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ON SPACELIKE BIHARMONIC NEW TYPE $\mathcal{B}$-SLANT HELICES WITH TIMELIKE $\mathbf{m}_2$ ACCORDING TO BISHOP FRAME IN LORENTZIAN HEISENBERG GROUP $\mathcal{H}^3$

ESSIN TURHAN AND TALAT KÖRPINAR

Abstract. In this paper, we study biharmonic spacelike new type $\mathcal{B}$-slant helices with timelike $\mathbf{m}_2$ according to Bishop frame in the Lorentzian Heisenberg group $\mathcal{H}^3$. We give necessary and sufficient conditions for new type $\mathcal{B}$-slant helices with timelike $\mathbf{m}_2$ to be biharmonic. We characterize this curves in the Lorentzian Heisenberg group $\mathcal{H}^3$.

1. Introduction

Helices arise in nanosprings, carbon nanotubes, α-helices, DNA double and collagen triple helix, the double helix shape is commonly associated with DNA, since the double helix is structure of DNA. They constructed a molecular model of DNA in which there were two complementary, antiparallel (side-by-side in opposite directions) strands of the bases guanine, adenine, thymine and cytosine, covalently linked through phosphodiester bonds. Each strand forms a helix and two helices are held together through hydrogen bonds, ionic forces, hydrophobic interactions and van der Waals forces forming a double helix, lipid bilayers, bacterial flagella in Salmonella and E. coli, aerial hyphae in actynomycetes, bacterial shape in spirochetes, horns, tendrils, vines, screws, springs, helical staircases and sea shells.

This study is organised as follows: Firstly, we give necessary and sufficient conditions for new type $\mathcal{B}$-slant helices with timelike $\mathbf{m}_2$ to be biharmonic. We characterize this curves in the Lorentzian Heisenberg group $\mathcal{H}^3$. Secondly, we study biharmonic spacelike new type $\mathcal{B}$-slant helices with timelike $\mathbf{m}_2$ according to Bishop frame in the Lorentzian Heisenberg group $\mathcal{H}^3$. Finally, we illustrate our results.

2. The Lorentzian Heisenberg Group $\mathcal{H}^3$

The Heisenberg group Heis^3 is a Lie group which is diffeomorphic to $\mathbb{R}^3$ and the group operation is defined as

$$(x,y,z)*(\overline{x},\overline{y},\overline{z})=(x+\overline{x},y+\overline{y},z+\overline{z}-\frac{1}{2}\overline{x}y+\frac{1}{2}x\overline{y}).$$

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The identity of the group is $(0,0,0)$ and the inverse of (x,y,z) is given by $(-x,-y,-z)$. The left-invariant Lorentz metric on Heis^3 is

$$g=-dx^2+dy^2+(xdy+dz)^2.$$

The following set of left-invariant vector fields forms an orthonormal basis for the corresponding Lie algebra:

$$\left\{\mathbf{e}_1=\frac{\partial}{\partial z},\ \mathbf{e}_2=\frac{\partial}{\partial y}-x\frac{\partial}{\partial z},\ \mathbf{e}_3=\frac{\partial}{\partial x}\right\}. \tag{2.1}$$

The characterising properties of this algebra are the following commutation relations, [15]:

$$g(\mathbf{e}_1,\mathbf{e}_1)=g(\mathbf{e}_2,\mathbf{e}_2)=1,\quad g(\mathbf{e}_3,\mathbf{e}_3)=-1.$$

Proposition 2.1. *For the covariant derivatives of the Levi-Civita connection of the left-invariant metric g, defined above the following is true:*

$$\nabla=\frac{1}{2}\begin{pmatrix}0 & \mathbf{e}_3 & \mathbf{e}_2\\ \mathbf{e}_3 & 0 & \mathbf{e}_1\\ \mathbf{e}_2 & -\mathbf{e}_1 & 0\end{pmatrix}, \tag{2.2}$$

where the (i,j)-element in the table above equals $\nabla_{\mathbf{e}_i}\mathbf{e}_j$ for our basis

$$\{\mathbf{e}_k, k=1,2,3\}.$$

3. Spacelike Biharmonic New Type $\mathcal{B}$–Slant Helices with Timelike $\mathbf{m}_2$ in the Lorentzian Heisenberg Group $\mathcal{H}^3$

Let $\gamma: I\longrightarrow\mathcal{H}^3$ be a non geodesic spacelike curve on the Lorentzian Heisenberg group $\mathcal{H}^3$ parametrized by arc length. Let $\{\mathbf{t},\mathbf{n},\mathbf{b}\}$ be the Frenet frame fields tangent to the Lorentzian Heisenberg group $\mathcal{H}^3$ along γ defined as follows:

$\mathbf{t}$ is the unit vector field γ' tangent to γ, $\mathbf{n}$ is the unit vector field in the direction of $\nabla_{\mathbf{t}}\mathbf{t}$ (normal to γ), and $\mathbf{b}$ is chosen so that $\{\mathbf{t},\mathbf{n},\mathbf{b}\}$ is a positively oriented orthonormal basis. Then, we have the following Frenet formulas:

$$\begin{aligned}\nabla_{\mathbf{t}}\mathbf{t}&=\kappa\mathbf{n},\\ \nabla_{\mathbf{t}}\mathbf{n}&=-\kappa\mathbf{t}+\tau\mathbf{b},\\ \nabla_{\mathbf{t}}\mathbf{b}&=\tau\mathbf{n},\end{aligned}$$

where κ is the curvature of γ and τ is its torsion and

$$\begin{aligned}&g(\mathbf{t},\mathbf{t})=1,\ g(\mathbf{n},\mathbf{n})=1,\ g(\mathbf{b},\mathbf{b})=-1,\\ &g(\mathbf{t},\mathbf{n})=g(\mathbf{t},\mathbf{b})=g(\mathbf{n},\mathbf{b})=0.\end{aligned}$$

In the rest of the paper, we suppose everywhere $\kappa\neq 0$ and $\tau\neq 0$.

The Bishop frame or parallel transport frame is an alternative approach to defining a moving frame that is well defined even when the curve has vanishing second derivative. The Bishop frame is expressed as

$$\begin{aligned}\nabla_{\mathbf{t}}\mathbf{t}&=\mathfrak{k}_1\mathbf{m}_1-\mathfrak{k}_2\mathbf{m}_2,\\ \nabla_{\mathbf{t}}\mathbf{m}_1&=-\mathfrak{k}_1\mathbf{t},\\ \nabla_{\mathbf{t}}\mathbf{m}_2&=-\mathfrak{k}_2\mathbf{t},\end{aligned} \tag{3.1}$$

where

$$g(\mathbf{t},\mathbf{t})=1,\ g(\mathbf{m}_1,\mathbf{m}_1)=1,\ g(\mathbf{m}_2,\mathbf{m}_2)=-1,$$
$$g(\mathbf{t},\mathbf{m}_1)=g(\mathbf{t},\mathbf{m}_2)=g(\mathbf{m}_1,\mathbf{m}_2)=0.$$

Here, we shall call the set $\{\mathbf{t},\mathbf{m}_1,\mathbf{m}_2\}$ as Bishop trihedra and $\mathfrak{k}_1$ and $\mathfrak{k}_2$ as Bishop curvatures, $\tau(s)=\theta'(s)$ and $\kappa(s)=\sqrt{|\mathfrak{k}_1^2-\mathfrak{k}_2^2|}$.

With respect to the orthonormal basis $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ we can write

$$\begin{aligned}\mathbf{t}&=t^1\mathbf{e}_1+t^2\mathbf{e}_2+t^3\mathbf{e}_3,\\ \mathbf{m}_1&=m_1^1\mathbf{e}_1+m_1^2\mathbf{e}_2+m_1^3\mathbf{e}_3,\\ \mathbf{m}_2&=m_2^1\mathbf{e}_1+m_2^2\mathbf{e}_2+m_2^3\mathbf{e}_3.\end{aligned} \tag{3.2}$$

Theorem 3.1. *$\gamma:I\longrightarrow\mathcal{H}^3$ is a spacelike biharmonic curve with timelike $\mathbf{m}_2$ according to Bishop frame if and only if*

$$\begin{aligned}&\mathfrak{k}_2^2-\mathfrak{k}_1^2=\text{constant}=C\neq0,\\ &\mathfrak{k}_1''+C\mathfrak{k}_1=k_1\left[-1+4\left(m_2^1\right)^2\right]+4k_2m_1^1m_2^1,\\ &\mathfrak{k}_2''+C\mathfrak{k}_2=4k_1m_1^1m_2^1-k_2\left[1+4\left(m_1^1\right)^2\right].\end{aligned} \tag{3.3}$$

Proof. Using Eq.(4), we have above system.

Definition 3.2. *A regular spacelike curve $\gamma:I\longrightarrow\mathcal{H}^3$ is called a new type slant helix provided the timelike unit vector $\mathbf{m}_2$ of the curve γ has constant angle $\mathcal{E}$ with some fixed spacelike unit vector u, that is*

$$g(\mathbf{m}_2(s),u)=\sinh\mathcal{E}\ \text{for all}\ s\in I. \tag{3.4}$$

The condition is not altered by reparametrization, so without loss of generality we may assume that slant helices have unit speed. The slant helices can be identified by a simple condition on natural curvatures.

To separate a spacelike new type slant helix according to Bishop frame from that of Frenet- Serret frame, in the rest of the paper, we shall use notation for the curve defined above as spacelike new type $\mathcal{B}$-slant helix.

We shall also use the following lemma.

Lemma 3.3. *Let $\gamma:I\longrightarrow\mathcal{H}^3$ be a unit speed spacelike curve with timelike $\mathbf{m}_2$. Then γ is a new type $\mathcal{B}$-slant helix if and only if*

$$\mathfrak{k}_1=\mathfrak{k}_2\tanh\mathcal{E}. \tag{3.5}$$

Proof. Using Eq.(3.4) and by using the Bishop frame Eq.(3.1), we find

$$\frac{\mathfrak{k}_1}{\mathfrak{k}_2}=\tanh\mathcal{E}=\text{constant}. \tag{3.6}$$

The converse statement is trivial. Thus, we obtain the lemma.

Corollary 3.4. *$\gamma : I \longrightarrow \mathcal{H}^3$ is spacelike biharmonic new type $\mathcal{B}-$slant helix with timelike $\mathbf{m}_2$ if and only if*

$$\begin{aligned} \mathfrak{k}_1 &= \text{constant} \neq 0, \\ \mathfrak{k}_2 &= \text{constant} \neq 0, \\ [-\mathfrak{k}_1^2 + \mathfrak{k}_2^2 + 1 - 4\left(m_2^1\right)^2] &= 4\coth \mathcal{E} m_1^1 m_2^1, \\ [-\mathfrak{k}_1^2 + \mathfrak{k}_2^2 + 1 + 4\left(m_1^1\right)^2] &= 4\tanh \mathcal{E} m_1^1 m_2^1. \end{aligned} \tag{3.7}$$

Theorem 3.5. *Let $\gamma : I \longrightarrow \mathcal{H}^3$ be a unit speed biharmonic spacelike new type $\mathcal{B}-$slant helix with non-zero Bishop curvatures. Then the equations of γ are*

$$\begin{aligned} \boldsymbol{x}_{\mathcal{B}}(s) &= \frac{1}{\mathcal{J}_0}\sinh \mathcal{E}\sinh[\mathcal{J}_0 s + \mathcal{J}_1] + \mathcal{J}_2, \\ \boldsymbol{y}_{\mathcal{B}}(s) &= \frac{1}{\mathcal{J}_0}\sinh \mathcal{E}\cosh[\mathcal{J}_0 s + \mathcal{J}_1] + \mathcal{J}_3, \\ \boldsymbol{z}_{\mathcal{B}}(s) &= \cosh \mathcal{E} s - \frac{\mathcal{J}_2}{\mathcal{J}_0}\sinh \mathcal{E}\cosh[\mathcal{J}_0 s + \mathcal{J}_1] \\ &\quad - \frac{1}{4\mathcal{J}_0}\sinh^2 \mathcal{E}(2[\mathcal{J}_0 s + \mathcal{J}_1] + \sinh 2[\mathcal{J}_0 s + \mathcal{J}_1]) + \mathcal{J}_4, \end{aligned} \tag{3.8}$$

where $\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3, \mathcal{J}_4$ are constants of integration and

$$\mathcal{J}_0 = \frac{\sqrt{\mathfrak{k}_1^2 - \mathfrak{k}_2^2}}{\cosh \mathcal{E}} - \sinh \mathcal{E}.$$

Proof. So, without loss of generality, we take the axis of γ is parallel to the spacelike vector $\mathbf{e}_1$. Then,

$$g(\mathbf{m}_2, \mathbf{e}_1) = m_2^1 = \sinh \mathcal{E}, \tag{3.9}$$

where $\mathcal{E}$ is constant angle.

On the other hand, the vector $\mathbf{m}_2$ is a unit timelike vector, we reach

$$\mathbf{m}_2 = \sinh \mathcal{E}\mathbf{e}_1 + \cosh \mathcal{E}\sinh \mathfrak{D}(s)\mathbf{e}_2 + \cosh \mathcal{E}\cosh \mathfrak{D}(s)\mathbf{e}_3. \tag{3.10}$$

Using Bishop formulas Eq.(3.1) and Eq.(2.1), we have

$$\mathbf{m}_1 = \cosh \mathfrak{D}(s)\mathbf{e}_2 + \sinh \mathfrak{D}(s)\mathbf{e}_3. \tag{3.11}$$

It is apparent that

$$\mathbf{t} = \cosh \mathcal{E}\mathbf{e}_1 + \sinh \mathcal{E}\sinh \mathfrak{D}(s)\mathbf{e}_2 + \sinh \mathcal{E}\cosh \mathfrak{D}(s)\mathbf{e}_3. \tag{3.12}$$

A straightforward computation shows that

$$\nabla_{\mathbf{t}}\mathbf{t} = (\mathfrak{t}_1')\mathbf{e}_1 + (\mathfrak{t}_2' + \mathfrak{t}_1\mathfrak{t}_3)\mathbf{e}_2 + (\mathfrak{t}_3' + \mathfrak{t}_1\mathfrak{t}_2)\mathbf{e}_3.$$

Therefore, we use Bishop formulas Eq.(3.1) and above equation we get

$$\mathfrak{D}(s) = [\frac{\sqrt{\mathfrak{k}_1^2 - \mathfrak{k}_2^2}}{\cosh \mathcal{E}} - \sinh \mathcal{E}]s + \mathcal{J}_1, \tag{3.13}$$

where $\mathcal{J}_1$ is a constant of integration.

From Eq.(3.12) and Eq.(3.13), we get

$$\mathbf{t} = (\sinh \mathcal{E}\cosh[\mathcal{J}_0 s + \mathcal{J}_1], \sinh \mathcal{E}\sinh[\mathcal{J}_0 s + \mathcal{J}_1], \cosh \mathcal{E} - x\sinh \mathcal{E}\sinh[\mathcal{J}_0 s + \mathcal{J}_1]), \tag{3.14}$$

where

$$\mathcal{J}_0 = \frac{\sqrt{\mathfrak{k}_1^2 - \mathfrak{k}_2^2}}{\cosh \mathcal{E}} - \sinh \mathcal{E}.$$

Therefore, by Eq(3.14) and taking into account Eq.(2.1), we obtain the system Eq.(3.8). This completes the proof.

Corollary 3.6. *Let $\gamma : I \longrightarrow \mathcal{H}^3$ be a unit speed biharmonic spacelike new type $\mathcal{B}-$slant helix with non-zero Bishop curvatures. Then the equation of γ is*

$$\begin{aligned}\gamma(s) = {} & [\cosh \mathcal{E} s - \frac{\mathcal{J}_2}{\mathcal{J}_0} \sinh \mathcal{E} \cosh [\mathcal{J}_0 s + \mathcal{J}_1] - \frac{1}{4\mathcal{J}_0} \sinh^2 \mathcal{E} (2 [\mathcal{J}_0 s + \mathcal{J}_1] + \sinh 2 [\mathcal{J}_0 s + \mathcal{J}_1]) \\ & + \mathcal{J}_4 + [\frac{1}{\mathcal{J}_0} \sinh \mathcal{E} \sinh [\mathcal{J}_0 s + \mathcal{J}_1] + \mathcal{J}_2][\frac{1}{\mathcal{J}_0} \sinh \mathcal{E} \cosh [\mathcal{J}_0 s + \mathcal{J}_1] + \mathcal{J}_3]]\mathbf{e}_1 \\ & + [\frac{1}{\mathcal{J}_0} \sinh \mathcal{E} \cosh [\mathcal{J}_0 s + \mathcal{J}_1] + \mathcal{J}_3]\mathbf{e}_2 + [\frac{1}{\mathcal{J}_0} \sinh \mathcal{E} \sinh [\mathcal{J}_0 s + \mathcal{J}_1] + \mathcal{J}_2]\mathbf{e}_3.\end{aligned}$$

where $\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3, \mathcal{J}_4$ are constants of integration and

$$\mathcal{J}_0 = \frac{\sqrt{\mathfrak{k}_1^2 - \mathfrak{k}_2^2}}{\cosh \mathcal{E}} - \sinh \mathcal{E}.$$

Proof. The proof is standard, so we omit it.

If we use Mathematica in (3.8), we get:

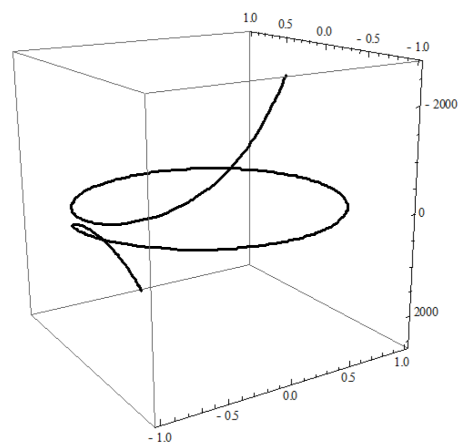

Fig.1.

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Firat University, Department of Mathematics,23119, Elaziğ, Turkey
E-mail address: essin.turhan@gmail.com, talatkorpinar@gmail.com