s^2}\right) - \left(\mathcal{A}_1^{\mathcal{D}}\kappa_g + \frac{\partial \mathcal{A}_2^{\mathcal{D}}}{\partial s} + \mathcal{A}_3^{\mathcal{D}}\tau_g\right)\tau_g]\mathbf{n}.
\end{aligned}
$$

On the other hand, from Darboux frame we have

$$
\begin{aligned}
\frac{\partial}{\partial t}\frac{\partial \mathbf{T}}{\partial s} &= \frac{\partial}{\partial t}(\kappa_g\mathbf{P} + \kappa_n\mathbf{n}) \\
&= [\kappa_g\left(\mathcal{A}_1^{\mathcal{D}}\kappa_g + \frac{\partial \mathcal{A}_2^{\mathcal{D}}}{\partial s} + \mathcal{A}_3^{\mathcal{D}}\tau_g\right) + \kappa_n\left(\mathcal{A}_1^{\mathcal{D}}\kappa_n + \mathcal{A}_2^{\mathcal{D}}\tau_g + \frac{\partial \mathcal{A}_3^{\mathcal{D}}}{\partial s}\right)]\mathbf{T} \\
&+ \left(\frac{\partial \kappa_g}{\partial t} - \psi\kappa_n\right)\mathbf{P} + \left(\frac{\partial \kappa_n}{\partial t} + \psi\kappa_g\right)\mathbf{n}.
\end{aligned}
$$

Hence we see that

$$
\frac{\partial \kappa_g}{\partial t} - \psi\kappa_n = \frac{\partial}{\partial s}(\mathcal{A}_1^{\mathcal{D}}\kappa_g) + \frac{\partial^2 \mathcal{A}_2^{\mathcal{D}}}{\partial s^2} + \frac{\partial}{\partial s}(\mathcal{A}_3^{\mathcal{D}}\tau_g) + \mathcal{A}_1^{\mathcal{D}}\kappa_n\tau_g - \mathcal{A}_2^{\mathcal{D}}\tau_g^2 + \frac{\partial \mathcal{A}_3^{\mathcal{D}}}{\partial s}\tau_g.
$$

and

$$
\frac{\partial \kappa_n}{\partial t} + \psi\kappa_g = \left(\frac{\partial}{\partial s}(\mathcal{A}_1^{\mathcal{D}}\kappa_n) - \frac{\partial}{\partial s}(\mathcal{A}_2^{\mathcal{D}}\tau_g) + \frac{\partial^2 \mathcal{A}_3^{\mathcal{D}}}{\partial s^2}\right) - \left(\mathcal{A}_1^{\mathcal{D}}\kappa_g + \frac{\partial \mathcal{A}_2^{\mathcal{D}}}{\partial s} + \mathcal{A}_3^{\mathcal{D}}\tau_g\right)\tau_g.
$$

Thus, we obtain the theorem.

Corollary 3.6.

$$
\left(\frac{\partial}{\partial s}(\mathcal{A}_1^{\mathcal{D}}\kappa_n) - \frac{\partial}{\partial s}\left(\mathcal{A}_2^{\mathcal{D}}\tau_g\right) + \frac{\partial^2 \mathcal{A}_3^{\mathcal{D}}}{\partial s^2}\right) - \psi\kappa_g = \frac{\partial \kappa_n}{\partial t} + \left(\mathcal{A}_1^{\mathcal{D}}\kappa_g\tau_g + \frac{\partial \mathcal{A}_2^{\mathcal{D}}}{\partial s}\tau_g + \mathcal{A}_3^{\mathcal{D}}\tau_g^2\right).
$$

Proof. Similarly, we have

$$
\begin{aligned}
\frac{\partial}{\partial s}\frac{\partial \mathbf{n}}{\partial t} &= \frac{\partial}{\partial s}\left[\left(\mathcal{A}_1^{\mathcal{D}}\kappa_n - \mathcal{A}_2^{\mathcal{D}}\tau_g + \frac{\partial \mathcal{A}_3^{\mathcal{D}}}{\partial s}\right)\mathbf{T} - \psi\mathbf{P}\right] \\
&= [\left(\frac{\partial}{\partial s}(\mathcal{A}_1^{\mathcal{D}}\kappa_n) - \frac{\partial}{\partial s}\left(\mathcal{A}_2^{\mathcal{D}}\tau_g\right) + \frac{\partial^2 \mathcal{A}_3^{\mathcal{D}}}{\partial s^2}\right) - \psi\kappa_g]\mathbf{T} \\
&+ [\kappa_g\left(\mathcal{A}_1^{\mathcal{D}}\kappa_n - \mathcal{A}_2^{\mathcal{D}}\tau_g + \frac{\partial \mathcal{A}_3^{\mathcal{D}}}{\partial s}\right) - \frac{\partial \psi}{\partial s}]\mathbf{P} \\
&+ [\kappa_n\left(\mathcal{A}_1^{\mathcal{D}}\kappa_n - \mathcal{A}_2^{\mathcal{D}}\tau_g + \frac{\partial \mathcal{A}_3^{\mathcal{D}}}{\partial s}\right) + \tau_g\psi]\mathbf{n}.
\end{aligned}
$$

On the other hand, a straight forward computation gives

$$\begin{aligned}\frac{\partial}{\partial t}\frac{\partial \mathbf{n}}{\partial s} &= \frac{\partial}{\partial t}\left(\kappa_n \mathbf{T} + \tau_g \mathbf{P}\right) \\ &= \frac{\partial \kappa_n}{\partial t} + \left(\mathcal{A}_1^{\mathcal{D}}\kappa_g\tau_g + \frac{\partial \mathcal{A}_2^{\mathcal{D}}}{\partial s}\tau_g + \mathcal{A}_3^{\mathcal{D}}\tau_g^2\right)]\mathbf{T} \\ &+[\left(\mathcal{A}_1^{\mathcal{D}}\kappa_g\kappa_n + \frac{\partial \mathcal{A}_2^{\mathcal{D}}}{\partial s}\kappa_n + \mathcal{A}_3^{\mathcal{D}}\kappa_n\tau_g + \frac{\partial \tau_g}{\partial t}\right)\mathbf{P} \\ &+ \left(\mathcal{A}_1^{\mathcal{D}}\kappa_n^2 - \mathcal{A}_2^{\mathcal{D}}\kappa_n\tau_g + \frac{\partial \mathcal{A}_3^{\mathcal{D}}}{\partial s}\kappa_n + \psi\tau_g\right)\mathbf{n}.\end{aligned}$$

Combining these we obtain the corollary.

In the light of Theorem 3.5, we express the following corollaries without proofs:

Corollary 3.7.

$$\kappa_g\left(\mathcal{A}_1^{\mathcal{D}}\kappa_n - \mathcal{A}_2^{\mathcal{D}}\tau_g + \frac{\partial \mathcal{A}_3^{\mathcal{D}}}{\partial s}\right) - \frac{\partial \psi}{\partial s} = \mathcal{A}_1^{\mathcal{D}}\kappa_g\kappa_n + \frac{\partial \mathcal{A}_2^{\mathcal{D}}}{\partial u}\kappa_n + \mathcal{A}_3^{\mathcal{D}}\kappa_n\tau_g + \frac{\partial \tau_g}{\partial t}.$$

Corollary 3.8. *Let $\mathcal{M}$ be an oriented time-like surface, α lying on $\mathcal{M}$ and the flow $\frac{\partial \alpha}{\partial t}$ is inextensible in Minkowski 3-space $\mathbb{M}_1^3$. If α is a geodesic curve, then*

$$\frac{\partial \psi}{\partial s} = -\frac{\partial \mathcal{A}_2^{\mathcal{D}}}{\partial s}\kappa_n - \mathcal{A}_3^{\mathcal{D}}\kappa_n\tau_g - \frac{\partial \tau_g}{\partial t}.$$

Proof. By using $\kappa_g = 0$ in Lemma 3.7, we get above equation. This completes the proof.

Corollary 3.9. *Let $\mathcal{M}$ be an oriented time-like surface, α lying on $\mathcal{M}$ and the flow $\frac{\partial \alpha}{\partial t}$ is inextensible in Minkowski 3-space $\mathbb{M}_1^3$. If α is a principal line, then*

$$\begin{aligned}\frac{\partial \psi}{\partial s}\kappa_g + \frac{\partial \mathcal{A}_3^{\mathcal{D}}}{\partial s} + \frac{\partial \mathcal{A}_2^{\mathcal{D}}}{\partial s}\kappa_n + \frac{\partial \tau_g}{\partial t} &= -\mathcal{A}_1^{\mathcal{D}}\kappa_g\kappa_n - \kappa_g\mathcal{A}_1^{\mathcal{D}}\kappa_n, \\ \mathcal{A}_1^{\mathcal{D}}\kappa_n + \frac{\partial \mathcal{A}_3^{\mathcal{D}}}{\partial s} + \kappa_n\frac{\partial \mathcal{A}_3^{\mathcal{D}}}{\partial s} &= -\kappa_n\mathcal{A}_1^{\mathcal{D}}\kappa_n.\end{aligned}$$

Proof. Substituting $\tau_g = 0$ in Lemma 3.7-3.8, we get above equation. This completes the proof.

Corollary 3.10. *Let $\mathcal{M}$ be an oriented time-like surface, α lying on $\mathcal{M}$ and the flow $\frac{\partial \alpha}{\partial t}$ is inextensible in Minkowski 3-space $\mathbb{M}_1^3$. If α is a asymptotic line, then*

$$\mathcal{A}_2^{\mathcal{D}}\tau_g + \frac{\partial \mathcal{A}_3^{\mathcal{D}}}{\partial s} + \psi\tau_g = 0.$$

Proof. By using $\kappa_n = 0$ in Lemma 3.8, we get above equation. This completes the proof.

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