POSITION VECTORS OF SMARANDACHE $D$–$Tg$ CURVES OF TIMELIKE BIHARMONIC $D$–HELICES ACCORDING TO DARBOUX FRAME ON NON-DEGENERATE TIMELIKE SURFACES IN THE LORENTZIAN HEISENBERG GROUP $\mathbb{H}$

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Abstract. In this paper, we study Smarandache $D$–$Tg$ curves of timelike biharmonic $D$-helices according to Darboux frame on non-degenerate timelike surfaces in the Lorentzian Heisenberg group $\mathbb{H}$. We obtain parametric equation Smarandache $D$–$Tg$ curves of timelike biharmonic $D$-helices in the Lorentzian Heisenberg group $\mathbb{H}$. Moreover, we illustrate the figure of our main theorem.

1. Introduction

Harmonic maps $f : (M, g) \rightarrow (N, h)$ between Riemannian manifolds are the critical points of the energy

$$E(f) = \frac{1}{2} \int_M |df|^2 v_g,$$

and they are therefore the solutions of the corresponding Euler–Lagrange equation. This equation is given by the vanishing of the tension field

$$\tau(f) = \text{trace} \nabla df.$$

As suggested by Eells and Sampson in [3], we can define the bienergy of a map $f$ by

$$E_2(f) = \frac{1}{2} \int_M |\tau(f)|^2 v_g,$$

and say that is biharmonic if it is a critical point of the bienergy.

Jiang derived the first and the second variation formula for the bienergy in [7], showing that the Euler–Lagrange equation associated to $E_2$ is

$$\tau_2(f) = -\mathcal{J}^f (\tau(f)) = -\Delta \tau(f) - \text{trace} R^{N} (df, \tau(f)) df = 0,$$

where $\mathcal{J}^f$ is the Jacobi operator of $f$. The equation $\tau_2(f) = 0$ is called the biharmonic equation. Since $\mathcal{J}^f$ is linear, any harmonic map is biharmonic. Therefore, we are interested in proper biharmonic maps, that is non-harmonic biharmonic maps.

In this paper, we study Smarandache $D$–$Tg$ curves of timelike biharmonic $D$-helices according to Darboux frame on non-degenerate timelike surfaces in the

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Lorentzian Heisenberg group $\mathbb{H}$. We obtain parametric equation Smarandache $D-T_g$ curves of timelike biharmonic $D$-helices in the Lorentzian Heisenberg group $\mathbb{H}$. Moreover, we illustrate the figure of our main theorem.

2. Lorentzian Heisenberg Group $\mathbb{H}$

Heisenberg group plays an important role in many branches of mathematics such as representation theory, harmonic analysis, PDEs or even quantum mechanics, where it was initially defined as a group of $3 \times 3$ matrices

$$\left\{ \begin{array}{ccc} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{array} \right\} : x, y, z \in \mathbb{R}$$

with the usual multiplication rule.

We will use the following complex definition of the Heisenberg group $\mathbb{H}$.

$$\mathbb{H} = \mathbb{C} \times \mathbb{R} = \{(w, z) : w \in \mathbb{C}, z \in \mathbb{R}\}$$

with

$$(w, z) \ast (\overline{w}, \overline{z}) = (w + \overline{w}, z + \overline{z} + \text{Im}((w, \overline{w})))$$

where $\langle, \rangle$ is the usual Hermitian product in $\mathbb{C}$.

The identity of the group is $(0,0,0)$ and the inverse of $(x,y,z)$ is given by $(-x,-y,-z)$.

Let $a = (w_1, z_1)$, $b = (w_2, z_2)$ and $c = (w_3, z_3)$. The commutator of the elements $a, b \in \mathbb{H}$ is equal to

$$[a, b] = a \ast b \ast a^{-1} \ast b^{-1}$$

$$= (w_1, z_1) \ast (w_2, z_2) \ast (-w_1, -z_1) \ast (-w_2, -z_2)$$

$$= (w_1 + w_2 - w_1 - w_2, z_1 + z_2 - z_1 - z_2)$$

$$= (0, \alpha),$$

where $\alpha \neq 0$ in general. For example

$$[(1,0), (i,0)] = (0,2) \neq (0,0).$$

Which shows that $\mathbb{H}$ is not abelian.

On the other hand, for any $a, b, c \in \mathbb{H}$, their double commutator is

$$[[a, b], c] = [(0, \alpha), (w_3, z_3)]$$

$$= (0,0).$$

This implies that $\mathbb{H}$ is a nilpotent Lie group with nilpotency 2.

The left-invariant Lorentz metric on $\mathbb{H}$ is

$$\rho = -dx^2 + dy^2 + (xdy + dz)^2.$$
The characterising properties of this algebra are the following commutation relations:

\[(2.2) \quad \rho(e_1, e_1) = \rho(e_2, e_2) = 1, \quad \rho(e_3, e_3) = -1.\]

3. Timelike Biharmonic $D$-Helices According to Darboux Frame on a Non-Degenerate Timelike Surface in the Lorentzian Heisenberg Group $\mathbb{H}$

Let $\Pi \subset \mathbb{H}$ be a timelike surface with the unit normal vector $n$ in the Lorentzian Heisenberg group $\mathbb{H}$. If $\gamma$ is a timelike curve on $\Pi \subset \mathbb{H}$, then we have the Frenet frame $\{T, N, B\}$ and Darboux frame $\{T, n, g\}$ with spacelike vector $g = T \wedge n$ along the curve $\gamma$. Let $\{T, N, B\}$ be the Frenet frame fields tangent to the Lorentzian Heisenberg group $\mathbb{H}$ along $\gamma$ defined as follows:

$T$ is the unit vector field $\gamma'$ tangent to $\gamma$, $N$ is the unit vector field in the direction of $\nabla_T T$ (normal to $\gamma$) and $B$ is chosen so that $\{T, N, B\}$ is a positively oriented orthonormal basis. Then, we have the following Frenet formulas:

\[
\begin{align*}
\nabla_T T &= \kappa N,
\n\nabla_T N &= \kappa T + \tau B,
\n\nabla_T B &= -\tau N,
\end{align*}
\]

where $\kappa$ is the curvature of $\gamma$ and $\tau$ is its torsion and

\[(3.2) \quad \rho(T, T) = -1, \quad \rho(N, N) = 1, \quad \rho(B, B) = 1, \quad \rho(T, N) = \rho(T, B) = \rho(N, B) = 0.\]

Let $\theta$ be the angle between $N$ and $n$. The relationships between $\{T, N, B\}$ and $\{T, n, g\}$ are as follows:

\[
\begin{align*}
T &= T,
N &= \cos \theta n + \sin \theta g,
B &= \sin \theta n - \cos \theta g,
\end{align*}
\]

and

\[
\begin{align*}
T &= T,

\nabla_T g &= \sin \theta N - \cos \theta B,
n &= \cos \theta N + \sin \theta B.
\end{align*}
\]

By differentiating (3.4), using (3.1), (3.3) and Frenet formulas we obtain

\[
\begin{align*}
\nabla_T T &= (\kappa \cos \theta) n + (\kappa \sin \theta) g,
\nabla_T g &= (-\kappa \sin \theta) T + \left( \tau + \frac{d\theta}{ds} \right) n,
\nabla_T n &= (-\kappa \cos \theta) T - \left( \tau + \frac{d\theta}{ds} \right) g.
\end{align*}
\]
If we represent $\kappa \cos \theta$, $\kappa \sin \theta$ and $\tau + \frac{d\theta}{ds}$ with the symbols $\kappa_n$, $\kappa_g$, and $\tau_g$ respectively, then the equations in (3.5) can be written as

$$
\begin{align*}
\nabla_T T &= \kappa_g g + \kappa_n n, \\
\nabla_T g &= -\kappa_g T + \tau_g n, \\
\nabla_T n &= -\kappa_n T - \tau_g g.
\end{align*}
$$

With respect to the orthonormal basis $\{e_1, e_2, e_3\}$, we can write

$$
\begin{align*}
T &= T_1 e_1 + T_2 e_2 + T_3 e_3, \\
g &= g_1 e_1 + g_2 e_2 + g_3 e_3, \\
n &= n_1 e_1 + n_2 e_2 + n_3 e_3,
\end{align*}
$$

To separate a curve according to Darboux frame from that of Frenet-Serret frame, in the rest of the paper, we shall use notation for the curve as $D$-curve.

First of all we recall the following well known result (cf. [8]).

**Theorem 3.1.** Let $\gamma : I \to \Pi \subset \mathbb{H}$ be a non-geodesic unit speed timelike curve on timelike surface $\Pi$ in the Lorentzian Heisenberg group $\mathbb{H}$. $\gamma$ is a unit speed timelike biharmonic curve on $\Pi$ if and only if

$$
\begin{align*}
\kappa_n^2 + \kappa_g^2 &= \text{constant} \neq 0, \\
\kappa_n'' - \kappa_n^3 + \kappa_n \tau_g - \kappa_g^2 \kappa_n + \kappa_g \tau_g + \kappa_n \tau_g' - \tau_g^2 \kappa_n &= \kappa_n(1 - 4\kappa_g^2) - 4\kappa_g n_1 g_1, \\
\kappa_g'' - \kappa_g^3 - 2\kappa_n \tau_g - \kappa_n \kappa_g - \kappa_n \tau_g' - \kappa_g \tau_g^2 &= 4\kappa_n n_1 g_1 + \kappa_g(1 - 4\kappa_n^2).
\end{align*}
$$

**Theorem 3.2.** Let $\gamma : I \to \Pi \subset \mathbb{H}$ be a non-geodesic unit speed timelike biharmonic $D$-helix on timelike surface $\Pi$ in the Lorentzian Heisenberg group $\mathbb{H}$. Then parametric equations of timelike biharmonic $D$-helix are

$$
\begin{align*}
x(s) &= \cosh P \left( \cosh[Ms] \sinh[N] + \cosh[N] \sinh[Ms] \right) + P_1, \\
y(s) &= \frac{\cosh P}{M} \left( \cosh[N] \cosh[Ms] + \sinh[N] \sinh[Ms] \right) + P_2, \\
z(s) &= \sinh P s + \frac{(N + Ms)}{2M^2} \cosh^2 P - \frac{P_1}{M} \cosh^2 P \cosh[N] \cosh[Ms] \\
&- \frac{P_1}{M} \cosh^2 P \sinh[N] \sinh[Ms] - \frac{1}{4M^2} \cosh^2 P \sinh[2Ms + N] + P_3,
\end{align*}
$$

where $P_1, P_2, P_3$ are constants of integration and

$$
M = \text{sech} P \sqrt{\kappa_n^2 + \kappa_g^2} - 2 \sinh P.
$$

Using Mathematica in above system we have Fig.1:
Corollary 3.3. Let \( \gamma : I \rightarrow \Pi \subset \mathbb{H} \) be a non-geodesic unit speed timelike biharmonic D- curve on timelike surface \( \Pi \) in the Lorentzian Heisenberg group \( \mathbb{H} \). Then,

\[ \kappa = \text{constant} \neq 0. \]

4. Smarandache D–Tg Curves According to Darboux Frame on a Non-Degenerate Timelike Surface in the Lorentzian Heisenberg Group \( \mathbb{H} \)

Definition 4.1. Let \( \gamma : I \rightarrow \Pi \subset \mathbb{H} \) be a non-geodesic unit speed timelike biharmonic D- curve on timelike surface \( \Pi \) in the Lorentzian Heisenberg group \( \mathbb{H} \) and \( \{T, n, g\} \) be its moving Darboux frame. Smarandache D–Tg curves are defined by

\[ \tilde{\gamma} = \frac{1}{\kappa_n + \tau_g} (T + g). \]

Theorem 4.2. Let \( \gamma : I \rightarrow \Pi \subset \mathbb{H} \) be a non-geodesic unit speed timelike biharmonic D- curve on timelike surface \( \Pi \) in the Lorentzian Heisenberg group \( \mathbb{H} \) and \( \tilde{\gamma} \) its Smarandache D–Tg curve. Then position vector of Smarandache D–Tg
curve is given by

\[
\hat{\gamma} = \left[ \sinh \mathcal{P} \right] \frac{\cos^2 \theta}{\kappa_n + \gamma_g} \left[ \cosh \mathcal{P} \sinh \mathcal{M} + \mathcal{N} \right] \cosh \mathcal{P} \sinh \left[ \mathcal{M} \sinh \mathcal{P} \right] \left( \frac{1}{\mathcal{M}} + 2 \sinh \mathcal{P} \right) \\
- \cosh \mathcal{P} \cosh \left[ \mathcal{M} \sinh \mathcal{P} \right] \cosh \mathcal{P} \cosh \left[ \mathcal{M} \sinh \mathcal{P} \right] \left( \frac{1}{\mathcal{M}} + 2 \sinh \mathcal{P} \right) | e_1 \\
+ \left[ \frac{\cosh \mathcal{P} \sinh \left[ \mathcal{M} \sinh \mathcal{P} \right]}{\kappa_n + \gamma_g} \right] + \frac{1}{\kappa_n + \gamma_g} \left[ \frac{\cos \theta \sin \theta}{\sinh \mathcal{P} \sinh \mathcal{M} + \mathcal{N}} \cosh \mathcal{P} \cosh \mathcal{P} \cosh \left[ \mathcal{M} \sinh \mathcal{P} \right] \left( \frac{1}{\mathcal{M}} + 2 \sinh \mathcal{P} \right) \\
+ \frac{\cos^2 \theta}{\kappa_n} \sinh \mathcal{P} \cosh \left[ \mathcal{M} \sinh \mathcal{P} \right] \left( \frac{1}{\mathcal{M}} + 2 \sinh \mathcal{P} \right) | e_2 \\
+ \left[ \frac{\cosh \mathcal{P} \cosh \left[ \mathcal{M} \sinh \mathcal{P} \right]}{\kappa_n + \gamma_g} \right] + \frac{1}{\kappa_n + \gamma_g} \left[ \frac{\cos \theta \sin \theta}{\sinh \mathcal{P} \sinh \mathcal{M} + \mathcal{N}} \cosh \mathcal{P} \cosh \mathcal{P} \cosh \left[ \mathcal{M} \sinh \mathcal{P} \right] \left( \frac{1}{\mathcal{M}} + 2 \sinh \mathcal{P} \right) \\
- \frac{\cos^2 \theta}{\kappa_n} \sinh \mathcal{P} \cosh \left[ \mathcal{M} \sinh \mathcal{P} \right] \left( \frac{1}{\mathcal{M}} + 2 \sinh \mathcal{P} \right) | e_3.
\]

where \( \mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3 \) are constants of integration and

\[\mathcal{M} = \text{sech} \mathcal{P} \sqrt{\kappa_n^2 + \gamma_g^2} - 2 \sinh \mathcal{P}.\]

**Proof.** From the assumption we get

\[\mathbf{T} = \sinh \mathcal{P} e_1 + \cosh \mathcal{P} \sinh \left[ \mathcal{M} \sinh \mathcal{P} \right] e_2 + \cosh \mathcal{P} \cosh \left[ \mathcal{M} \sinh \mathcal{P} \right] e_3.\]

The covariant derivative of the vector field \( \mathbf{T} \) is:

\[\nabla_{\mathbf{T}} \mathbf{T} = \cosh \mathcal{P} \cosh \left[ \mathcal{M} \sinh \mathcal{P} \right] \left( \frac{1}{\mathcal{M}} + 2 \sinh \mathcal{P} \right) e_2 \\
+ \cosh \mathcal{P} \cosh \left[ \mathcal{M} \sinh \mathcal{P} \right] \left( \frac{1}{\mathcal{M}} + 2 \sinh \mathcal{P} \right) e_3.\]

From the above equation, it is seen that

\[\mathbf{g} = -\frac{\cos^2 \theta}{\kappa_n} \left[ \cosh \mathcal{P} \sinh \mathcal{M} \sinh \mathcal{P} \right] \cosh \mathcal{P} \sinh \left[ \mathcal{M} \sinh \mathcal{P} \right] \left( \frac{1}{\mathcal{M}} + 2 \sinh \mathcal{P} \right) \\
- \cosh \mathcal{P} \cosh \left[ \mathcal{M} \sinh \mathcal{P} \right] \cosh \mathcal{P} \cosh \left[ \mathcal{M} \sinh \mathcal{P} \right] \left( \frac{1}{\mathcal{M}} + 2 \sinh \mathcal{P} \right) | e_1 \\
+ \frac{\cos \theta \sin \theta}{\kappa_n} \cosh \mathcal{P} \cosh \left[ \mathcal{M} \sinh \mathcal{P} \right] \left( \frac{1}{\mathcal{M}} + 2 \sinh \mathcal{P} \right) | e_2 \\
+ \frac{\cos^2 \theta}{\kappa_n} \sinh \mathcal{P} \cosh \left[ \mathcal{M} \sinh \mathcal{P} \right] \left( \frac{1}{\mathcal{M}} + 2 \sinh \mathcal{P} \right) | e_3.\]

Substituting Eq.(4.4) and Eq.(4.3) into Eq.(4.1), we obtain Eq.(4.2). This completes the proof.

**Theorem 4.3.** Let \( \gamma : I \to \Pi \subset \mathbb{H} \) be a non-geodesic unit speed timelike biharmonic \( \mathcal{D} \)-curve on timelike surface \( \Pi \) in the Lorentzian Heisenberg group \( \mathbb{H} \).
and \( \gamma \) its Smarandache \( \mathcal{D}-\mathcal{T}_g \) curve. Then parametric equations of Smarandache \( \mathcal{D}-\mathcal{T}_g \) curve are given by

\[
\begin{align*}
    x_\gamma(s) &= \frac{\cosh P \cosh [Ms + N]}{\kappa_n + \tau_g} + \frac{1}{\kappa_n + \tau_g} \left[ \frac{\cos \theta \sin \theta}{\kappa_n} \cosh P \sinh [Ms + N] \left( \frac{1}{\mathcal{M}} + 2 \sinh P \right) 
    \right. \\
    &\quad \left. - \frac{\cos^2 \theta}{\kappa_n} \sinh P \cosh [Ms + N] \left( \frac{1}{\mathcal{M}} + 2 \sinh P \right) \right], \\
    y_\gamma(s) &= \frac{\cosh P \sinh [Ms + N]}{\kappa_n + \tau_g} + \frac{1}{\kappa_n + \tau_g} \left[ \frac{\cos \theta \sin \theta}{\kappa_n} \cosh P \cosh [Ms + N] \left( \frac{1}{\mathcal{M}} + 2 \sinh P \right) 
    \right. \\
    &\quad \left. + \frac{\cos^2 \theta}{\kappa_n} \sinh P \cosh [Ms + N] \left( \frac{1}{\mathcal{M}} + 2 \sinh P \right) \right], \\
    z_\gamma(s) &= \frac{\sinh P}{\kappa_n + \tau_g} \left[ \cosh P \sinh [Ms + N] \cosh P \sinh [Ms + N] \left( \frac{1}{\mathcal{M}} + 2 \sinh P \right) 
    \right. \\
    &\quad \left. - \cosh P \cosh [Ms + N] \cosh P \cosh [Ms + N] \left( \frac{1}{\mathcal{M}} + 2 \sinh P \right) \right] \\
    &\quad - \frac{\cos^2 \theta}{\kappa_n} \sinh P \cosh [Ms + N] \left( \frac{1}{\mathcal{M}} + 2 \sinh P \right),
\end{align*}
\]

(4.6)

where \( N \) is constant of integration and

\[
\mathcal{M} = \text{sech} P \sqrt{\kappa_n^2 + \kappa_g^2} - 2 \sinh P.
\]

**Proof.** Omitted.

**References**


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