AMO — Advanced Modeling and Optimization, Volume 14, Number 2, 2012

CONSTRUCTION OF DUAL FOCAL CURVES IN $\mathbb{D}^3$

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Abstract. In this paper, we study dual focal curves in the Dual 3-space $\mathbb{D}^3$. We characterize dual focal curves in terms of their dual focal curvatures.

1. Introduction

The application of dual numbers to the lines of the 3-space is carried out by the principle of transference which has been formulated by Study and Kotelnikov. It allows a complete generalization of the mathematical expression for the spherical point geometry to the spatial line geometry by means of dual-number extension, i.e. replacing all ordinary quantities by the corresponding dual-number quantities.

In this paper, we study dual focal curves in the Dual 3-space $\mathbb{D}^3$. We characterize dual focal curves in terms of their dual focal curvatures.

2. Preliminaries

In the Euclidean 3-Space $\mathbb{E}^3$, lines combined with one of their two directions can be represented by unit dual vectors over the the ring of dual numbers. The important properties of real vector analysis are valid for the dual vectors. The oriented lines $\mathbb{E}^3$ are in one to one correspondence with the points of the dual unit sphere $\mathbb{D}^3$.

A dual point on $\mathbb{D}^3$ corresponds to a line in $\mathbb{E}^3$, two different points of $\mathbb{D}^3$ represents two skew lines in $\mathbb{E}^3$. A differentiable curve on $\mathbb{D}^3$ represents a ruled surface $\mathbb{E}^3$. If φ and φ^* are real numbers and $\varepsilon^2 = 0$ the combination $\hat{\varphi} = \varphi + \varepsilon\varphi^*$ is called a dual number. The symbol ε designates the dual unit with the property $\varepsilon^2 = 0$. In analogy with the complex numbers W.K. Clifford defined the dual numbers and showed that they form an algebra, not a field. Later, E.Study introduced the dual angle subtended by two nonparallel lines $\mathbb{E}^3$, and defined it as $\hat{\varphi} = \varphi + \varepsilon\varphi^*$ in which φ and φ^* are, respectively, the projected angle and the shortest distance between the two lines.

Let

Date: September 25, 2011.
2000 *Mathematics Subject Classification.* Primary 53A04; Secondary 53A10.
Key words and phrases. Frenet frame, Dual 3-space, Focal curve.

***AMO - Advanced Modeling and Optimization. ISSN: 1841-4311**

$$\begin{aligned} &\hat{\gamma} : I \subset \mathbb{R} \to \mathbb{D}^3 \\ &s \to \hat{\gamma}(s) = \gamma(s) + \varepsilon\gamma^*(s) \end{aligned}$$

be differential unit speed dual curve in dual space $\mathbb{D}^3$. Denote by $\{\hat{\mathbf{T}}, \hat{\mathbf{N}}, \hat{\mathbf{B}}\}$ the moving dual frenet frame along the dual space curve $\hat{\gamma}(s)$ in the dual space $\mathbb{D}^3$. Then $\hat{\mathbf{T}}, \hat{\mathbf{N}}$ and $\hat{\mathbf{B}}$ are the dual tangent, the dual principal normal and the dual binormal vector fields, respectively. Then for the dual curve $\hat{\alpha}$ the frenet formulae are given by,

$$\begin{aligned} \hat{\mathbf{T}}' &= \kappa\mathbf{N} + \varepsilon(\kappa^*\mathbf{N} + \kappa\mathbf{N}^*), \\ \mathbf{N}' &= -\kappa\mathbf{T} + \tau\mathbf{B} + \varepsilon\left(-\kappa^*\mathbf{T} - \kappa\mathbf{T}^* + \tau^*\mathbf{B} + \tau\mathbf{B}^*\right), \\ \hat{\mathbf{B}}' &= -\tau\mathbf{N} - \varepsilon\left(\tau^*\mathbf{N} + \tau\mathbf{N}^*\right), \end{aligned}$$

where $\hat{\kappa} = \kappa + \varepsilon\kappa^*$ is nowhere pure dual natural curvatures and $\hat{\tau} = \tau + \varepsilon\tau^*$ is nowhere pure dual torsion.

3. Dual Focal Curves According To Dual Frenet Frame In $\mathbb{D}^3$

Denoting the dual focal curve by $\hat{\wp}$ we can write

$$\hat{\wp}(s) = (\hat{\gamma} + \hat{\mathfrak{q}}_1\hat{\mathbf{N}} + \hat{\mathfrak{q}}_2\hat{\mathbf{B}})(s), \tag{3.1}$$

where the coefficients $\hat{\mathfrak{q}}_1$, $\hat{\mathfrak{q}}_2$ are smooth functions of the parameter of the curve $\hat{\gamma}$, called the first and second dual focal curvatures of $\hat{\gamma}$, respectively.

The formula (3.1) is separed into the real and dual part, we have

$$\begin{aligned} \wp(s) &= (\gamma + \mathfrak{q}_1\mathbf{N} + \mathfrak{q}_2\mathbf{B})(s), \\ \wp^*(s) &= \left(\gamma^* + \mathfrak{q}_1\mathbf{N}^* + \mathfrak{q}_1^*\mathbf{N} + \mathfrak{q}_2\mathbf{B}^* + \mathfrak{q}_2^*\mathbf{B}\right)(s). \end{aligned} \tag{3.2}$$

Theorem 3.1. *Let $\hat{\gamma} : I \longrightarrow \mathbb{D}^3$ be a unit speed dual curve and $\hat{\wp}$ its dual focal curve on $\mathbb{D}^3$. Then,*

$$\wp = \gamma + \frac{1}{\kappa}\mathbf{N} - \frac{\kappa'}{\kappa^2\tau}\mathbf{B}, \tag{3.3}$$

$$\begin{aligned} \wp^* = \gamma^* + \frac{1}{\kappa}\mathbf{N}^* - \frac{\kappa^*}{\kappa^2}\mathbf{N} - \frac{\kappa'}{\kappa^{\mathbf{2}}\tau}\mathbf{B}^* + \\ \left(\frac{-(\kappa^*)'\kappa^2 + 2\kappa\kappa^*\kappa'}{\kappa^4\tau} + \frac{\tau^*\kappa'}{\kappa^2\tau^2}\right)\mathbf{B}. \end{aligned} \tag{3.4}$$

Proof. Assume that $\hat{\gamma}$ is a unit speed dual curve and $\hat{\wp}$ its dual focal curve on $\mathbb{D}^3$.

So, by differentiating of the formula (3.1), we get

$$\hat{\wp}(s)' = (1 - \hat{\kappa}\hat{\mathfrak{q}}_1)\hat{\mathbf{T}} + (\hat{\mathfrak{q}}_1' - \hat{\tau}\hat{\mathfrak{q}}_2)\hat{\mathbf{N}} + (\hat{\tau}\hat{\mathfrak{q}}_1 + \hat{\mathfrak{q}}_2')\hat{\mathbf{B}}. \tag{3.5}$$

Using above equation, the first 2 components vanish, we have

$$1 - \hat{\kappa}\hat{\mathfrak{q}}_1 = 0, \tag{3.6}$$

$$\hat{\mathfrak{q}}_1' - \hat{\tau}\hat{\mathfrak{q}}_2 = 0. \tag{3.7}$$

Equations (3.6) and (3.7) are separed into the real and dual part, we have

$$\begin{aligned}
\kappa\mathfrak{q}_1 &= 1,\\
\kappa\mathfrak{q}_1^* + \kappa^*\mathfrak{q}_1 &= 0,\\
\mathfrak{q}_1' - \tau\mathfrak{q}_2 &= 0,\\
(\mathfrak{q}_1^*)' - \tau\mathfrak{q}_2^* - \tau^*\mathfrak{q}_2 &= 0.
\end{aligned}$$

Since,

$$\begin{aligned}
\mathfrak{q}_1 &= \frac{1}{\kappa},\\
\mathfrak{q}_1^* &= -\frac{\kappa^*}{\kappa^2},\\
\mathfrak{q}_2 &= -\frac{\kappa'}{\kappa^2\tau},\\
\mathfrak{q}_2^* &= \frac{-(\kappa^*)'\kappa^2 + 2\kappa\kappa^*\kappa'}{\kappa^4\tau} + \frac{\tau^*\kappa'}{\kappa^2\tau^2}.
\end{aligned}$$

From above system, we express (3.3) and (3.4). This completes the proof.

Corollary 3.2. *Let $\hat{\gamma} : I \longrightarrow \mathbb{D}^3$ be a unit speed dual curve and $\hat{\wp}$ its dual focal curve on $\mathbb{D}^3$. Then, real and dual part of dual focal curvatures of $\hat{\wp}$ are*

$$\begin{aligned}
\mathfrak{q}_1 &= \frac{1}{\kappa},\\
\mathfrak{q}_1^* &= -\frac{\kappa^*}{\kappa^2},\\
\mathfrak{q}_2 &= -\frac{\kappa'}{\kappa^2\tau},\\
\mathfrak{q}_2^* &= \frac{-(\kappa^*)'\kappa^2 + 2\kappa\kappa^*\kappa'}{\kappa^4\tau} + \frac{\tau^*\kappa'}{\kappa^2\tau^2}.
\end{aligned}$$

In the light of Theorem 3.1 and Corollary 3.2, we express the following corollary without proof:

Corollary 3.3. *Let $\hat{\gamma} : I \longrightarrow \mathbb{D}^3$ be a unit speed dual curve and $\hat{\wp}$ its dual focal curve on $\mathbb{D}^3$. If $\hat{\kappa}$ and $\hat{\tau}$ are constant then, treal and dual part of dual focal curvatures of $\hat{\wp}$ are*

$$\begin{aligned}
\mathfrak{q}_1 &= \textit{constant} \neq 0,\\
\mathfrak{q}_1^* &= \textit{constant} \neq 0,\\
\mathfrak{q}_2 &= 0,\\
\mathfrak{q}_2^* &= 0.
\end{aligned}$$

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