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CHARACTERIZATION OF RICCI TENSOR IN THE LORENTZIAN HEISENBERG GROUP HEIS^3

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Abstract. In this paper, we consider the Lorentzian Heisenberg left-invariant metric and use some results on Levi-Civita connection. We characterize the components of the Ricci tensor.

1. Introduction

Heisenberg group and their subelliptic Laplacians are at the cross-roads of many domains of analysis and geometry: Nilpotent Lie group theory, hypoelliptic second order partial differential equations, strictly pseudoconvex domains in complex analysis, probability theory of degenerate diffusion processes, sub-Riemannian geometry, control theory and semiclassical analysis of quantum mechanics, [2,3,11].

On the other hand, Heisenberg group has a rich geometric structure. In fact its group of isometries is of dimension 4, which is the maximal possible dimension for a non-constant curvature metric on a 3-manifold. Also, from the algebraic point of view, Heis^3 is a 2-step nilpotent Lie group, i.e. "almost Abelian".

In this paper, we consider the Lorentzian Heisenberg left-invariant metric and use some results on Levi-Civita connection. We characterize the components of the Ricci tensor.

2. Ricci Tensor in the Lorentzian Heisenberg Group Heis^3

Heisenberg group plays an important role in many branches of mathematics such as representation theory, harmonic analysis, PDE's or even quantum mechanic. Heis^3 has a rich geometric structure. In fact its group of isometries is of dimension 4, which is the maximal possible dimension for a non-constant curvature metric on a 3-manifold. Also, from the algebraic point of view, Heis^3 is a 2-step nilpotent Lie group, i.e. "almost Abelian".

This group is formed by all matrices of the form

$$Heis^3 = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\},$$

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with the group multiplication induced by the standard matrix product.

The Lorentzian Heisenberg group Heis^3 can be seen as the space $\mathbb{R}^3$ endowed with multiplication

$$(\overline{x},\overline{y},\overline{z})(x,y,z)=(\overline{x}+x,\overline{y}+y,\overline{z}+z-\overline{x}y+x\overline{y}).$$

The orthonormal basis

$$\mathbf{E}_1=\begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix},\quad \mathbf{E}_2=\begin{pmatrix}0&0&0\\0&0&1\\0&0&0\end{pmatrix},\quad \mathbf{E}_3=\begin{pmatrix}0&0&1\\0&0&0\\0&0&0\end{pmatrix}$$

of tangent space at the identity, determines on Heis^3 a left-invariant Lorentzian metric

$$ds^2=(\omega^1)^2+(\omega^2)^2-(\omega^3)^2, \tag{2.1}$$

where

$$\omega^1=dz+xdy,\qquad \omega^2=dy,\qquad \omega^3=dx$$

is the left-invariant orthonormal coframe associated with the orthonormal left-invariant frame,

$$\mathbf{e}_1=\frac{\partial}{\partial z},\quad \mathbf{e}_2=\frac{\partial}{\partial y}-x\frac{\partial}{\partial z},\quad \mathbf{e}_3=\frac{\partial}{\partial x}. \tag{2.2}$$

The corresponding Lie brackets are

$$[\mathbf{e}_2,\mathbf{e}_3]=\ 2\mathbf{e}_1,\qquad [\mathbf{e}_1,\mathbf{e}_3]=[\mathbf{e}_1,\mathbf{e}_2]=0.$$

Now let C_{ij}^k be the structure's constants of the Lie algebra g of G [14] that is,

$$\left[\mathbf{e}_i,\mathbf{e}_j\right]=C_{ij}^k\mathbf{e}_k.$$

The Coshul formula for the Levi–Civita connection is:

$$2g(\nabla_{\mathbf{e}_i}\mathbf{e}_j,\mathbf{e}_k)=C_{ij}^k-C_{jk}^i+C_{ki}^j:=L_{ij}^k,$$

where the non zero L_{ij}^k 's are

$$L_{12}^3=1,\quad L_{21}^3=1,\quad L_{13}^2=1,\quad L_{31}^2=1,\quad L_{23}^1=1,\quad L_{32}^1=-1. \tag{2.3}$$

The vector fields (2.2) define a unique Riemannian metric g such that

$$g\left(\mathbf{e}_1,\mathbf{e}_1\right)=\lambda,\quad g\left(\mathbf{e}_2,\mathbf{e}_2\right)=1,\quad g\left(\mathbf{e}_3,\mathbf{e}_3\right)=-1,\qquad \lambda>0.$$

The coefficients are given by

$$g_{ij}^{(\lambda)}=\begin{pmatrix}-1&0&0\\0&1+\lambda x^2&\lambda x\\0&\lambda x&\lambda\end{pmatrix}. \tag{2.4}$$

To avoid confusion we shall denote the above metric by $g^{(\lambda)}$.

Theorem 2.1. *If $R_{ij}^{(\lambda)}$ are the components of the Ricci tensor with respect to the metric $g_{ij}^{(\lambda)}$ on Heis^3, then*

$$R_{11}=-2\lambda,\quad R_{22}=2\lambda,\quad R_{33}=-2\lambda^2.$$

Proof. A computation shows that the non-zero components of the 4-covariant Riemann tensor of curvature are

$$R_{2323} = \lambda^2, \quad R_{1313} = -\lambda^2,$$
$$R_{1213} = x\lambda^2, \quad R_{1212} = 3\lambda + x^2\lambda^2.$$

Using the contraction formula

$$R_{ij} = \sum_{k,l} R_{ijkl} \left(g^{\lambda}\right)^{kl},$$

where

$$\left(g^{\lambda}\right)^{kl} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & -\lambda x \\ 0 & -x & \frac{1}{\lambda} + x^2 \end{pmatrix}.$$

The non-zero components of the Ricci tensor are

$$R_{11} = -2\lambda, \quad R_{22} = 2\lambda, \quad R_{33} = -2\lambda^2.$$

This proves the claim.

We start with the properties of the Levi-Civita connection on Heis^3. For each metric g^{λ} one has a natural Levi-Civita connection ∇^{λ} defined by

$$\nabla^{\lambda}_{\partial_i} \partial_j = \Gamma^k_{ij}(\lambda)\partial_k, \tag{2.5}$$

where the Christoffel symbols are defined by the metric (2.4). They depend linearly on λ and are given by

$$\begin{array}{lll} \Gamma^2_{12} = \lambda x, & \Gamma^3_{12} = -\lambda x^2, & \Gamma^2_{13} = \lambda, \\ \Gamma^3_{13} = -\lambda x & \Gamma^1_{23} = \lambda, & \Gamma^1_{22} = 2\lambda x. \end{array} \tag{2.6}$$

Theorem 2.2. *For every $\lambda > 0$ we obtain*

$$\nabla^{\lambda}_{\partial_1}\partial_2 = \lambda x \mathbf{e}_2,\ \nabla^{\lambda}_{\partial_2}\partial_3 = \nabla^{\lambda}_{\partial_3}\partial_2 = \lambda \mathbf{e}_3,$$

$$\nabla^{\lambda}_{\partial_2}\partial_1 = \mathbf{e}_1 + \lambda x \mathbf{e}_2, \quad \nabla^{\lambda}_{\partial_1}\partial_3 = \nabla^{\lambda}_{\partial_3}\partial_1 = \lambda \mathbf{e}_2,$$

$$\nabla^{\lambda}_{\partial_1}\partial_1 = 0,\ \nabla^{\lambda}_{\partial_2}\partial_2 = 2\lambda x \mathbf{e}_3, \quad \nabla^{\lambda}_{\partial_3}\partial_3 = 0.$$

Proof. Using (2.6), this proves the claim.

Corollary 2.3. *The vector fields $\mathbf{e}_1$ and $\mathbf{e}_3$ are geodesic vector fields.*

Proof. By Proposition 2.2, we have

$$\nabla^{\lambda}_{\mathbf{e}_1}\mathbf{e}_1 = 0, \quad \nabla^{\lambda}_{\mathbf{e}_3}\mathbf{e}_3 = 0.$$

The corollary is proved.

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