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$\mathcal{B}-$TANGENT DEVELOPABLE SURFACES OF SPACELIKE BIHARMONIC NEW TYPE $\mathcal{B}-$SLANT HELICES WITH TIMELIKE $\mathbf{m}_2$ ACCORDING TO BISHOP FRAME IN LORENTZIAN HEISENBERG GROUP $\mathcal{H}^3$

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Abstract. In this paper, we study $\mathcal{B}-$tangent developable surfaces of spacelike biharmonic new type $\mathcal{B}$-slant helices with timelike $\mathbf{m}_2$ according to Bishop frame in the Lorentzian Heisenberg group $\mathcal{H}^3$. We give necessary and sufficient conditions for new type $\mathcal{B}$-slant helices with timelike $\mathbf{m}_2$ to be biharmonic. We characterize $\mathcal{B}-$tangent developable surfaces of spacelike biharmonic new type $\mathcal{B}$-slant helices with timelike $\mathbf{m}_2$ according to Bishop frame in the Lorentzian Heisenberg group $\mathcal{H}^3$.

1. Introduction

Modeling developable surfaces through approximation is attractive as designersdo not have to concern themselves with developability constraints during the modeling process. Ideally, they can freely utilize all sorts of modeling tools (e.g., blends, fillets) and then rely on an approximation algorithm to yield a developable result. In practice though, the approximation approach is highly restricted since the methods can only succeed if the original input surfaces already have fairly small Gaussian curvature. Moreover, in most cases the final result is not analytically developable. While this is not a problem for applications such as texture-mapping, it can be problematic for manufacturing, where the surfaces need to be realised from planar patterns (e.g., sewing).

This study is organised as follows: Firstly, study $\mathcal{B}-$tangent developable surfaces of spacelike biharmonic new type $\mathcal{B}$-slant helices with timelike $\mathbf{m}_2$ according to Bishop frame in the Lorentzian Heisenberg group $\mathcal{H}^3$. Secondly, We give necessary and sufficient conditions for new type $\mathcal{B}$-slant helices with timelike $\mathbf{m}_2$ to be biharmonic. Finally, we characterize $\mathcal{B}-$tangent developable surfaces of spacelike biharmonic new type $\mathcal{B}$-slant helices with timelike $\mathbf{m}_2$ according to Bishop frame in the Lorentzian Heisenberg group $\mathcal{H}^3$.

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2. The Lorentzian Heisenberg Group $\mathcal{H}^3$

The Heisenberg group Heis^3 is a Lie group which is diffeomorphic to $\mathbb{R}^3$ and the group operation is defined as

$$(x, y, z) * (\overline{x}, \overline{y}, \overline{z}) = (x + \overline{x}, y + \overline{y}, z + \overline{z} - \frac{1}{2}\overline{x}y + \frac{1}{2}x\overline{y}).$$

The identity of the group is $(0, 0, 0)$ and the inverse of (x, y, z) is given by $(-x, -y, -z)$. The left-invariant Lorentz metric on Heis^3 is

$$g = -dx^2 + dy^2 + (xdy + dz)^2.$$

The following set of left-invariant vector fields forms an orthonormal basis for the corresponding Lie algebra:

$$\left\{ \mathbf{e}_1 = \frac{\partial}{\partial z},\ \mathbf{e}_2 = \frac{\partial}{\partial y} - x\frac{\partial}{\partial z},\ \mathbf{e}_3 = \frac{\partial}{\partial x} \right\}. \tag{2.1}$$

The characterising properties of this algebra are the following commutation relations, [15]:

$$g(\mathbf{e}_1, \mathbf{e}_1) = g(\mathbf{e}_2, \mathbf{e}_2) = 1, \quad g(\mathbf{e}_3, \mathbf{e}_3) = -1.$$

Proposition 2.1. *For the covariant derivatives of the Levi-Civita connection of the left-invariant metric g, defined above the following is true:*

$$\nabla = \frac{1}{2}\begin{pmatrix} 0 & \mathbf{e}_3 & \mathbf{e}_2 \\ \mathbf{e}_3 & 0 & \mathbf{e}_1 \\ \mathbf{e}_2 & -\mathbf{e}_1 & 0 \end{pmatrix}, \tag{2.2}$$

where the (i, j)-element in the table above equals $\nabla_{\mathbf{e}_i}\mathbf{e}_j$ for our basis

$$\{\mathbf{e}_k, k = 1, 2, 3\}.$$

3. Spacelike Biharmonic New Type $\mathcal{B}$–Slant Helices with Bishop Frame In The Lorentzian Heisenberg Group $\mathcal{H}^3$

Let $\gamma : I \longrightarrow \mathcal{H}^3$ be a non geodesic spacelike curve on the Lorentzian Heisenberg group $\mathcal{H}^3$ parametrized by arc length. Let $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ be the Frenet frame fields tangent to the Lorentzian Heisenberg group $\mathcal{H}^3$ along γ defined as follows:

$\mathbf{t}$ is the unit vector field γ' tangent to γ, $\mathbf{n}$ is the unit vector field in the direction of $\nabla_{\mathbf{t}}\mathbf{t}$ (normal to γ), and $\mathbf{b}$ is chosen so that $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ is a positively oriented orthonormal basis. Then, we have the following Frenet formulas:

$$\begin{aligned} \nabla_{\mathbf{t}}\mathbf{t} &= \kappa\mathbf{n}, \\ \nabla_{\mathbf{t}}\mathbf{n} &= -\kappa\mathbf{t} + \tau\mathbf{b}, \\ \nabla_{\mathbf{t}}\mathbf{b} &= \tau\mathbf{n}, \end{aligned}$$

where κ is the curvature of γ and τ is its torsion and

$$\begin{aligned} &g(\mathbf{t}, \mathbf{t}) = 1,\ g(\mathbf{n}, \mathbf{n}) = 1,\ g(\mathbf{b}, \mathbf{b}) = -1, \\ &g(\mathbf{t}, \mathbf{n}) = g(\mathbf{t}, \mathbf{b}) = g(\mathbf{n}, \mathbf{b}) = 0. \end{aligned}$$

In the rest of the paper, we suppose everywhere $\kappa \neq 0$ and $\tau \neq 0$.

The Bishop frame or parallel transport frame is an alternative approach to defining a moving frame that is well defined even when the curve has vanishing second derivative. The Bishop frame is expressed as

$$\begin{aligned} \nabla_{\mathbf{t}}\mathbf{t} &= \mathfrak{k}_1\mathbf{m}_1 - \mathfrak{k}_2\mathbf{m}_2, \\ \nabla_{\mathbf{t}}\mathbf{m}_1 &= -\mathfrak{k}_1\mathbf{t}, \\ \nabla_{\mathbf{t}}\mathbf{m}_2 &= -\mathfrak{k}_2\mathbf{t}, \end{aligned} \tag{3.1}$$

where

$$g(\mathbf{t},\mathbf{t}) = 1,\ g(\mathbf{m}_1,\mathbf{m}_1) = 1,\ g(\mathbf{m}_2,\mathbf{m}_2) = -1,$$
$$g(\mathbf{t},\mathbf{m}_1) = g(\mathbf{t},\mathbf{m}_2) = g(\mathbf{m}_1,\mathbf{m}_2) = 0.$$

Here, we shall call the set $\{\mathbf{t},\mathbf{m}_1,\mathbf{m}_2\}$ as Bishop trihedra and $\mathfrak{k}_1$ and $\mathfrak{k}_2$ as Bishop curvatures, $\tau(s) = \theta'(s)$ and $\kappa(s) = \sqrt{|\mathfrak{k}_1^2 - \mathfrak{k}_2^2|}$.

With respect to the orthonormal basis $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ we can write

$$\begin{aligned} \mathbf{t} &= t^1\mathbf{e}_1 + t^2\mathbf{e}_2 + t^3\mathbf{e}_3, \\ \mathbf{m}_1 &= m_1^1\mathbf{e}_1 + m_1^2\mathbf{e}_2 + m_1^3\mathbf{e}_3, \\ \mathbf{m}_2 &= m_2^1\mathbf{e}_1 + m_2^2\mathbf{e}_2 + m_2^3\mathbf{e}_3. \end{aligned} \tag{3.2}$$

Theorem 3.1. *$\gamma : I \longrightarrow \mathcal{H}^3$ is a spacelike biharmonic curve with timelike $\mathbf{m}_2$ according to Bishop frame if and only if*

$$\begin{aligned} \mathfrak{k}_2^2 - \mathfrak{k}_1^2 &= \text{constant} = C \neq 0, \\ \mathfrak{k}_1'' + C\mathfrak{k}_1 &= k_1\left[-1 + 4\left(m_2^1\right)^2\right] + 4k_2m_1^1m_2^1, \\ \mathfrak{k}_2'' + C\mathfrak{k}_2 &= 4k_1m_1^1m_2^1 - k_2\left[1 + 4\left(m_1^1\right)^2\right]. \end{aligned} \tag{3.3}$$

Proof. Using Eq.(4), we have above system.

Definition 3.2. *A regular spacelike curve $\gamma : I \longrightarrow \mathcal{H}^3$ is called a new type slant helix provided the timelike unit vector $\mathbf{m}_2$ of the curve γ has constant angle $\mathcal{E}$ with some fixed spacelike unit vector u, that is*

$$g(\mathbf{m}_2(s), u) = \sinh\mathcal{E} \text{ for all } s \in I. \tag{3.4}$$

The condition is not altered by reparametrization, so without loss of generality we may assume that slant helices have unit speed. The slant helices can be identified by a simple condition on natural curvatures.

To separate a spacelike new type slant helix according to Bishop frame from that of Frenet- Serret frame, in the rest of the paper, we shall use notation for the curve defined above as spacelike new type $\mathcal{B}$-slant helix.

Theorem 3.3. ([15]) *Let $\gamma : I \longrightarrow \mathcal{H}^3$ be a unit speed biharmonic spacelike new type $\mathcal{B}-$slant helix with non-zero Bishop curvatures. Then the equations of γ are*

$$
\begin{aligned}
\boldsymbol{x}_{\mathcal{B}}(s) &= \frac{1}{\mathcal{J}_0}\sinh\mathcal{E}\sinh\left[\mathcal{J}_0 s+\mathcal{J}_1\right]+\mathcal{J}_2,\\
\boldsymbol{y}_{\mathcal{B}}(s) &= \frac{1}{\mathcal{J}_0}\sinh\mathcal{E}\cosh\left[\mathcal{J}_0 s+\mathcal{J}_1\right]+\mathcal{J}_3,\\
\boldsymbol{z}_{\mathcal{B}}(s) &= \cosh\mathcal{E}s-\frac{\mathcal{J}_2}{\mathcal{J}_0}\sinh\mathcal{E}\cosh\left[\mathcal{J}_0 s+\mathcal{J}_1\right]\\
&\quad -\frac{1}{4\mathcal{J}_0}\sinh^2\mathcal{E}\left(2\left[\mathcal{J}_0 s+\mathcal{J}_1\right]+\sinh 2\left[\mathcal{J}_0 s+\mathcal{J}_1\right]\right)+\mathcal{J}_4,
\end{aligned}
\tag{3.5}
$$

where $\mathcal{J}_1,\mathcal{J}_2,\mathcal{J}_3,\mathcal{J}_4$ *are constants of integration and*

$$
\mathcal{J}_0=\frac{\sqrt{\mathfrak{k}_1^2-\mathfrak{k}_2^2}}{\cosh\mathcal{E}}-\sinh\mathcal{E}.
$$

If we use Mathematica in above system, we get:

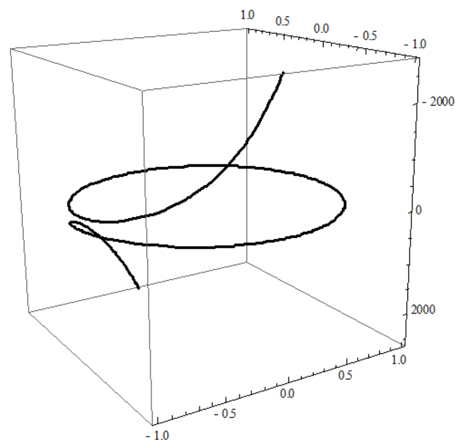

Fig.1.

4. $\mathcal{B}-$Tangent Developable Surfaces of Spacelike Biharmonic New Type $\mathcal{B}-$Slant Helices with Bishop Frame In The Lorentzian Heisenberg Group $\mathcal{H}^3$

To separate a tangent developable according to Bishop frame from that of Frenet-Serret frame, in the rest of the paper, we shall use notation for this surface as $\mathcal{B}-$tangent developable.

The purpose of this section is to study $\mathcal{B}-$tangent developable of biharmonic spacelike new type $\mathcal{B}-$slant helix in $\mathcal{H}^3$.

The $\mathcal{B}-$tangent developable of γ is a ruled surface

$$
\mathcal{O}_{new}(s,u)=\gamma(s)+u\gamma'(s). \tag{4.1}
$$

Theorem 4.1. *Let $\mathcal{O}_{new}$ be $\mathcal{B}-$tangent developable of a unit speed non-geodesic biharmonic spacelike new type $\mathcal{B}-$slant helix with timelike $\mathbf{m}_2$ in $\mathcal{H}^3$. Then, the parametric equations of $\mathcal{B}-$tangent developable are*

$$\begin{aligned}
\boldsymbol{x}_{\mathcal{B}}(s,u) &= \frac{1}{\mathcal{J}_0}\sinh\mathcal{E}\sinh\left[\mathcal{J}_0 s+\mathcal{J}_1\right]+u\sinh\mathcal{E}\cosh\left[\mathcal{J}_0 s+\mathcal{J}_1\right]+\mathcal{J}_2,\\
\boldsymbol{y}_{\mathcal{B}}(s,u) &= \frac{1}{\mathcal{J}_0}\sinh\mathcal{E}\cosh\left[\mathcal{J}_0 s+\mathcal{J}_1\right]+u\sinh\mathcal{E}\sinh\left[\mathcal{J}_0 s+\mathcal{J}_1\right]+\mathcal{J}_3,\\
\boldsymbol{z}_{\mathcal{B}}(s,u) &= \cosh\mathcal{E}s-\frac{\mathcal{J}_2}{\mathcal{J}_0}\sinh\mathcal{E}\cosh\left[\mathcal{J}_0 s+\mathcal{J}_1\right]\\
&\quad-\frac{1}{4\mathcal{J}_0}\sinh^2\mathcal{E}(2\left[\mathcal{J}_0 s+\mathcal{J}_1\right]+\sinh 2\left[\mathcal{J}_0 s+\mathcal{J}_1\right])+\mathcal{J}_4\\
&\quad+u\cosh\mathcal{E}-u[\frac{1}{\mathcal{J}_0}\sinh\mathcal{E}\sinh\left[\mathcal{J}_0 s+\mathcal{J}_1\right]+\mathcal{J}_2]\sinh\mathcal{E}\sinh\left[\mathcal{J}_0 s+\mathcal{J}_1\right],
\end{aligned}\tag{4.2}$$

where $\mathcal{J}_1,\mathcal{J}_2,\mathcal{J}_3,\mathcal{J}_4$ are constants of integration and

$$\mathcal{J}_0=\frac{\sqrt{\mathfrak{k}_1^2-\mathfrak{k}_2^2}}{\cosh\mathcal{E}}-\sinh\mathcal{E}.$$

Proof. From Definition 3.2, we have the following equation

$$\mathbf{t}=\cosh\mathcal{E}\mathbf{e}_1+\sinh\mathcal{E}\sinh\left[\mathcal{J}_0 s+\mathcal{J}_1\right]\mathbf{e}_2+\sinh\mathcal{E}\cosh\left[\mathcal{J}_0 s+\mathcal{J}_1\right]\mathbf{e}_3.\tag{4.3}$$

In terms of Eq.(2.1) and Eq.(4.3), we may give:

$$\mathbf{t}=(\sinh\mathcal{E}\cosh\left[\mathcal{J}_0 s+\mathcal{J}_1\right],\sinh\mathcal{E}\sinh\left[\mathcal{J}_0 s+\mathcal{J}_1\right],\cosh\mathcal{E}-x\sinh\mathcal{E}\sinh\left[\mathcal{J}_0 s+\mathcal{J}_1\right]),\tag{4.4}$$

Consequently, the parametric equations of $\mathcal{O}_{new}$ can be found from Eq.(4.1), Eq.(4.4). This concludes the proof of Theorem.

We can prove the following interesting main result.

Theorem 4.2. *Let $\gamma: I\longrightarrow\mathcal{H}^3$ be a unit speed non-geodesic biharmonic spacelike new type $\mathcal{B}-$slant helix. Then, the position vector of $\mathcal{B}-$tangent developable of biharmonic spacelike new type $\mathcal{B}-$slant helix is*

$$\begin{aligned}
\mathcal{O}_{new}(s,u) &= [\cosh\mathcal{E}s-\frac{\mathcal{J}_2}{\mathcal{J}_0}\sinh\mathcal{E}\cosh\left[\mathcal{J}_0 s+\mathcal{J}_1\right]-\frac{1}{4\mathcal{J}_0}\sinh^2\mathcal{E}(2\left[\mathcal{J}_0 s+\mathcal{J}_1\right]+\sinh 2\left[\mathcal{J}_0 s+\mathcal{J}_1\right])\\
&\quad+\mathcal{J}_4+[\frac{1}{\mathcal{J}_0}\sinh\mathcal{E}\sinh\left[\mathcal{J}_0 s+\mathcal{J}_1\right]+\mathcal{J}_2][\frac{1}{\mathcal{J}_0}\sinh\mathcal{E}\cosh\left[\mathcal{J}_0 s+\mathcal{J}_1\right]+\mathcal{J}_3]+u\cosh\mathcal{E}]\mathbf{e}_1\\
&\quad+[\frac{1}{\mathcal{J}_0}\sinh\mathcal{E}\cosh\left[\mathcal{J}_0 s+\mathcal{J}_1\right]+\mathcal{J}_3+u\sinh\mathcal{E}\sinh\left[\mathcal{J}_0 s+\mathcal{J}_1\right]]\mathbf{e}_2\\
&\quad+[\frac{1}{\mathcal{J}_0}\sinh\mathcal{E}\sinh\left[\mathcal{J}_0 s+\mathcal{J}_1\right]+\mathcal{J}_2+u\sinh\mathcal{E}\cosh\left[\mathcal{J}_0 s+\mathcal{J}_1\right]]\mathbf{e}_3.
\end{aligned}\tag{4.5}$$

where $\mathcal{J}_1,\mathcal{J}_2,\mathcal{J}_3,\mathcal{J}_4$ are constants of integration and

$$\mathcal{J}_0=\frac{\sqrt{\mathfrak{k}_1^2-\mathfrak{k}_2^2}}{\cosh\mathcal{E}}-\sinh\mathcal{E}.$$

Proof. We assume that γ is a unit speed new type $\mathcal{B}-$slant helix.

Substituting Eq.(2.1) to Eq.(4.2), we have Eq.(4.5). Thus, the proof is completed.

Thus, we proved the following:

We may use Mathematica in Theorem 4.1, yields

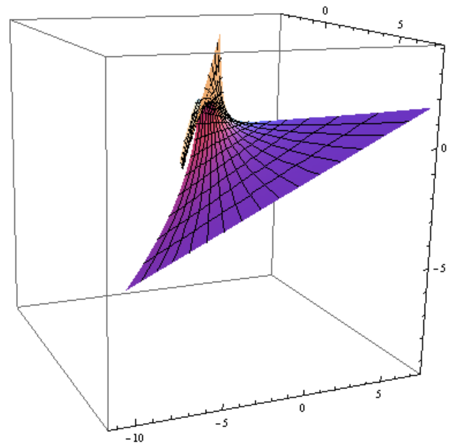

Fig.2.

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