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POSITION VECTORS OF SMARANDACHE $\mathcal{D}-$Tg CURVES OF TIMELIKE BIHARMONIC $\mathcal{D}-$HELICES ACCORDING TO DARBOUX FRAME ON NON-DEGENERATE TIMELIKE SURFACES IN THE LORENTZIAN HEISENBERG GROUP $\mathbb{H}$

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Abstract. In this paper, we study Smarandache $\mathcal{D}-\mathbf{Tg}$ curves of timelike biharmonic $\mathcal{D}$-helices according to Darboux frame on non-degenerate timelike surfaces in the Lorentzian Heisenberg group $\mathbb{H}$. We obtain parametric equation Smarandache $\mathcal{D}-\mathbf{Tg}$ curves of timelike biharmonic $\mathcal{D}$-helices in the Lorentzian Heisenberg group $\mathbb{H}$. Moreover, we illustrate the figure of our main theorem.

1. Introduction

Harmonic maps $f : (M, g) \longrightarrow (N, h)$ between Riemannian manifolds are the critical points of the energy

$$E(f) = \frac{1}{2}\int_M |df|^2 v_g, \tag{1.1}$$

and they are therefore the solutions of the corresponding Euler–Lagrange equation. This equation is given by the vanishing of the tension field

$$\tau(f) = \mathrm{trace}\nabla df. \tag{1.2}$$

As suggested by Eells and Sampson in [3], we can define the bienergy of a map f by

$$E_2(f) = \frac{1}{2}\int_M |\tau(f)|^2 v_g, \tag{1.3}$$

and say that is biharmonic if it is a critical point of the bienergy.

Jiang derived the first and the second variation formula for the bienergy in [7], showing that the Euler–Lagrange equation associated to E_2 is

$$\begin{aligned} \tau_2(f) &= -\mathcal{J}^f(\tau(f)) = -\Delta\tau(f) - \mathrm{trace}R^N(df, \tau(f))\,df \\ &= 0, \end{aligned} \tag{1.4}$$

where $\mathcal{J}^f$ is the Jacobi operator of f . The equation $\tau_2(f) = 0$ is called the biharmonic equation. Since $\mathcal{J}^f$ is linear, any harmonic map is biharmonic. Therefore, we are interested in proper biharmonic maps, that is non-harmonic biharmonic maps.

In this paper, we study Smarandache $\mathcal{D}-\mathbf{Tg}$ curves of timelike biharmonic $\mathcal{D}$-helices according to Darboux frame on non-degenerate timelike surfaces in the

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Lorentzian Heisenberg group $\mathbb{H}$. We obtain parametric equation Smarandache $\mathcal{D}-\mathbf{Tg}$ curves of timelike biharmonic $\mathcal{D}$-helices in the Lorentzian Heisenberg group $\mathbb{H}$. Moreover, we illustrate the figure of our main theorem.

2. Lorentzian Heisenberg Group $\mathbb{H}$

Heisenberg group plays an important role in many branches of mathematics such as representation theory, harmonic analysis, PDEs or even quantum mechanics, where it was initially defined as a group of 3×3 matrices

$$\left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : \ x, y, z \in \mathbb{R} \right\}$$

with the usual multiplication rule.

We will use the following complex definition of the Heisenberg group $\mathbb{H}$.

$$\mathbb{H} = \mathbb{C} \times \mathbb{R} = \{(w, z) : w \in \mathbb{C}, z \in \mathbb{R}\}$$

with

$$(w, z) * (\overline{w}, \overline{z}) = (w + \overline{w}, z + \overline{z} + \operatorname{Im}(\langle w, \overline{w} \rangle)),$$

where $\langle , \rangle$ is the usual Hermitian product in $\mathbb{C}$.

The identity of the group is $(0, 0, 0)$ and the inverse of (x, y, z) is given by $(-x, -y, -z)$.

Let $a = (w_1, z_1)$, $b = (w_2, z_2)$ and $c = (w_3, z_3)$. The commutator of the elements $a, b \in \mathbb{H}$ is equal to

$$\begin{aligned} [a, b] &= a * b * a^{-1} * b^{-1} \\ &= (w_1, z_1) * (w_2, z_2) * (-w_1, -z_1) * (-w_2, -z_2) \\ &= (w_1 + w_2 - w_1 - w_2, z_1 + z_2 - z_1 - z_2) \\ &= (0, \alpha), \end{aligned}$$

where $\alpha \neq 0$ in general. For example

$$[(1, 0), (i, 0)] = (0, 2) \neq (0, 0).$$

Which shows that $\mathbb{H}$ is not abelian.

On the other hand, for any $a, b, c \in \mathbb{H}$, their double commutator is

$$\begin{aligned} [[a, b], c] &= [(0, \alpha), (w_3, z_3)] \\ &= (0, 0). \end{aligned}$$

This implies that $\mathbb{H}$ is a nilpotent Lie group with nilpotency 2.

The left-invariant Lorentz metric on $\mathbb{H}$ is

$$\rho = -dx^2 + dy^2 + (xdy + dz)^2.$$

The following set of left-invariant vector fields forms an orthonormal basis for the corresponding Lie algebra:

$$\left\{ \mathbf{e}_1 = \frac{\partial}{\partial z}, \ \mathbf{e}_2 = \frac{\partial}{\partial y} - x\frac{\partial}{\partial z}, \ \mathbf{e}_3 = \frac{\partial}{\partial x} \right\}. \tag{2.1}$$

The characterising properties of this algebra are the following commutation relations:

$$\rho(\mathbf{e}_1,\mathbf{e}_1)=\rho(\mathbf{e}_2,\mathbf{e}_2)=1,\quad \rho(\mathbf{e}_3,\mathbf{e}_3)=-1. \tag{2.2}$$

3. Timelike Biharmonic $\mathcal{D}$-Helices According to Darboux Frame on a Non-Degenerate Timelike Surface in the Lorentzian Heisenberg Group $\mathbb{H}$

Let $\Pi\subset\mathbb{H}$ be a timelike surface with the unit normal vector $\mathbf{n}$ in the Lorentzian Heisenberg group $\mathbb{H}$. If γ is a timelike curve on $\Pi\subset\mathbb{H}$, then we have the Frenet frame $\{\mathbf{T},\mathbf{N},\mathbf{B}\}$ and Darboux frame $\{\mathbf{T},\mathbf{n},\mathbf{g}\}$ with spacelike vector $\mathbf{g}=\mathbf{T}\wedge\mathbf{n}$ along the curve γ. Let $\{\mathbf{T},\mathbf{N},\mathbf{B}\}$ be the Frenet frame fields tangent to the Lorentzian Heisenberg group $\mathbb{H}$ along γ defined as follows:

$\mathbf{T}$ is the unit vector field γ' tangent to γ, $\mathbf{N}$ is the unit vector field in the direction of $\nabla_{\mathbf{T}}\mathbf{T}$ (normal to γ) and $\mathbf{B}$ is chosen so that $\{\mathbf{T},\mathbf{N},\mathbf{B}\}$ is a positively oriented orthonormal basis. Then, we have the following Frenet formulas:

$$\begin{aligned}\nabla_{\mathbf{T}}\mathbf{T}&=\kappa\mathbf{N},\\ \nabla_{\mathbf{T}}\mathbf{N}&=\kappa\mathbf{T}+\tau\mathbf{B},\\ \nabla_{\mathbf{T}}\mathbf{B}&=-\tau\mathbf{N},\end{aligned} \tag{3.1}$$

where κ is the curvature of γ and τ is its torsion and

$$\begin{aligned}&\rho(\mathbf{T},\mathbf{T})=-1,\ \rho(\mathbf{N},\mathbf{N})=1,\ \rho(\mathbf{B},\mathbf{B})=1,\\ &\rho(\mathbf{T},\mathbf{N})=\rho(\mathbf{T},\mathbf{B})=\rho(\mathbf{N},\mathbf{B})=0.\end{aligned} \tag{3.2}$$

Let θ be the angle between $\mathbf{N}$ and $\mathbf{n}$. The relationships between $\{\mathbf{T},\mathbf{N},\mathbf{B}\}$ and $\{\mathbf{T},\mathbf{n},\mathbf{g}\}$ are as follows:

$$\begin{aligned}&\mathbf{T}{=}\mathbf{T},\\ &\mathbf{N}=\cos\theta\mathbf{n}+\sin\theta\mathbf{g},\\ &\mathbf{B}=\sin\theta\mathbf{n}-\cos\theta\mathbf{g},\end{aligned} \tag{3.3}$$

and

$$\begin{aligned}&\mathbf{T}{=}\mathbf{T},\\ &\mathbf{g}=\sin\theta\mathbf{N}-\cos\theta\mathbf{B},\\ &\mathbf{n}=\cos\theta\mathbf{N}+\sin\theta\mathbf{B}.\end{aligned} \tag{3.4}$$

By differentiating (3.4), using (3.1), (3.3) and Frenet formulas we obtain

$$\begin{aligned}\nabla_{\mathbf{T}}\mathbf{T}&=(\kappa\cos\theta)\,\mathbf{n}+(\kappa\sin\theta)\,\mathbf{g},\\ \nabla_{\mathbf{T}}\mathbf{g}&=(-\kappa\sin\theta)\,\mathbf{T}+\left(\tau+\frac{d\theta}{ds}\right)\mathbf{n},\\ \nabla_{\mathbf{T}}\mathbf{n}&=(-\kappa\cos\theta)\,\mathbf{T}-\left(\tau+\frac{d\theta}{ds}\right)\mathbf{g}.\end{aligned} \tag{3.5}$$

If we represent $\kappa\cos\theta$, $\kappa\sin\theta$ and $\tau+\frac{d\theta}{ds}$ with the symbols κ_n, κ_g, and τ_g respectively, then the equations in (3.5) can be written as

$$\begin{aligned}\nabla_{\mathbf{T}}\mathbf{T} &= \kappa_g\mathbf{g}+\kappa_n\mathbf{n},\\ \nabla_{\mathbf{T}}\mathbf{g} &= -\kappa_g\mathbf{T}+\tau_g\mathbf{n},\\ \nabla_{\mathbf{T}}\mathbf{n} &= -\kappa_n\mathbf{T}-\tau_g\mathbf{g}.\end{aligned} \tag{3.6}$$

With respect to the orthonormal basis $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$, we can write

$$\begin{aligned}\mathbf{T} &= T_1\mathbf{e}_1+T_2\mathbf{e}_2+T_3\mathbf{e}_3,\\ \mathbf{g} &= g_1\mathbf{e}_1+g_2\mathbf{e}_2+g_3\mathbf{e}_3.\\ \mathbf{n} &= n_1\mathbf{e}_1+n_2\mathbf{e}_2+n_3\mathbf{e}_3,\end{aligned} \tag{3.7}$$

To separate a curve according to Darboux frame from that of Frenet- Serret frame, in the rest of the paper, we shall use notation for the curve as $\mathcal{D}$-curve.

First of all we recall the following well known result (cf. [8]).

Theorem 3.1. *Let $\gamma : I \longrightarrow \Pi\subset\mathbb{H}$ be a non-geodesic unit speed timelike curve on timelike surface Π in the Lorentzian Heisenberg group $\mathbb{H}$. γ is a unit speed timelike biharmonic curve on Π if and only if*

$$\begin{aligned}&\kappa_n^2+\kappa_g^2 = constant \neq 0,\\ &\kappa_n''-\kappa_n^3+\kappa_g\tau_g-\kappa_g^2\kappa_n+\kappa_g'\tau_g+\kappa_g\tau_g'-\tau_g^2\kappa_n = \kappa_n(1-4g_1^2)-4\kappa_g n_1 g_1,\\ &\kappa_g''-\kappa_g^3-2\kappa_n'\tau_g-\kappa_n^2\kappa_g-\kappa_n\tau_g'-\kappa_g\tau_g^2 = 4\kappa_n n_1 g_1+\kappa_g(1-4n_1^2).\end{aligned} \tag{3.8}$$

Theorem 3.2. *Let $\gamma : I \longrightarrow \Pi\subset\mathbb{H}$ be a non-geodesic unit speed timelike biharmonic $\mathcal{D}$- helix on timelike surface Π in the Lorentzian Heisenberg group $\mathbb{H}$. Then parametric equations of timelike biharmonic $\mathcal{D}$- helix are*

$$\begin{aligned}\boldsymbol{x}(s) &= \frac{\cosh\mathcal{P}}{\mathcal{M}}(\cosh[\mathcal{M}s]\sinh[\mathcal{N}]+\cosh[\mathcal{N}]\sinh[\mathcal{M}s])+\mathcal{P}_1,\\ \boldsymbol{y}(s) &= \frac{\cosh\mathcal{P}}{\mathcal{M}}(\cosh[\mathcal{N}]\cosh[\mathcal{M}s]+\sinh[\mathcal{N}]\sinh[\mathcal{M}s])+\mathcal{P}_2,\\ \boldsymbol{z}(s) &= \sinh\mathcal{P}s+\frac{(\mathcal{N}+\mathcal{M}s)}{2\mathcal{M}^2}\cosh^2\mathcal{P}-\frac{\mathcal{P}_1}{\mathcal{M}}\cosh^2\mathcal{P}\cosh[\mathcal{N}]\cosh[\mathcal{M}s]\\ &\quad -\frac{\mathcal{P}_1}{\mathcal{M}}\cosh^2\mathcal{P}\sinh[\mathcal{N}]\sinh[\mathcal{M}s]-\frac{1}{4\mathcal{M}^2}\cosh^2\mathcal{P}\sinh 2[\mathcal{M}s+\mathcal{N}]+\mathcal{P}_3,\end{aligned} \tag{3.9}$$

where $\mathcal{P}_1,\mathcal{P}_2,\mathcal{P}_3$ are constants of integration and

$$\mathcal{M} = \operatorname{sech}\mathcal{P}\sqrt{\kappa_n^2+\kappa_g^2}-2\sinh\mathcal{P}.$$

Using Mathematica in above system we have Fig.1:

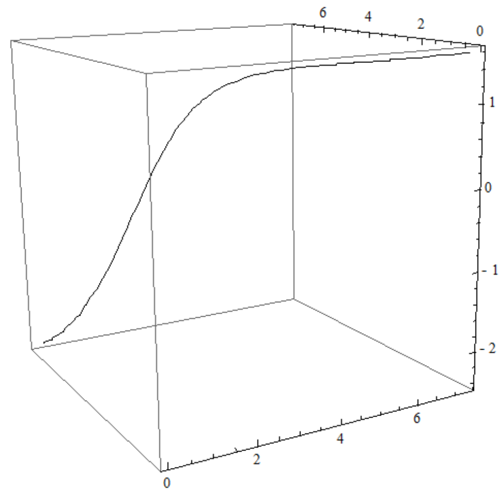

Fig.1

Corollary 3.3. *Let $\gamma : I \longrightarrow \Pi \subset \mathbb{H}$ be a non-geodesic unit speed timelike biharmonic $\mathcal{D}$- curve on timelike surface Π in the Lorentzian Heisenberg group $\mathbb{H}$. Then,*

$$\kappa = constant \neq 0.$$

4. Smarandache $\mathcal{D}-\mathbf{Tg}$ curves According to Darboux Frame on a Non-Degenerate Timelike Surface in the Lorentzian Heisenberg Group $\mathbb{H}$

Definition 4.1. *Let $\gamma : I \longrightarrow \Pi \subset \mathbb{H}$ be a non-geodesic unit speed timelike biharmonic $\mathcal{D}$- curve on timelike surface Π in the Lorentzian Heisenberg group $\mathbb{H}$ and $\{\mathbf{T}, \mathbf{n}, \mathbf{g}\}$ be its moving Darboux frame. Smarandache $\mathcal{D}-\mathbf{Tg}$ curves are defined by*

$$\tilde{\gamma} = \frac{1}{\kappa_n + \tau_g} (\mathbf{T} + \mathbf{g}) . \tag{4.1}$$

Theorem 4.2. *Let $\gamma : I \longrightarrow \Pi \subset \mathbb{H}$ be a non-geodesic unit speed timelike biharmonic $\mathcal{D}$- curve on timelike surface Π in the Lorentzian Heisenberg group $\mathbb{H}$ and $\tilde{\gamma}$ its Smarandache $\mathcal{D}-\mathbf{Tg}$ curve. Then position vector of Smarandache $\mathcal{D}-\mathbf{Tg}$*

curve is given by

$$\tilde{\gamma} = [\frac{\sinh \mathcal{P}}{\kappa_n + \tau_g} - \frac{\cos^2 \theta}{\kappa_n (\kappa_n + \tau_g)}[\cosh \mathcal{P} \sinh \mathcal{M}s + \mathcal{N}] \cosh \mathcal{P} \sinh [\mathcal{M}s + \mathcal{N}] (\frac{1}{\mathcal{M}} + 2\sinh \mathcal{P})$$

(4.2)

$$- \cosh \mathcal{P} \cosh [\mathcal{M}s + \mathcal{N}] \cosh \mathcal{P} \cosh [\mathcal{M}s + \mathcal{N}] (\frac{1}{\mathcal{M}} + 2\sinh \mathcal{P})]]\mathbf{e}_1$$

$$+ [\frac{\cosh \mathcal{P} \sinh [\mathcal{M}s + \mathcal{N}]}{\kappa_n + \tau_g} + \frac{1}{\kappa_n + \tau_g}[\frac{\cos \theta \sin \theta}{\kappa_n} \cosh \mathcal{P} \cosh [\mathcal{M}s + \mathcal{N}] (\frac{1}{\mathcal{M}} + 2\sinh \mathcal{P})$$

$$+ \frac{\cos^2 \theta}{\kappa_n} \sinh \mathcal{P} \cosh \mathcal{P} \sinh [\mathcal{M}s + \mathcal{N}] (\frac{1}{\mathcal{M}} + 2\sinh \mathcal{P})]]\mathbf{e}_2$$

$$+ [\frac{\cosh \mathcal{P} \cosh [\mathcal{M}s + \mathcal{N}]}{\kappa_n + \tau_g} + \frac{1}{\kappa_n + \tau_g}[\frac{\cos \theta \sin \theta}{\kappa_n} \cosh \mathcal{P} \sinh [\mathcal{M}s + \mathcal{N}] (\frac{1}{\mathcal{M}} + 2\sinh \mathcal{P})$$

$$- \frac{\cos^2 \theta}{\kappa_n} \sinh \mathcal{P} \cosh \mathcal{P} \cosh [\mathcal{M}s + \mathcal{N}] (\frac{1}{\mathcal{M}} + 2\sinh \mathcal{P})]]\mathbf{e}_3,$$

where $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$ *are constants of integration and*

$$\mathcal{M} = \operatorname{sech} \mathcal{P} \sqrt{\kappa_n^2 + \kappa_g^2} - 2\sinh \mathcal{P}.$$

Proof. From the assumption we get

$$\mathbf{T} = \sinh \mathcal{P} \mathbf{e}_1 + \cosh \mathcal{P} \sinh [\mathcal{M}s + \mathcal{N}]\, \mathbf{e}_2 + \cosh \mathcal{P} \cosh [\mathcal{M}s + \mathcal{N}]\, \mathbf{e}_3. \tag{4.3}$$

The covariant derivative of the vector field $\mathbf{T}$ is:

$$\nabla_{\mathbf{T}} \mathbf{T} = \cosh \mathcal{P} \cosh [\mathcal{M}s + \mathcal{N}] (\frac{1}{\mathcal{M}} + 2\sinh \mathcal{P}) \mathbf{e}_2$$
$$+ \cosh \mathcal{P} \sinh [\mathcal{M}s + \mathcal{N}] (\frac{1}{\mathcal{M}} + 2\sinh \mathcal{P}) \mathbf{e}_3.$$

From the above equation, it is seen that

$$\mathbf{g} = -\frac{\cos^2 \theta}{\kappa_n}[\cosh \mathcal{P} \sinh \mathcal{M}s + \mathcal{N}] \cosh \mathcal{P} \sinh [\mathcal{M}s + \mathcal{N}] (\frac{1}{\mathcal{M}} + 2\sinh \mathcal{P})$$
$$- \cosh \mathcal{P} \cosh [\mathcal{M}s + \mathcal{N}] \cosh \mathcal{P} \cosh [\mathcal{M}s + \mathcal{N}] (\frac{1}{\mathcal{M}} + 2\sinh \mathcal{P})]\mathbf{e}_1$$

(4.4)

$$+ [\frac{\cos \theta \sin \theta}{\kappa_n} \cosh \mathcal{P} \cosh [\mathcal{M}s + \mathcal{N}] (\frac{1}{\mathcal{M}} + 2\sinh \mathcal{P})$$
$$+ \frac{\cos^2 \theta}{\kappa_n} \sinh \mathcal{P} \cosh \mathcal{P} \sinh [\mathcal{M}s + \mathcal{N}] (\frac{1}{\mathcal{M}} + 2\sinh \mathcal{P})]\mathbf{e}_2$$
$$[\frac{\cos \theta \sin \theta}{\kappa_n} \cosh \mathcal{P} \sinh [\mathcal{M}s + \mathcal{N}] (\frac{1}{\mathcal{M}} + 2\sinh \mathcal{P})$$
$$- \frac{\cos^2 \theta}{\kappa_n} \sinh \mathcal{P} \cosh \mathcal{P} \cosh [\mathcal{M}s + \mathcal{N}] (\frac{1}{\mathcal{M}} + 2\sinh \mathcal{P})]\mathbf{e}_3.$$

Substituting Eq.(4.4) and Eq.(4.3) into Eq.(4.1) , we obtain Eq.(4.2). This completes the proof.

Theorem 4.3. *Let* $\gamma : I \longrightarrow \Pi \subset \mathbb{H}$ *be a non-geodesic unit speed timelike biharmonic* $\mathcal{D}$*- curve on timelike surface* Π *in the Lorentzian Heisenberg group* $\mathbb{H}$

and $\tilde{\gamma}$ its Smarandache $\mathcal{D}-\mathbf{Tg}$ curve. Then parametric equations of Smarandache $\mathcal{D}-\mathbf{Tg}$ curve are given by

$$x_{\tilde{\gamma}}(s) = [\frac{\cosh\mathcal{P}\cosh[\mathcal{M}s+\mathcal{N}]}{\kappa_n+\tau_g} + \frac{1}{\kappa_n+\tau_g}[\frac{\cos\theta\sin\theta}{\kappa_n}\cosh\mathcal{P}\sinh[\mathcal{M}s+\mathcal{N}](\frac{1}{\mathcal{M}}+2\sinh\mathcal{P})$$
$$-\frac{\cos^2\theta}{\kappa_n}\sinh\mathcal{P}\cosh\mathcal{P}\cosh[\mathcal{M}s+\mathcal{N}](\frac{1}{\mathcal{M}}+2\sinh\mathcal{P})]],$$

(4.6)

$$y_{\tilde{\gamma}}(s) = [\frac{\cosh\mathcal{P}\sinh[\mathcal{M}s+\mathcal{N}]}{\kappa_n+\tau_g} + \frac{1}{\kappa_n+\tau_g}[\frac{\cos\theta\sin\theta}{\kappa_n}\cosh\mathcal{P}\cosh[\mathcal{M}s+\mathcal{N}](\frac{1}{\mathcal{M}}+2\sinh\mathcal{P})$$
$$+\frac{\cos^2\theta}{\kappa_n}\sinh\mathcal{P}\cosh\mathcal{P}\sinh[\mathcal{M}s+\mathcal{N}](\frac{1}{\mathcal{M}}+2\sinh\mathcal{P})]],$$

$$z_{\tilde{\gamma}}(s) = [\frac{\sinh\mathcal{P}}{\kappa_n+\tau_g} - \frac{\cos^2\theta}{\kappa_n(\kappa_n+\tau_g)}[\cosh\mathcal{P}\sinh\mathcal{M}s+\mathcal{N}]\cosh\mathcal{P}\sinh[\mathcal{M}s+\mathcal{N}](\frac{1}{\mathcal{M}}+2\sinh\mathcal{P})$$
$$-\cosh\mathcal{P}\cosh[\mathcal{M}s+\mathcal{N}]\cosh\mathcal{P}\cosh[\mathcal{M}s+\mathcal{N}](\frac{1}{\mathcal{M}}+2\sinh\mathcal{P})]]$$

(4.6)

$$-[\frac{\cosh\mathcal{P}\sinh[\mathcal{M}s+\mathcal{N}]}{\kappa_n+\tau_g} + \frac{1}{\kappa_n+\tau_g}[\frac{\cos\theta\sin\theta}{\kappa_n}\cosh\mathcal{P}\cosh[\mathcal{M}s+\mathcal{N}](\frac{1}{\mathcal{M}}+2\sinh\mathcal{P})$$
$$+\frac{\cos^2\theta}{\kappa_n}\sinh\mathcal{P}\cosh\mathcal{P}\sinh[\mathcal{M}s+\mathcal{N}](\frac{1}{\mathcal{M}}+2\sinh\mathcal{P})]]$$
$$[\frac{\cosh\mathcal{P}\cosh[\mathcal{M}s+\mathcal{N}]}{\kappa_n+\tau_g} + \frac{1}{\kappa_n+\tau_g}[\frac{\cos\theta\sin\theta}{\kappa_n}\cosh\mathcal{P}\sinh[\mathcal{M}s+\mathcal{N}](\frac{1}{\mathcal{M}}+2\sinh\mathcal{P})$$
$$-\frac{\cos^2\theta}{\kappa_n}\sinh\mathcal{P}\cosh\mathcal{P}\cosh[\mathcal{M}s+\mathcal{N}](\frac{1}{\mathcal{M}}+2\sinh\mathcal{P})]],$$

where $\mathcal{N}$ is constant of integration and

$$\mathcal{M} = \operatorname{sech}\mathcal{P}\sqrt{\kappa_n^2+\kappa_g^2} - 2\sinh\mathcal{P}.$$

Proof. Omitted.

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