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NEW CHARACTERIZATION OF INEXTENSIBLE FLOWS OF DUAL CURVES ACCORDING TO DUAL BISHOP FRAME IN THE DUAL SPACE $\mathbb{D}^3$

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Abstract. In this paper, we study inxtensible flows of dual curves in dual space $\mathbb{D}^3$.

1. Introduction

The application of dual numbers to the lines of the 3-space is carried out by the principle of transference which has been formulated by Study and Kotelnikov. It allows a complete generalization of the mathematical expression for the spherical point geometry to the spatial line geometry by means of dual-number extension, i.e. replacing all ordinary quantities by the corresponding dual-number quantities.

In this paper, we study inxtensible flows of dual curves in dual space $\mathbb{D}^3$. We research inextensible flows of dual curves according to dual Bishop frame in dual space $\mathbb{D}^3$.

2. Preliminaries

In the Euclidean 3-Space $\mathbb{E}^3$, lines combined with one of their two directions can be represented by unit dual vectors over the the ring of dual numbers. The important properties of real vector analysis are valid for the dual vectors. The oriented lines $\mathbb{E}^3$ are in one to one correspondence with the points of the dual unit sphere $\mathbb{D}^3$.

There is a tight connection between spatial kinematics and the geometry of line in the three-dimensional Euclidean space $\mathbb{E}^3$ Therefore we start with recalling the use of appropriate line coordinates: An oriented line L in the three threedimensional Euclidean space $\mathbb{E}^3$ can be determined by a point $p \in L$ and a normalized direction vector x of L, i.e. $\|x\| = 1$.To obtain components for L [3], one forms the moment vector

$$x^* = p \times x, \tag{2.1}$$

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with respect to the origin point in $\mathbb{E}^3$.If p is substituted by any point

$$q = p + \mu x, \mu \in \mathbb{R}, \tag{2.2}$$

on L, Eq. (2.1) implies that $x^*$is independent of p on L.The two vectors x and x^* are not independent of one another; they satisfy the following relationships:

$$\langle x, x \rangle = 1, \quad \langle x^*, x \rangle = 0. \tag{2.3}$$

The six components x_i and x_i^* $(i = 1, 2, 3)$ of x and x^* are called the normalized Plücker coordinates of the line L. Hence the two vectors x and x^* determine the oriented line L.

Conversely, any six-tuple x_i , $x_i^*(i = 1, 2, 3)$ with

$$x_1^2 + x_2^2 + x_3^2 = 1, \quad x_1 x_1^* + x_2 x_2^* + x_3 x_3^* = 0, \tag{2.4}$$

represents a line in the three-dimensional Euclidean space $\mathbb{E}^3$.Thus, the set of all oriented lines in the three-dimensional Euclidean space $\mathbb{E}^3$ is in one-to-one correspondence with pairs of vectors in $\mathbb{E}^3$ subject to the relationships in Eq. (2.3).

For all pairs $(x, x^*) \in \mathbb{E}^3 \times \mathbb{E}^3$ the set

$$\mathbb{D}^3 = \left\{ X = x + \varepsilon x^*, \varepsilon \neq 0, \varepsilon^2 = 0 \right\}, \tag{2.5}$$

together with the scalar product

$$\langle X, Y \rangle = \langle x, y \rangle + \varepsilon \left(\langle y, x^* \rangle + \langle y^*, x \rangle \right), \tag{2.6}$$

forms the dual 3-space $\mathbb{D}^3$.Thereby a point $X = (X_1 + X_2 + X_3)^t$ has dual coordinates $X_i = (x_i + \varepsilon x_i^*) \in \mathbb{D}$.The norm is defined by

$$\langle X, X \rangle^{\frac{1}{2}} = \|X\| = \|x\| \left(1 + \varepsilon \frac{\langle x, x^* \rangle}{\|x\|^2}\right). \tag{2.7}$$

In the dual 3-space $\mathbb{D}^3$ the dual unit sphere is defined by

$$K = \left\{ A \in \mathbb{D}^3 : \|X\|^2 = X_1^2 + X_2^2 + X_3^2 = 1 \right\}. \tag{2.8}$$

The set of all oriented lines in the Euclidean 3-space $\mathbb{E}^3$ is in one-to-one correspondence with the set of points of dual unit sphere in the dual lines in the Euclidean 3-space $\mathbb{E}^3$ is in one-to-one correspondence with the set of points of dual unit sphere in the dual 3-space $\mathbb{D}^3$.The representation of directed lines in $\mathbb{E}^3$ by dual unit vectors brings about several advantages and from now on we do not distinguish between oriented lines and their representing dual unit vectors.If every $x_i(s)$ and $x_i^*(s)$, $1 \leq i \leq 3$, real valued functions are differentiable, the dual space curve

$$\begin{aligned} &\hat{x} : I \subset \mathbb{R} \to \mathbb{D}^3 \\ &t \to \hat{x}(s) = \left(x_1(s) + \varepsilon x_1^*(s), x_2(s) + \varepsilon x_2^*(s), x_3(s) + \varepsilon x_3^*(s) \right), \end{aligned}$$

in $\mathbb{D}^3$ is differentiable.

Let $\left\{\hat{\mathbf{T}}, \hat{\mathbf{M}}_1, \hat{\mathbf{M}}_2\right\}$ be the dual Bishop frame of the differentiable dual space curve in the dual space $\mathbb{D}^3$. Then the dual Bishop frame equations are

$$\begin{aligned} \hat{\mathbf{T}}' &= \hat{k}_1\hat{\mathbf{M}}_1 + \hat{k}_2\hat{\mathbf{M}}_2, \\ \hat{\mathbf{M}}_1' &= -\hat{k}_1\hat{\mathbf{T}}, \\ \hat{\mathbf{M}}_2' &= -\hat{k}_2\hat{\mathbf{T}}. \end{aligned} \tag{2.1}$$

where $\hat{k}_1 = k_1 + \varepsilon k_1^*$ and $\hat{k}_2 = k_2 + \varepsilon k_2^*$ are nowhere pure dual natural curvatures and

$$\hat{k} = k + \varepsilon k^* = \sqrt{k_1^2 + k_2^2 + 2\varepsilon\left(k_1k_1^* + k_2k_2^*\right)},$$

$$\hat{\theta}(s) = \theta + \varepsilon\theta^* = \arctan\left(\frac{\hat{k}_2}{\hat{k}_1}\right) = \arctan\left(\frac{k_2}{k_1} + \varepsilon\frac{(k_1k_2^* - k_1k_2^*)}{k_1^2}\right),$$

$$\hat{\tau}(s) = \frac{d\hat{\theta}(t)}{ds}.$$

3. Inextensible Flows of Dual Curves according to Dual Bishop Frame in $\mathbb{D}^3$

Throughout this article, we assume that $\hat{\eta} : [0, l] \times [0, \omega] \to \mathbb{D}^3$ is a one parameter family of smooth dual curves in dual space $\mathbb{D}^3$. Let u be the curve parametrization variable, $0 \leq u \leq l$.

The arclength of $\hat{\eta}$ is given by

$$s(u) = \int_0^u \left|\frac{\partial\hat{\eta}}{\partial u}\right| du, \tag{3.1}$$

where

$$\left|\frac{\partial\hat{\eta}}{\partial u}\right| = \left|\left\langle\frac{\partial\hat{\eta}}{\partial u}, \frac{\partial\hat{\eta}}{\partial u}\right\rangle\right|^{\frac{1}{2}}. \tag{3.2}$$

The operator $\dfrac{\partial}{\partial s}$ is given in terms of u by

$$\frac{\partial}{\partial s} = \frac{1}{v}\frac{\partial}{\partial u},$$

where $v = \left|\dfrac{\partial\hat{\eta}}{\partial u}\right|$. The arclength parameter is $ds = vdu$.

Any flow of $\hat{\eta}$ can be represented as

$$\frac{\partial\hat{\eta}}{\partial u} = \hat{\mathfrak{f}}_1^{\mathcal{B}}\hat{\mathbf{T}} + \hat{\mathfrak{f}}_2^{\mathcal{B}}\hat{\mathbf{M}}_1 + \hat{\mathfrak{f}}_3^{\mathcal{B}}\hat{\mathbf{M}}_2. \tag{3.3}$$

Letting the arclength variation be

$$s(u,t) = \int_0^u vdu.$$

In the dual space the requirement that the curve not be subject to any elongation or compression can be expressed by the condition

$$\frac{\partial}{\partial t}s(u,t)=\int_0^u\frac{\partial v}{\partial t}du=0, \tag{3.4}$$

for all $u\in[0,l]$.

Definition 3.1. *A dual curve evolution $\hat{\eta}(u,t)$ and its flow $\frac{\partial\hat{\eta}}{\partial t}$ in $\mathbb{D}^3$ are said to be inextensible if*

$$\frac{\partial}{\partial t}\left|\frac{\partial\hat{\eta}}{\partial u}\right|=0.$$

Lemma 3.2. *Let $\frac{\partial\hat{\eta}}{\partial t}$ be a smooth flow of the dual curve $\hat{\eta}$. The flow is inextensible if and only if*

$$\frac{\partial v}{\partial t}=\frac{\partial\mathfrak{f}_1^{\mathcal{B}}}{\partial u}-\mathfrak{f}_2^{\mathcal{B}}vk_1-\mathfrak{f}_3^{\mathcal{B}}vk_2, \tag{3.5}$$

$$\frac{\partial\mathfrak{f}_1^{*\mathcal{B}}}{\partial t}=(\mathfrak{f}_2^{\mathcal{B}}k_1^*+\mathfrak{f}_2^{*\mathcal{B}}k_1+\mathfrak{f}_3^{\mathcal{B}}k_2^*+\mathfrak{f}_3^{*\mathcal{B}}k_2. \tag{3.6}$$

Proof. Suppose that $\frac{\partial\hat{\eta}}{\partial t}$ be a smooth flow of the curve $\hat{\eta}$. Using definition of $\hat{\eta}$, we have

$$v^2=\left\langle\frac{\partial\hat{\eta}}{\partial u},\frac{\partial\hat{\eta}}{\partial u}\right\rangle. \tag{3.7}$$

$\frac{\partial}{\partial u}$ and $\frac{\partial}{\partial t}$ commute since and are independent coordinates. So, by differentiating of the formula (3.7), we get

$$2v\frac{\partial v}{\partial t}=\frac{\partial}{\partial t}\left\langle\frac{\partial\hat{\eta}}{\partial u},\frac{\partial\hat{\eta}}{\partial u}\right\rangle.$$

On the other hand, changing $\frac{\partial}{\partial u}$ and $\frac{\partial}{\partial t}$, we have

$$v\frac{\partial v}{\partial t}=\left\langle\frac{\partial\hat{\eta}}{\partial u},\frac{\partial}{\partial u}(\frac{\partial\hat{\eta}}{\partial t})\right\rangle.$$

From Eq. (3.3), we obtain

$$v\frac{\partial v}{\partial t}=<\frac{\partial\hat{\eta}}{\partial u},\frac{\partial}{\partial u}(\hat{\mathfrak{f}}_1^{\mathcal{B}}\hat{\mathbf{T}}+\hat{\mathfrak{f}}_2^{\mathcal{B}}\hat{\mathbf{M}}_1+\hat{\mathfrak{f}}_3^{\mathcal{B}}\hat{\mathbf{M}}_2)>.$$

By the formula of dual the Bishop, we have

$$\begin{aligned}\frac{\partial v}{\partial t}=<\hat{\mathbf{T}},(&\frac{\partial\mathfrak{f}_1^{\mathcal{B}}}{\partial u}-\mathfrak{f}_2^{\mathcal{B}}vk_1-\mathfrak{f}_3^{\mathcal{B}}vk_2+\varepsilon(\frac{\partial\mathfrak{f}_1^{*\mathcal{B}}}{\partial u}-(\mathfrak{f}_2^{\mathcal{B}}vk_1^*+\mathfrak{f}_2^{*\mathcal{B}}vk_1+\mathfrak{f}_3^{\mathcal{B}}vk_2^*+\mathfrak{f}_3^{*\mathcal{B}}vk_2))\hat{\mathbf{T}}\\&+(\mathfrak{f}_1^{\mathcal{B}}vk_1+\frac{\partial\mathfrak{f}_2^{\mathcal{B}}}{\partial u}+\varepsilon(\frac{\partial\mathfrak{f}_2^{*\mathcal{B}}}{\partial u}+\mathfrak{f}_1^{\mathcal{B}}vk_1^*+\mathfrak{f}_1^{*\mathcal{B}}vk_1))\hat{\mathbf{M}}_1\\&+(\mathfrak{f}_1^{\mathcal{B}}vk_2+\frac{\partial\mathfrak{f}_3^{\mathcal{B}}}{\partial u}+\varepsilon(\frac{\partial\mathfrak{f}_3^{*\mathcal{B}}}{\partial u}+\mathfrak{f}_1^{\mathcal{B}}vk_2^*+\mathfrak{f}_1^{*\mathcal{B}}vk_2))\hat{\mathbf{M}}_2>.\end{aligned}$$

Making necessary calculations from above equation, we have Eq. (3.5) and Eq. (3.6), which proves the lemma.

Theorem 3.3. *Let* $\frac{\partial \hat{\eta}}{\partial t}$ *be a smooth flow of the dual curve* $\hat{\eta}$. *The flow is inextensible if and only if*

$$\frac{\partial \mathfrak{f}_1^{\mathcal{B}}}{\partial s} = \mathfrak{f}_2^{\mathcal{B}} v k_1 + \mathfrak{f}_3^{\mathcal{B}} v k_2, \tag{3.8}$$
$$\frac{\partial \mathfrak{f}_1^{*\mathcal{B}}}{\partial s} = (\mathfrak{f}_2^{\mathcal{B}} v k_1^* + \mathfrak{f}_2^{*\mathcal{B}} v k_1 + \mathfrak{f}_3^{\mathcal{B}} v k_2^* + \mathfrak{f}_3^{*\mathcal{B}} v k_2.$$

Proof. Now let $\frac{\partial \hat{\eta}}{\partial t}$ be extensible. From Eq. (3.4), we have

$$\begin{aligned}\frac{\partial}{\partial t} s(u,t) &= \int_0^u \frac{\partial v}{\partial t} du = \int_0^u \left(\frac{\partial \mathfrak{f}_1^{\mathcal{B}}}{\partial u} - \mathfrak{f}_2^{\mathcal{B}} v k_1 - \mathfrak{f}_3^{\mathcal{B}} v k_2\right) du \\ &+ \int_0^u \varepsilon \left(\frac{\partial \mathfrak{f}_1^{*\mathcal{B}}}{\partial u} - (\mathfrak{f}_2^{\mathcal{B}} v k_1^* + \mathfrak{f}_2^{*\mathcal{B}} v k_1 + \mathfrak{f}_3^{\mathcal{B}} v k_2^* + \mathfrak{f}_3^{*\mathcal{B}} v k_2)\right) du \\ &= 0,\end{aligned} \tag{3.9}$$

Substituting Eq. (3.5) and Eq. (3.6) in Eq. (3.9) complete the proof of the theorem.

We now restrict ourselves to arc length parametrized curves.

Lemma 3.4.

$$\frac{\partial \mathbf{T}}{\partial t} = (\mathfrak{f}_1^{\mathcal{B}} k_1 + \frac{\partial \mathfrak{f}_2^{\mathcal{B}}}{\partial s})\mathbf{M}_1 + (\mathfrak{f}_1^{\mathcal{B}} k_2 + \frac{\partial \mathfrak{f}_3^{\mathcal{B}}}{\partial s})\mathbf{M}_2$$
$$\begin{aligned}\frac{\partial \mathbf{T}^*}{\partial t} &= (\frac{\partial \mathfrak{f}_2^{*\mathcal{B}}}{\partial s} + \mathfrak{f}_1^{\mathcal{B}} k_1^* + \mathfrak{f}_1^{*\mathcal{B}} k_1)\mathbf{M}_1 + (\mathfrak{f}_1^{\mathcal{B}} k_1 + \frac{\partial \mathfrak{f}_2^{\mathcal{B}}}{\partial s})\mathbf{M}_1^* \\ &+ (\frac{\partial \mathfrak{f}_3^{*\mathcal{B}}}{\partial s} + \mathfrak{f}_1^{\mathcal{B}} k_2^* + \mathfrak{f}_1^{*\mathcal{B}} k_2)\mathbf{M}_2 + (\mathfrak{f}_1^{\mathcal{B}} k_2 + \frac{\partial \mathfrak{f}_3^{\mathcal{B}}}{\partial s})\mathbf{M}_2^*,\end{aligned}$$
$$\frac{\partial \mathbf{M}_1}{\partial t} = -(\mathfrak{f}_1^{\mathcal{B}} k_1 + \frac{\partial \mathfrak{f}_2^{\mathcal{B}}}{\partial s})\mathbf{T} + \psi \mathbf{M}_2, \tag{3.10}$$
$$\frac{\partial \mathbf{M}_1^*}{\partial t} = [-((\frac{\partial \mathfrak{f}_2^{*\mathcal{B}}}{\partial s} + \mathfrak{f}_1^{\mathcal{B}} k_1^* + \mathfrak{f}_1^{*\mathcal{B}} k_1)\mathbf{T} + (\mathfrak{f}_1^{\mathcal{B}} k_1 + \frac{\partial \mathfrak{f}_2^{\mathcal{B}}}{\partial s})\mathbf{T}^*)) + \psi \mathbf{M}_2^* + \psi^* \mathbf{M}_2],$$
$$\frac{\partial \mathbf{M}_2}{\partial t} = -(\mathfrak{f}_1^{\mathcal{B}} k_2 + \frac{\partial \mathfrak{f}_3^{\mathcal{B}}}{\partial s})\mathbf{T} - \psi \mathbf{M}_1, \tag{3.11}$$
$$\frac{\partial \mathbf{M}_2}{\partial t} = -(\frac{\partial \mathfrak{f}_3^{*\mathcal{B}}}{\partial s} + \mathfrak{f}_1^{\mathcal{B}} k_2^* + \mathfrak{f}_1^{*\mathcal{B}} k_2)\mathbf{T} + (\mathfrak{f}_1^{\mathcal{B}} k_2 + \frac{\partial \mathfrak{f}_3^{\mathcal{B}}}{\partial s})\mathbf{T}^* + \psi \mathbf{M}_1^* + \psi^* \mathbf{M}_1,$$

where $\hat{\psi} = \psi + \varepsilon \psi^* = \left\langle \frac{\partial \hat{\mathbf{M}}_1}{\partial t}, \hat{\mathbf{M}}_2 \right\rangle = \left\langle \frac{\partial (\mathbf{M}_1 + \varepsilon \mathbf{M}_1^*)}{\partial t}, \mathbf{M}_2 + \varepsilon \mathbf{M}_2^* \right\rangle.$

Proof. Using definition of $\hat{F}$, we have

$$\frac{\partial \hat{\mathbf{T}}}{\partial t} = \frac{\partial}{\partial t}\frac{\partial \hat{F}}{\partial s} = \frac{\partial}{\partial s}(\hat{\mathfrak{f}}_1^{\mathcal{B}} \hat{\mathbf{T}} + \hat{\mathfrak{f}}_2^{\mathcal{B}} \hat{\mathbf{M}}_1 + \hat{\mathfrak{f}}_3^{\mathcal{B}} \hat{\mathbf{M}}_2).$$

Using the dual Bishop equations, we have

$$\frac{\partial\hat{\mathbf{T}}}{\partial t}=(\frac{\partial\hat{\mathfrak{f}}_1^{\mathcal{B}}}{\partial s}-\hat{\mathfrak{f}}_2^{\mathcal{B}}\hat{k}_1-\hat{\mathfrak{f}}_3^{\mathcal{B}}\hat{k}_2)\hat{\mathbf{T}}+(\hat{\mathfrak{f}}_1^{\mathcal{B}}\hat{k}_1+\frac{\partial\hat{\mathfrak{f}}_2^{\mathcal{B}}}{\partial s})\hat{\mathbf{M}}_1+(\hat{\mathfrak{f}}_1^{\mathcal{B}}\hat{k}_2+\frac{\partial\hat{\mathfrak{f}}_3^{\mathcal{B}}}{\partial s})\hat{\mathbf{M}}_2. \tag{3.12}$$

Substituting (3.8) in (3.10), we get

$$\begin{aligned}\frac{\partial\hat{\mathbf{T}}}{\partial t}&=(\mathfrak{f}_1^{\mathcal{B}}k_1+\frac{\partial\mathfrak{f}_2^{\mathcal{B}}}{\partial s})\hat{\mathbf{M}}_1+(\mathfrak{f}_1^{\mathcal{B}}k_2+\frac{\partial\mathfrak{f}_3^{\mathcal{B}}}{\partial s})\hat{\mathbf{M}}_2\\&+\varepsilon\left[(\frac{\partial\mathfrak{f}_2^{*\mathcal{B}}}{\partial s}+\mathfrak{f}_1^{\mathcal{B}}k_1^*+\mathfrak{f}_1^{*\mathcal{B}}k_1)\hat{\mathbf{M}}_1+(\frac{\partial\mathfrak{f}_3^{*\mathcal{B}}}{\partial s}+\mathfrak{f}_1^{\mathcal{B}}k_2^*+\mathfrak{f}_1^{*\mathcal{B}}k_2)\hat{\mathbf{M}}_2\right].\end{aligned}$$

Also, using above equation we obtain

$$\begin{aligned}\frac{\partial\mathbf{T}}{\partial t}&=(\mathfrak{f}_1^{\mathcal{B}}k_1+\frac{\partial\mathfrak{f}_2^{\mathcal{B}}}{\partial s})\mathbf{M}_1+(\mathfrak{f}_1^{\mathcal{B}}k_2+\frac{\partial\mathfrak{f}_3^{\mathcal{B}}}{\partial s})\mathbf{M}_2,\\\frac{\partial\mathbf{T}^*}{\partial t}&=(\frac{\partial\mathfrak{f}_2^{*\mathcal{B}}}{\partial s}+\mathfrak{f}_1^{\mathcal{B}}k_1^*+\mathfrak{f}_1^{*\mathcal{B}}k_1)\mathbf{M}_1+(\mathfrak{f}_1^{\mathcal{B}}k_1+\frac{\partial\mathfrak{f}_2^{\mathcal{B}}}{\partial s})\mathbf{M}_1^*+\\&(\frac{\partial\mathfrak{f}_3^{*\mathcal{B}}}{\partial s}+\mathfrak{f}_1^{\mathcal{B}}k_2^*+\mathfrak{f}_1^{*\mathcal{B}}k_2)\mathbf{M}_2+(\mathfrak{f}_1^{\mathcal{B}}k_2+\frac{\partial\mathfrak{f}_3^{\mathcal{B}}}{\partial s})\mathbf{M}_2^*.\end{aligned}$$

Now differentiate the dual Bishop frame by t :

$$\begin{aligned}0&=\frac{\partial}{\partial t}\left\langle\hat{\mathbf{T}},\hat{\mathbf{M}}_1\right\rangle=\left\langle\frac{\partial\hat{\mathbf{T}}}{\partial t},\hat{\mathbf{M}}_1\right\rangle+\left\langle\hat{\mathbf{T}},\frac{\partial\hat{\mathbf{M}}_1}{\partial t}\right\rangle,\\&=\mathfrak{f}_1^{\mathcal{B}}k_1+\frac{\partial\mathfrak{f}_2^{\mathcal{B}}}{\partial s}+\varepsilon(\frac{\partial\mathfrak{f}_2^{*\mathcal{B}}}{\partial s}+\mathfrak{f}_1^{\mathcal{B}}k_1^*+\mathfrak{f}_1^{*\mathcal{B}}k_1)+\left\langle\hat{\mathbf{T}},\frac{\partial\hat{\mathbf{N}}}{\partial t}\right\rangle.\\0&=\frac{\partial}{\partial t}\left\langle\hat{\mathbf{T}},\hat{\mathbf{M}}_2\right\rangle=\left\langle\frac{\partial\hat{\mathbf{T}}}{\partial t},\hat{\mathbf{M}}_2\right\rangle+\left\langle\hat{\mathbf{T}},\frac{\partial\hat{\mathbf{M}}_2}{\partial t}\right\rangle,\\&=\mathfrak{f}_1^{\mathcal{B}}k_2+\frac{\partial\mathfrak{f}_3^{\mathcal{B}}}{\partial s}+\varepsilon(\frac{\partial\mathfrak{f}_3^{*\mathcal{B}}}{\partial s}+\mathfrak{f}_1^{\mathcal{B}}k_2^*+\mathfrak{f}_1^{*\mathcal{B}}k_2)+\left\langle\hat{\mathbf{T}},\frac{\partial\hat{\mathbf{M}}_2}{\partial t}\right\rangle.\\0&=\frac{\partial}{\partial t}\left\langle\hat{\mathbf{M}}_1,\hat{\mathbf{M}}_2\right\rangle=\left\langle\frac{\partial\hat{\mathbf{M}}_1}{\partial t},\hat{\mathbf{M}}_2\right\rangle+\left\langle\hat{\mathbf{M}}_1,\frac{\partial\hat{\mathbf{M}}_2}{\partial t}\right\rangle\\&=\hat{\psi}+\left\langle\hat{\mathbf{M}}_1,\frac{\partial\hat{\mathbf{M}}_2}{\partial t}\right\rangle.\end{aligned}$$

From the above and using $\left\langle\frac{\partial\hat{\mathbf{M}}_1}{\partial t},\hat{\mathbf{M}}_1\right\rangle=\left\langle\frac{\partial\hat{\mathbf{M}}_2}{\partial t},\hat{\mathbf{M}}_2\right\rangle=0$, we obtain

$$\begin{aligned}\frac{\partial\mathbf{M}_1}{\partial t}&=-(\mathfrak{f}_1^{\mathcal{B}}k_1+\frac{\partial\mathfrak{f}_2^{\mathcal{B}}}{\partial s})\mathbf{T}+\psi\mathbf{M}_1,\\\frac{\partial\mathbf{M}_1^*}{\partial t}&=[-((\frac{\partial\mathfrak{f}_2^{*\mathcal{B}}}{\partial s}+\mathfrak{f}_1^{\mathcal{B}}k_1^*+\mathfrak{f}_1^{*\mathcal{B}}k_1)\mathbf{T}+(\mathfrak{f}_1^{\mathcal{B}}k_1+\frac{\partial\mathfrak{f}_2^{\mathcal{B}}}{\partial s})\mathbf{T}^*))+\psi\mathbf{M}_1^*+\psi^*\mathbf{M}_1],\\\frac{\partial\mathbf{M}_2}{\partial t}&=-(\mathfrak{f}_1^{\mathcal{B}}k_2+\frac{\partial\mathfrak{f}_3^{\mathcal{B}}}{\partial s})\mathbf{T}-\psi\mathbf{M}_1,\\\frac{\partial\mathbf{M}_2^*}{\partial t}&=-[(\frac{\partial\mathfrak{f}_3^{*\mathcal{B}}}{\partial s}+\mathfrak{f}_1^{\mathcal{B}}k_2^*+\mathfrak{f}_1^{*\mathcal{B}}k_2)\mathbf{T}+(\mathfrak{f}_1^{\mathcal{B}}k_2+\frac{\partial\mathfrak{f}_3^{\mathcal{B}}}{\partial s})\mathbf{T}^*+\psi\mathbf{M}_1^*+\psi^*\mathbf{M}_1].\end{aligned}$$

where $\psi = \left\langle \dfrac{\partial \mathbf{M}_1}{\partial t}, \mathbf{M}_2 \right\rangle, \psi^* = \left\langle \dfrac{\partial \mathbf{M}_1}{\partial t}, \mathbf{M}_2^* \right\rangle + \left\langle \dfrac{\partial \mathbf{M}_1^*}{\partial t}, \mathbf{M}_2 \right\rangle$.

The following theorem states the conditions on the curvature and torsion for the dual curve flow $\hat{F}(s,t)$ to be inextensible.

Theorem 3.5. *Suppose the flow* $\dfrac{\partial \hat{F}}{\partial t} = \hat{\mathfrak{f}}_1^{\mathcal{B}} \hat{\mathbf{T}} + \hat{\mathfrak{f}}_2^{\mathcal{B}} \hat{\mathbf{M}}_1 + \hat{\mathfrak{f}}_3^{\mathcal{B}} \hat{\mathbf{M}}_2$ *is inextensible. Then, the following system of partial differential equations holds:*

$$\begin{aligned}
\frac{\partial k_1}{\partial t} &= \frac{\partial}{\partial s}\left(\mathfrak{f}_1^{\mathcal{B}} k_1\right) + \frac{\partial^2 \mathfrak{f}_2^{\mathcal{B}}}{\partial s^2} - \psi k_1 + \psi k_2 \\
\frac{\partial k_1^*}{\partial t} &= \left(\frac{\partial}{\partial s}\left(\mathfrak{f}_1^{*\mathcal{B}} k_1 + \mathfrak{f}_1^{\mathcal{B}} k_1^*\right) + \frac{\partial^2 \mathfrak{f}_2^{*\mathcal{B}}}{\partial s^2} - \psi k_1^* - \psi^* k_1 + \psi k_2^* + \psi^* k_2)\right).
\end{aligned} \tag{3.11}$$

$$\begin{aligned}
\frac{\partial k_2}{\partial t} &= -\frac{\partial}{\partial s}(\mathfrak{f}_1^{\mathcal{B}} k_2) - \frac{\partial^2 \mathfrak{f}_3^{\mathcal{B}}}{\partial s^2} + \psi k_1 \\
\frac{\partial k_2^*}{\partial t} &= -\frac{\partial}{\partial s}(\mathfrak{f}_1^{\mathcal{B}} k_2^* + \mathfrak{f}_1^{*\mathcal{B}} k_2) - \frac{\partial^2 \mathfrak{f}_3^{*\mathcal{B}}}{\partial s^2} + \psi k_1^* + \psi^* k_1.
\end{aligned} \tag{3.12}$$

Proof. Using (3.10), we have

$$\begin{aligned}
\frac{\partial}{\partial s}\frac{\partial \hat{\mathbf{T}}}{\partial t} &= \frac{\partial}{\partial s}[(\hat{\mathfrak{f}}_1^{\mathcal{B}} \hat{k}_1 + \frac{\partial \hat{\mathfrak{f}}_2^{\mathcal{B}}}{\partial s})\hat{\mathbf{M}}_1 + (\hat{\mathfrak{f}}_1^{\mathcal{B}} \hat{k}_2 + \frac{\partial \hat{\mathfrak{f}}_3^{\mathcal{B}}}{\partial s})\hat{\mathbf{M}}_2] \\
&= (\frac{\partial}{\partial s}(\hat{\mathfrak{f}}_1^{\mathcal{B}} \hat{k}_1) + \frac{\partial^2 \hat{\mathfrak{f}}_2^{\mathcal{B}}}{\partial s^2})\hat{\mathbf{M}}_1 - (\hat{\mathfrak{f}}_1^{\mathcal{B}} \hat{k}_1 + \frac{\partial \hat{\mathfrak{f}}_2^{\mathcal{B}}}{\partial s})\hat{k}_1 \hat{\mathbf{T}} \\
&+ (\frac{\partial}{\partial s}(\hat{\mathfrak{f}}_1^{\mathcal{B}} \hat{k}_2) + \frac{\partial^2 \hat{\mathfrak{f}}_3^{\mathcal{B}}}{\partial s^2})\hat{\mathbf{M}}_2 - (\hat{\mathfrak{f}}_1^{\mathcal{B}} \hat{k}_2 + \frac{\partial \hat{\mathfrak{f}}_3^{\mathcal{B}}}{\partial s})\hat{k}_2 \hat{\mathbf{T}}.
\end{aligned}$$

On the other hand, from dual Bishop frame we have

$$\begin{aligned}
\frac{\partial}{\partial t}\frac{\partial \hat{\mathbf{T}}}{\partial s} &= \frac{\partial}{\partial t}(\hat{k}_1 \hat{\mathbf{M}}_1 + \hat{k}_2 \hat{\mathbf{M}}_2) \\
&= \frac{\partial \hat{k}_1}{\partial t}\hat{\mathbf{M}}_1 + \hat{k}_1(-(\hat{\mathfrak{f}}_1^{\mathcal{B}} \hat{k}_1 + \frac{\partial \hat{\mathfrak{f}}_2^{\mathcal{B}}}{\partial s})\hat{\mathbf{T}} + \hat{\psi}\hat{\mathbf{M}}_1) + \frac{\partial \hat{k}_2}{\partial t}\hat{\mathbf{M}}_2 + \hat{k}_2(-(\hat{\mathfrak{f}}_1^{\mathcal{B}} \hat{k}_2 + \frac{\partial \hat{\mathfrak{f}}_3^{\mathcal{B}}}{\partial s})\hat{\mathbf{T}} - \hat{\psi}\hat{\mathbf{M}}_1). \\
&= \frac{\partial \hat{k}_1}{\partial t}\hat{\mathbf{M}}_1 + (-(\hat{\mathfrak{f}}_1 \hat{k}_1^2 + \frac{\partial \hat{\mathfrak{f}}_2}{\partial s}\hat{k}_1)\hat{\mathbf{T}} + \hat{\psi}\hat{k}_1 \hat{\mathbf{M}}_1) + \frac{\partial \hat{k}_2}{\partial t}\hat{\mathbf{M}}_2 - (\hat{\mathfrak{f}}_1 \hat{k}_2^2 + \frac{\partial \hat{\mathfrak{f}}_3}{\partial s}\hat{k}_2)\hat{\mathbf{T}} - \hat{\psi}\hat{k}_2 \hat{\mathbf{M}}_1) \\
&(\frac{\partial \hat{k}_1}{\partial t} + \hat{\psi}\hat{k}_1 - \hat{\psi}\hat{k}_2)\hat{\mathbf{M}}_1 + (-(\hat{\mathfrak{f}}_1^{\mathcal{B}} \hat{k}_1^2 + \frac{\partial \hat{\mathfrak{f}}_2^{\mathcal{B}}}{\partial s}\hat{k}_1) - (\hat{\mathfrak{f}}_1^{\mathcal{B}} \hat{k}_2^2 + \frac{\partial \hat{\mathfrak{f}}_3^{\mathcal{B}}}{\partial s}\hat{k}_2))\hat{\mathbf{T}} + \frac{\partial \hat{k}_2}{\partial t}\hat{\mathbf{M}}_2.
\end{aligned}$$

Hence, we see that

$$\begin{aligned}
\frac{\partial \hat{k}_1}{\partial t} &= \frac{\partial}{\partial s}\left(\mathfrak{f}_1^{\mathcal{B}} k_1\right) + \frac{\partial^2 \mathfrak{f}_2^{\mathcal{B}}}{\partial s^2} - \psi k_1 + \psi k_2 \\
&+ \varepsilon(\frac{\partial}{\partial s}\left(\mathfrak{f}_1^{*\mathcal{B}} k_1 + \mathfrak{f}_1^{\mathcal{B}} k_1^*\right) + \frac{\partial^2 \mathfrak{f}_2^{*\mathcal{B}}}{\partial s^2} - \psi k_1^* - \psi^* k_1 + \psi k_2^* + \psi^* k_2)).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}\frac{\partial}{\partial s}\frac{\partial \hat{\mathbf{M}}_2}{\partial t} &= \frac{\partial}{\partial s}[-(\hat{\mathfrak{f}}_1^{\mathcal{B}}\hat{k}_2 + \frac{\partial \hat{\mathfrak{f}}_3^{\mathcal{B}}}{\partial s})\hat{\mathbf{T}} - \hat{\psi}\hat{\mathbf{M}}_1] \\ &= (-\frac{\partial}{\partial s}(\hat{\mathfrak{f}}_1^{\mathcal{B}}\hat{k}_2) - \frac{\partial^2 \hat{\mathfrak{f}}_3^{\mathcal{B}}}{\partial s^2} + \hat{\psi}\hat{k}_1)\hat{\mathbf{T}} - (\hat{\mathfrak{f}}_1^{\mathcal{B}}\hat{k}_1\hat{k}_2 + \frac{\partial \hat{\mathfrak{f}}_3^{\mathcal{B}}}{\partial s}\hat{k}_1 + \frac{\partial \hat{\psi}}{\partial s})\hat{\mathbf{M}}_1 \\ &- (\hat{\mathfrak{f}}_1^{\mathcal{B}}\hat{k}_2 + \frac{\partial \hat{\mathfrak{f}}_3^{\mathcal{B}}}{\partial s})(\hat{k}_2\hat{\mathbf{M}}_2),\end{aligned}$$

Using above equation, we obtain

$$\begin{aligned}\frac{\partial}{\partial t}\frac{\partial \hat{\mathbf{M}}_2}{\partial s} &= \frac{\partial}{\partial t}(-\hat{k}_2\hat{\mathbf{T}}) \\ &= \frac{\partial \hat{k}_2}{\partial t}\hat{\mathbf{T}} - \hat{k}_2[(\hat{\mathfrak{f}}_1^{\mathcal{B}}\hat{k}_1 + \frac{\partial \hat{\mathfrak{f}}_2^{\mathcal{B}}}{\partial s})\hat{\mathbf{M}}_1 + (\hat{\mathfrak{f}}_1^{\mathcal{B}}\hat{k}_2 + \frac{\partial \hat{\mathfrak{f}}_3^{\mathcal{B}}}{\partial s})\hat{\mathbf{M}}_2].\end{aligned}$$

Thus, we get

$$\begin{aligned}\frac{\partial \hat{k}_2}{\partial t} &= -\frac{\partial}{\partial s}(\mathfrak{f}_1^{\mathcal{B}}k_2) - \frac{\partial^2 \mathfrak{f}_3^{\mathcal{B}}}{\partial s^2} + \psi k_1 \\ &+ \varepsilon(-\frac{\partial}{\partial s}(\mathfrak{f}_1^{\mathcal{B}}k_2^* + \mathfrak{f}_1^{*\mathcal{B}}k_2) - \frac{\partial^2 \mathfrak{f}_3^{*\mathcal{B}}}{\partial s^2} + \psi k_1^* + \psi^* k_1)).\end{aligned}$$

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