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ON SURFACE OF REVOLUTION WITH 1-TYPE GAUSS MAP IN THE 3-DIMENSIONAL DUAL MINKOWSKI SPACE

MUSTAFA YENEROGLU AND VEDAT ASIL

Abstract. In this paper, surface of revolution with 1-type Gauss map are studied in the 3-dimensional Dual Minkowski space. By the use of the concept of 1-type Gauss map, a characterisation theorem concerning surfaces of revolution and constancy of the mean curvature of certain open subsets on this surface are obtained.

1. Introduction

Dual numbers were introduced in the 19 th. century by Cliford [1]. Dual quantities, the differential geometry of dual curves and application to the theoretical space kinematic were given by Veldkamp [2]. V. Brodsky and M. Shoham examined dual numbers representation of rigid body dynamics [3]. A. Parkın studied orthogonal matrix transformations [4].

Y.H. Kim and D.W. Yoon studied ruled surfaces with pointwise 1-type Gauss map, i.e... They also classify all submanifolds in an m-Euclidean space E^m satisfying the following equation

$$\Delta G = fG,$$

where Δ in the Laplacian of the induced metric and G the Gauss map for the submanifold, for some function f on the submanifold [5]. A Niang investigate rotation surfaces in the Minkowski 3-dimensional space with pointwise 1-type Gauss map [6]. M. Choi and Y. H. Kim characterised the helicoidal surfaces with pointwise 1-type Gauss map [11]. M. Yeneroglu and V.Asil studied the rotation surfaces with pointwise 1-type Gauss map in the 3-dimensional Dual Space[10].

In this study, the condition (1.1) will be expanded in D^3_1, i.e,

$$\hat{\Delta}\hat{G} = \hat{f}\hat{G}, \tag{1.2}$$

where $\hat{\Delta} = \Delta + \varepsilon\Delta^*$ is the Laplacian in D^3_1, $\hat{G} = G + \varepsilon G^*$ is dual Gauss map, $\hat{f} = f + \varepsilon f^*$ is a dual functions.

The main goal of this article is to prove the following theorem:

Theorem 1.1. Let $\hat{M}$ be a connected surfaces of revolution in a D^3_1 whose axis of the rotation is $\hat{L}$ Let $\hat{M}'$ be any connected component of the subset $\hat{M} - \hat{L}$. Then $\hat{M}'$ is ponintwise 1-type Gauss map if and only if a constant mean curvature.

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2. Preliminaries

A dual number can be defined as an ordered pair combining a real part , a and a dual part a^*,

$$\hat{a} = a + \varepsilon a^*, \tag{2.1}$$

where ε is the dual unit with multiplication rule $\varepsilon^2 = 0$. A ordered triple of dual numbers (x_1, x_2, x_3) is called dual vector, we write $(\hat{x}_1, \hat{x}_2, \hat{x}_3) = \hat{x}$. The numbers x_1, x_2, x_3 are called the coordinates of $\hat{x}$.

Let $\hat{x} = (\hat{x}_1, \hat{x}_2, \hat{x}_3)$ and $\hat{y} = (\hat{y}_1, \hat{y}_2, \hat{y}_3)$ be two dual vector, Lorentzian inner product and cross-product of this two dual vectors are defined as follows:

$$< \hat{x}, \hat{y} >= \hat{x}_1\hat{y}_1 + \hat{x}_2\hat{y}_2 - \hat{x}_3\hat{y}_3, \tag{2.2}$$

$$\hat{x} \times \hat{y} = (\hat{x}_2\hat{y}_3 - \hat{x}_3\hat{y}_2, -\hat{x}_1\hat{y}_3 + \hat{x}_3\hat{y}_1, -\hat{x}_1\hat{y}_2 + \hat{x}_2\hat{y}_1). \tag{2.3}$$

A dual vector $\hat{x}$ is said to be time-like if $< \hat{x}, \hat{x} >< 0$, space-like if $< \hat{x}, \hat{x} >> 0$ and null if $< \hat{x}, \hat{x} >= 0$. The norm of a dual vector $\hat{x}$ is defined to be $\|\hat{x}\| = \sqrt{|< \hat{x}, \hat{x} >|}$, [9].

A dual function of dual space is given by

$$\hat{f}(\hat{t}) = f(t, t^*) + \varepsilon f^*(t, t^*), \tag{2.4}$$

where $\hat{t} = t + \varepsilon t^*$ is a dual variable, f and f^* are two, generally different, functions of the two real variables. This type of function is referred to simply as the dual functions throughout the paper.

Properties of dual functions were thoroughly investigated by H.Hacısalihoğlu, [7]. He derived the general expression for dual analytic (differentiable) function as follows

$$\hat{f}(\hat{t}) = f(t, t^*) + \varepsilon f^*(t, t^*) = f(t) + \varepsilon t^* f'(t). \tag{2.5}$$

The condition for dual fonctions of being analytic is

$$\frac{\partial f^*}{\partial t^*} = \frac{\partial f}{\partial t}. \tag{2.6}$$

The derivative of such a dual function with respect to a dual variable is

$$\frac{d\hat{f}(\hat{t})}{d\hat{t}} = \frac{\partial f}{\partial t} + \varepsilon \frac{\partial f^*}{\partial t} = f' + \varepsilon t^* f''. \tag{2.7}$$

A dual function of two dual parameters is given by

$$\hat{F}(\hat{u}, \hat{v}) = F(\hat{u}, \hat{v}) + \varepsilon F^*(\hat{u}, \hat{v}). \tag{2.8}$$

where $\hat{u} = u + \varepsilon u^*$ and $\hat{v} = v + \varepsilon v^*$ are two dual variable, F and F^* are two functions of two dual parameters. A dual analytic function of two dual parameters is expressed as follows;

$$\hat{F}(\hat{u}, \hat{v}) = F(u, v) + \varepsilon(u^* F_u(u, v) + v^* F_v(u, v)). \tag{(2.9)}$$

The partial derivatives of (2.9) are given by

$$\hat{F}_{\hat{u}}(\hat{u}, \hat{v}) = F_{\hat{u}}(\hat{u}, \hat{v}) + \varepsilon F^*_{\hat{u}}(\hat{u}, \hat{v}) = F_u(u, v) + \varepsilon(u^* F_{uu}(, v) + v^* F_{vu}(u, v)), \tag{2.10}$$

$$\hat{F}_{\hat{v}}(\hat{u}, \hat{v}) = F_{\hat{v}}(\hat{u}, \hat{v}) + \varepsilon F^*_{\hat{v}}(\hat{u}, \hat{v}) = F_v(u, v) + \varepsilon(u^* F_{uv}(u, v) + v^* F_{vv}(u, v)). \tag{2.11}$$

The surface $\hat{M}$ in D_1^3 is described locally by

$$\begin{aligned} \hat{X} &: \hat{U} \subset D_1^2 \to D_1^3 \\ (\hat{u},\hat{v}) &\to \hat{X}(\hat{u},\hat{v}) = X(\hat{u},\hat{v}) + \varepsilon X^*(\hat{u},\hat{v}) \\ \hat{X}(\hat{u},\hat{v}) &= X(u,v) + \varepsilon(u^* X_u(u,v) + v^* X_v(u,v)), \end{aligned} \tag{2.12}$$

where (u, v) are local coordinates on the open set $\hat{U}$ of D_1^2.

The Gauss map $\hat{G} = G + \varepsilon G^*$ on $\hat{U}$ is given by the followings formula:

$$\begin{aligned} \hat{G} &= \frac{\hat{X}_{\hat{u}} \times \hat{X}_{\hat{v}}}{\left\| \hat{X}_{\hat{u}} \times \hat{X}_{\hat{v}} \right\|} = G + \varepsilon(u^* G_u + v^* G_v), \\ G &= \frac{X_u \times X_v}{\|X_u \times X_v\|}. \end{aligned} \tag{2.13}$$

The first and second fundamental forms $\hat{I} = I + \varepsilon I^*$ and $I\hat{I} = II + \varepsilon II^*$, respectively, are obtained by

$$\begin{aligned} \hat{I} &=< \hat{X}_{\hat{u}}, \hat{X}_{\hat{u}} > (\acute{\hat{u}})^2 + 2 < \hat{X}_{\hat{u}}, \hat{X}_{\hat{v}} > \acute{\hat{u}}\acute{\hat{v}} + < \hat{X}_{\hat{v}}, \hat{X}_{\hat{v}} > (\acute{\hat{v}})^2, \\ \hat{I} &= I + \varepsilon(u^* I_u + v^* I_v), \\ I &=< X_u, X_u > (\acute{u}) + 2 < X_u, X_v > \acute{u}\acute{v} + < X_v, X_v > (\acute{v})^2, \end{aligned} \tag{2.14}$$

$$\begin{aligned} I\hat{I} &=< \hat{G}, \hat{X}_{\hat{u}\hat{u}} > (\acute{\hat{u}})^2 + 2 < \hat{G}, \hat{X}_{\hat{u}\hat{v}} > \acute{\hat{u}}\acute{\hat{v}} + < \hat{G}, \hat{X}_{\hat{v}\hat{v}} > (\acute{\hat{v}})^2, \\ I\hat{I} &= II + \varepsilon(u^* II_u + v^* II_v), \\ II &=< G, X_{uu} > (\acute{u})^2 + 2 < G, X_{uv} > \acute{u}\acute{v} + < G, X_{vv} > (\acute{v})^2. \end{aligned} \tag{2.15}$$

The mean curvature $\hat{H} = H + \varepsilon H^*$ over $\hat{U}$ is given by the following formula:

(2.16)

$$2\hat{H} = \frac{< \hat{G}, \hat{X}_{\hat{u}\hat{u}} >< \hat{X}_{\hat{u}}, \hat{X}_{\hat{u}} > -2 < \hat{G}, \hat{X}_{\hat{u}\hat{v}} >< \hat{X}_{\hat{u}}, \hat{X}_{\hat{v}} > + < \hat{G}, \hat{X}_{\hat{v}\hat{v}} >< \hat{X}_{\hat{v}}, \hat{X}_{\hat{v}} >}{< \hat{X}_{\hat{u}}, \hat{X}_{\hat{u}} >< \hat{X}_{\hat{v}}, \hat{X}_{\hat{v}} > -(< \hat{X}_{\hat{u}}, \hat{X}_{\hat{v}} >)^2},$$

$$2\hat{H} = 2H + 2\varepsilon(u^* H_u + v^* H_v),$$

$$2H = \frac{< G, X_{uu} >< X_u, X_u > -2 < G, X_{uv} >< X_u, X_v > + < G, X_{vv} >< X_v, X_v >}{< X_u, X_u >< X_u, X_v > -(< X_v, X_v >)^2}.$$

The Laplacian with respect to local coordinates $(\hat{x}_1, \hat{x}_2)$ in $\hat{U}$ for surface $\hat{M}$ is

$$\hat{\Delta} = -\frac{1}{\sqrt{\det(\hat{g}_{ij})}} \sum \frac{\hat{\partial}}{\partial \hat{x}_i} \left(\sqrt{\det(\hat{g}_{ij})} \hat{g}^{ij} \frac{\hat{\partial}}{\partial \hat{x}_j} \right), \tag{2.17}$$

where $(\hat{g}_{ij})$ is a dual matrix with entries $\hat{g}_{ij} =< \hat{X}_{\hat{x}_i}, \hat{X}_{\hat{x}_j} >$; the dual matrix $(\hat{g}^{ij})$ is the inverse matrix of $(\hat{g}_{ij})$.

3. Surfaces of revolution in $\mathbf{D}_1^3$

The subgroups of rotations around a time-like axis, a spce-like axis and light-like axis consist of, respectively,

$$\begin{bmatrix} \cos\hat{\theta} & -\sin\hat{\theta} & 0 \\ \sin\hat{\theta} & \cos\hat{\theta} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \hat{\theta} = \theta + \varepsilon\theta^*, \tag{3.1}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cosh\varphi & \sinh\varphi \\ 0 & \sinh\varphi & \cosh\varphi \end{bmatrix}, \qquad \varphi = \varphi + \varepsilon\varphi^*.$$

A surface of revolution is generated by revolving a profile curve about an axis of revolution.

$\hat{L}$ be the axis of rotation of the surface. Let $\hat{M}'$ be any connected component of the subset $\hat{M} - \hat{L}$. We the following lemma.

Lemma 3.1: ı) If $\hat{L}$ is space-like; then $\hat{M}'$ is expressed in the form $\hat{x} = \hat{g}(\hat{s})$, $\hat{y} = \hat{r}(\hat{s})\sinh\hat{\varphi}$, $\hat{z} = \hat{r}(\hat{s})\cosh\hat{\varphi}$; $\hat{\varphi} = \varphi + \varepsilon\varphi^*$, $\hat{s} = s + \varepsilon s^*$ with metric

$$\hat{I} = (\hat{g}'^2 - \hat{r}'^2)d\hat{s}^2 + \hat{r}d\hat{\varphi} \tag{3.2}$$

where $\hat{g}(\hat{s})$ and $\hat{r}(\hat{s})$ are smooth functions of the parameter $\hat{s}$ such that

$$\hat{g}'^2 - \hat{r}^2 = \zeta, \tag{3.3}$$

and $\hat{r}(\hat{s}) \neq 0$ and for all $\hat{s}, \zeta = \pm 1$.

ıı) If $\hat{L}$ is time-like; then $\hat{M}'$ is expressed in the form $\hat{x} = \hat{r}(\hat{s})\cos\hat{\theta}$, $\hat{y} = \hat{r}(\hat{s})\sin\hat{\theta}$, $\hat{z} = \hat{h}(\hat{s})$; $\hat{\theta} = \theta + \varepsilon\theta^*$, $\hat{s} = s + \varepsilon s^*$ with metric

$$\hat{I} = (\hat{r}^2 - \hat{h}^2)d\hat{s}^2 + \hat{r}^2 d\hat{\theta}^2, \tag{3.4}$$

where $\hat{r}(\hat{s})$ and $\hat{h}(\hat{s})$ are dual smooth functions of parameter $\hat{s}$ such that

$$\hat{r}^2 - \hat{h}^2 = \zeta, \tag{3.5}$$

and $\hat{r}(\hat{s}) \neq 0$ and for all $\hat{s}, \zeta = \pm 1$.

Conversely, a surface given in the above form is a surface of revolution, the profile curve is $\hat{s} \to \hat{x} = \hat{r}(\hat{s})$, $\hat{z} = \hat{h}(\hat{s})$, where $\hat{s}$ is an arcparameter.

In addition to the lemma 3.1 we have the followings result.

Lemma 3.2: a) For a surface of revolution in (ı) given in the lemma 3.1 and expressed in the form

$$\hat{X}(\hat{s}, \hat{\varphi}) = (\hat{g}(\hat{s}), \hat{r}(\hat{s})\sinh\hat{\varphi}, \hat{r}(\hat{s})\cosh\hat{\varphi}), \tag{3.6}$$

where $\hat{s}$ is a dual te arc lenght.

a1) The first and second fundemental forms are given by

$$\hat{I} = \zeta d\hat{s}^2 + \hat{r}^2 d\hat{\varphi}^2, \tag{3.7}$$

$$I\hat{I} = \zeta(\hat{g}\hat{r}' - \hat{r}\hat{g}).$$

a2) The mean curvature $\hat{H}$ satisfy

$$2\hat{H} = \zeta(\hat{g}'\hat{r}'' - \hat{r}''\hat{g}') + \frac{\hat{g}'}{\hat{r}}, \tag{3.8}$$

$$2\hat{H}' = \zeta(\hat{g}'\hat{r}''' - \hat{g}''\hat{r}'') + (\frac{\hat{g}'}{\hat{r}})'.$$

a3) The Laplacian is given by

$$\hat{\Delta} = -(\zeta[\frac{\hat{\partial}}{\partial\hat{s}^2} + \frac{\hat{r}'}{\hat{r}}\frac{\hat{\partial}}{\partial\hat{s}} + \frac{1}{\hat{r}^2}\frac{\hat{\partial}}{\partial\hat{\varphi}^2}]), \tag{3.9}$$

where $\zeta = \pm 1$.

b) A surface of revolution in D_1^3 whose axis is space-like is expressed as follows:

$$\hat{X}(\hat{s}, \hat{\theta}) = (\hat{r}(\hat{s})\cos\hat{\theta}, \hat{r}(\hat{s})\sin\hat{\theta}, \hat{h}(\hat{s})), \tag{3.10}$$

b1) The first and second fundemental forms are given by

$$\hat{I} = \zeta d\hat{s}^2 + \hat{r}d\hat{\theta}^2, \tag{3.11}$$

$$I\hat{I} = \zeta(-\hat{r}''\hat{h}' + \hat{h}''\hat{r}')d\hat{s}^2 + \hat{h}'\hat{r}d\hat{\theta}^2.$$

b2) The mean curvature $\hat{H}$ satisfy

$$2\hat{H} = \zeta(\hat{r}'\hat{h}'' - \hat{r}''\hat{h}') + \frac{\hat{h}'}{\hat{r}}, \tag{3.12}$$

$$2\hat{H}' = \zeta(\hat{r}'\hat{h}''' - \hat{h}''\hat{r}'') + (\frac{\hat{h}'}{\hat{r}})'.$$

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b3) The Laplacian is given by

$$\hat{\Delta} = -(\zeta[\frac{\hat{\partial}}{\partial\hat{s}^2} + \frac{\hat{r}'}{\hat{r}}\frac{\hat{\partial}}{\partial\hat{s}} + \frac{1}{\hat{r}^2}\frac{\hat{\partial}}{\partial\hat{\theta}^2}]), \tag{3.13}$$

where $\zeta = \pm 1$.

Proof: The proof of lemma 3.2 follows immendiately from equations (2.13), (2.14), (2.15), (2,16) and (2.17).

4. The Proof of Theorem 1.1

In order to prof the theorem we have to prove a first part, that is, we will prove that $\hat{H}$ is a constant. We will agree to prove here the complete result for types (ı) and (ıı) in lemma 3.1, for (ıı) in lemma 3.1 it is not difficult to see that things work the same.

case(ı): We consider a surface of revolution $\hat{M}'$ in this of type (a) in lemma 3.2; $\hat{M}'$ is assumed to be a connected component of the set $\hat{M} - \hat{L}$. Let's express condition $\hat{\Delta}\hat{G} = \hat{f}\hat{G}$ on $\hat{M}'$ for the Gauss map $\hat{G} = -(\hat{g}', \hat{r}'\sinh\hat{\varphi}, \hat{r}'\cosh\hat{\varphi})$.

We get from the following three vectors

$$\hat{G}_{\hat{s}} = -(\hat{g}', \hat{r}'\sinh\hat{\varphi}, \hat{r}'\cosh\hat{\varphi}), \tag{4.1}$$

$$\hat{G}_{\hat{s}\hat{s}} = -(\hat{g}', \hat{r}'\sinh\hat{\varphi}, \hat{r}'\cosh\hat{\varphi}),$$

$$\hat{G}_{\hat{\varphi}\hat{\varphi}} = -(0, \hat{r}'\sinh\hat{\varphi}, \hat{r}'\cosh\hat{\varphi}).$$

Then the Laplacian of the Gauss map by applying the formula (2.17) is the vector;

$$\hat{\Delta}\hat{G} = -[\zeta(\hat{G}_{\hat{s}\hat{s}} + \frac{\hat{r}'}{\hat{r}}\hat{G}_{\hat{s}} + \frac{1}{\hat{r}^2}\hat{G}_{\hat{\varphi}\hat{\varphi}}]). \tag{4.2}$$

So that

$$\hat{\Delta}\hat{G} = \begin{bmatrix} \zeta(\hat{g}'' + \frac{\hat{g}'}{\hat{g}}\hat{g}') \\ [\zeta(\hat{r}'' + \frac{\hat{g}'}{\hat{g}}\hat{r}') + \frac{1}{\hat{g}^2}\hat{r}'] \sinh\hat{\varphi} \\ [\zeta(\hat{r}'' + \frac{\hat{g}'}{\hat{g}}\hat{r}') + \frac{1}{\hat{g}^2}\hat{r}'] \cosh\hat{\varphi}) \end{bmatrix}. \tag{4.3}$$

By using the following functions

$$\hat{A} = \zeta(\hat{g}'' + \frac{\hat{g}'}{\hat{g}}\hat{g}'), \qquad \hat{B} = \zeta(\hat{r}'' + \frac{\hat{g}'}{\hat{g}}\hat{r}') + \frac{1}{\hat{g}^2}\hat{r}'$$

as abbreviation in (4.3) we write

$$\hat{\Delta}\hat{G} = (\zeta\hat{A}, \hat{B}\sinh\hat{\varphi}, \hat{B}\cosh\hat{\varphi})), \tag{4.4}$$

so we have

$$< \hat{\Delta}\hat{G}, \hat{G} >= -(\zeta\hat{g}'\hat{A} - \hat{r}'\hat{B}). \tag{4.5}$$

In fact, the condition $\hat{\Delta}\hat{G} = \hat{f}\hat{G}$ is equivalent to condition

$$\hat{\Delta}\hat{G} + \zeta < \hat{\Delta}\hat{G}, \hat{G} > \hat{G} = 0. \tag{4.6}$$

This condition is then equivalent to the following system of equations:

$$\begin{aligned} \zeta\hat{A} - \zeta(\zeta\hat{g}'\hat{A} - \hat{r}')(\hat{g}') &= 0, \\ [\hat{B} - \zeta(\zeta\hat{g}'\hat{A} - \hat{r}')(\hat{r}')]\sinh\hat{\varphi} &= 0, \\ [\hat{B} - \zeta(\zeta\hat{g}'\hat{A} - \hat{r}')(\hat{r}')]\sinh\hat{\varphi} &= 0. \end{aligned} \tag{4.7}$$

These are equivalent to the two followings equations

$$\begin{aligned} (1 + \zeta\hat{g}'^2)\hat{A} - \hat{B}\hat{r}'\hat{g}' &= 0, \\ \hat{g}'\hat{r}'\hat{A} + (1 - \zeta\hat{r}'^2)\hat{B} &= 0. \end{aligned} \tag{4.8}$$

Hence we obtain

$$\begin{aligned} \hat{r}'(\hat{r}'\hat{A} - \zeta\hat{B}) &= 0, \\ \hat{g}'(\hat{A}\hat{r}' - \zeta\hat{B}\hat{g}') &= 0. \end{aligned} \tag{4.9}$$

By using once again the (3.3) and its derivative, we get

$$\hat{r}'\hat{A} - \zeta\hat{g}'\hat{B} = (\hat{r}'\hat{g}'' - \hat{g}'\hat{r}'') + \zeta(\frac{\hat{g}'}{\hat{r}})'. \tag{4.10}$$

On the other hand, from the second formula in (3.8) for the derivative of mean curvature $\hat{H}$ we get

$$(\hat{r}''\hat{g}' - \hat{r}'\hat{g}'') = 2\zeta\hat{H}' - \zeta(\frac{\hat{g}'}{\hat{r}})'. \tag{4.11}$$

Thus, from (4.10) and (4.11) is given by

$$\hat{r}'\hat{A} - \zeta\hat{B}\hat{g}' = 2\zeta\hat{H}'.$$

Now the conditions (4.9) become

$$\tilde{r}\hat{H}' = 0, \qquad (4.13)$$
$$\hat{g}\hat{H}' = 0.$$

From this follows by (3.3) that $\hat{H}'$ vanishes identically on $\hat{M}'$. This proves the theorem.

Corollary 4.1: The surface of revolution with constant mean curvature in D_1^3 has screw motion.

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Department of Mathematics Faculty of Art and Science, Firat University 23119 Elaziğ, TURKEY,
E-mail address: mustafayeneroglu@gmail.com, vasil@firat.edu.tr