AMO — Advanced Modeling and Optimization, Volume 14, Number 2, 2012

PEDAL CURVES OF TANGENT DEVELOPABLE SURFACES OF BIHARMONIC CURVES IN $\widetilde{\mathcal{SL}_2}(\mathbb{R})$

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Abstract. In this paper, we study pedal curves of tangent developable of biharmonic curves in the $\widetilde{\mathcal{SL}_2}(\mathbb{R})$. Finally, we find explicit parametric equations of this curves in the $\widetilde{\mathcal{SL}_2}(\mathbb{R})$.

1. Introduction

Developable surfaces are defined as the surfaces on which the Gaussian curvature is 0 everywhere. The developable surfaces are useful since they can be made out of sheet metal or paper by rolling a flat sheet of material without stretching it. Most large-scale objects such as airplanes or ships are constructed using un-stretched sheet metals, since sheet metals are easy to model and they have good stability and vibration properties. Moreover, sheet metals provide good fluid dynamic properties. In ship or airplane design, the problems usually stem from engineering concerns and in engineering design there has been a strong interest in developable surfaces.

In this paper, we study pedal curves of tangent developable of biharmonic curves in the $\widetilde{\mathcal{SL}_2}(\mathbb{R})$. Finally, we find explicit parametric equations of this curves in the $\widetilde{\mathcal{SL}_2}(\mathbb{R})$.

2. $\widetilde{\mathcal{SL}_2}(\mathbb{R})$

We identify $\widetilde{\mathcal{SL}_2}(\mathbb{R})$ with

$$\mathbb{R}^3_+ = \left\{(x,y,z) \in \mathbb{R}^3 : z > 0\right\}$$

endowed with the metric

$$g = ds^2 = (dx + \frac{dy}{z})^2 + \frac{dy^2 + dz^2}{z^2}.$$

The following set of left-invariant vector fields forms an orthonormal basis for $\widetilde{\mathcal{SL}_2}(\mathbb{R})$

$$\mathbf{e}_1 = \frac{\partial}{\partial x}, \mathbf{e}_2 = z\frac{\partial}{\partial y} - \frac{\partial}{\partial x}, \mathbf{e}_3 = z\frac{\partial}{\partial z}. \tag{2.1}$$

Date: January 21, 2012.
2000 *Mathematics Subject Classification.* Primary 53A04; Secondary 53A10.
Key words and phrases. Biharmonic curve, $\widetilde{\mathcal{SL}_2}(\mathbb{R})$, pedal curve.

***AMO - Advanced Modeling and Optimization. ISSN: 1841-4311**

The characterising properties of g defined by

$$g(\mathbf{e}_1,\mathbf{e}_1)=g(\mathbf{e}_2,\mathbf{e}_2)=g(\mathbf{e}_3,\mathbf{e}_3)=1,$$
$$g(\mathbf{e}_1,\mathbf{e}_2)=g(\mathbf{e}_2,\mathbf{e}_3)=g(\mathbf{e}_1,\mathbf{e}_3)=0.$$

The Riemannian connection ∇ of the metric g is given by

$$\begin{aligned}2g(\nabla_X Y,Z)&=Xg(Y,Z)+Yg(Z,X)-Zg(X,Y)\\&\quad-g(X,[Y,Z])-g(Y,[X,Z])+g(Z,[X,Y]),\end{aligned}$$

which is known as Koszul's formula.

Using the Koszul's formula, we obtain

$$\begin{aligned}&\nabla_{\mathbf{e}_1}\mathbf{e}_1=0, &&\nabla_{\mathbf{e}_1}\mathbf{e}_2=\frac{1}{2}\mathbf{e}_3, &&\nabla_{\mathbf{e}_1}\mathbf{e}_3=-\frac{1}{2}\mathbf{e}_2,\\&\nabla_{\mathbf{e}_2}\mathbf{e}_1=\frac{1}{2}\mathbf{e}_3, &&\nabla_{\mathbf{e}_2}\mathbf{e}_2=\mathbf{e}_3, &&\nabla_{\mathbf{e}_2}\mathbf{e}_3=-\frac{1}{2}\mathbf{e}_1-\mathbf{e}_2,\\&\nabla_{\mathbf{e}_3}\mathbf{e}_1=-\frac{1}{2}\mathbf{e}_2, &&\nabla_{\mathbf{e}_3}\mathbf{e}_2=\frac{1}{2}\mathbf{e}_1, &&\nabla_{\mathbf{e}_3}\mathbf{e}_3=0.\end{aligned}\tag{2.2}$$

Moreover we put

$$R_{ijk}=R(\mathbf{e}_i,\mathbf{e}_j)\mathbf{e}_k,\quad R_{ijkl}=R(\mathbf{e}_i,\mathbf{e}_j,\mathbf{e}_k,\mathbf{e}_l),$$

where the indices i,j,k and l take the values $1,2$ and 3

$$R_{1212}=\ R_{1313}=\frac{1}{4},\quad R_{2323}=-\frac{7}{4}.\tag{2.3}$$

3. Biharmonic Curves in $\widetilde{\mathcal{SL}_2(\mathbb{R})}$

Biharmonic equation for the curve γ reduces to

$$\nabla^3_{\mathbf{T}}\mathbf{T}-R(\mathbf{T},\nabla_{\mathbf{T}}\mathbf{T})\mathbf{T}=0,\tag{3.1}$$

that is, γ is called a biharmonic curve if it is a solution of the equation (3.1).

Let us consider biharmonicity of curves in $\widetilde{\mathcal{SL}_2(\mathbb{R})}$. Let $\{\mathbf{T},\mathbf{N},\mathbf{B}\}$ be the Frenet frame field along γ. Then, the Frenet frame satisfies the following Frenet–Serret equations:

$$\begin{aligned}\nabla_{\mathbf{T}}\mathbf{T}&=\kappa\mathbf{N},\\\nabla_{\mathbf{T}}\mathbf{N}&=-\kappa\mathbf{T}+\tau\mathbf{B},\\\nabla_{\mathbf{T}}\mathbf{B}&=-\tau\mathbf{N},\end{aligned}\tag{3.2}$$

where $\kappa=|\mathcal{T}(\gamma)|=|\nabla_{\mathbf{T}}\mathbf{T}|$ is the curvature of γ and τ its torsion and

$$g(\mathbf{T},\mathbf{T})=1,\ g(\mathbf{N},\mathbf{N})=1,\ g(\mathbf{B},\mathbf{B})=1,$$
$$g(\mathbf{T},\mathbf{N})=g(\mathbf{T},\mathbf{B})=g(\mathbf{N},\mathbf{B})=0.$$

With respect to the orthonormal basis $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$, we can write

$$\begin{aligned}\mathbf{T}&=T_1\mathbf{e}_1+T_2\mathbf{e}_2+T_3\mathbf{e}_3,\\\mathbf{N}&=N_1\mathbf{e}_1+N_2\mathbf{e}_2+N_3\mathbf{e}_3,\\\mathbf{B}&=\mathbf{T}\times\mathbf{N}=B_1\mathbf{e}_1+B_2\mathbf{e}_2+B_3\mathbf{e}_3.\end{aligned}\tag{3.3}$$

Theorem 3.1. *$\gamma : I \longrightarrow \widetilde{\mathcal{SL}_2(\mathbb{R})}$ is a biharmonic curve if and only if*

$$\begin{aligned} &\kappa = \text{constant} \neq 0, \\ &\kappa^2 + \tau^2 = -\frac{1}{4} + \frac{15}{4} B_1^2, \\ &\tau' = 2N_1 B_1. \end{aligned} \tag{3.4}$$

Proof. Using (3.1) and Frenet formulas (3.2), we have (3.4).

Theorem 3.2. ([9]) *Let $\gamma : I \longrightarrow \widetilde{\mathcal{SL}_2(\mathbb{R})}$ be a unit speed non-geodesic biharmonic curve. Then, the parametric equations of γ are*

$$x(s) = \frac{1}{\aleph} \sin\varphi \sin[\aleph s + C] + \frac{1}{\aleph} \sin\varphi \cos[\aleph s + C] + \wp_2, \tag{3.5}$$

$$y(s) = \frac{1}{\aleph^2 + \cos^2\varphi} \sin\varphi \wp_1 e^{\cos\varphi s} (-\aleph \cos[\aleph s + C] + \cos\varphi \sin[\aleph s + C]),$$

$$z(s) = \wp_1 e^{\cos\varphi s},$$

where $\aleph$, C, $\wp_1$, $\wp_2$ are constants of integration.

4. Pedal Curves of Tangent Developable Surfaces of Biharmonic Curves in $\widetilde{\mathcal{SL}_2(\mathbb{R})}$

The purpose of this section is to study pedal curves of tangent developable surfaces of biharmonic curves in $\widetilde{\mathcal{SL}_2(\mathbb{R})}$.

The tangent developable of γ is a ruled surface

$$\mathfrak{F}(s,u) = \gamma(s) + u\gamma'(s). \tag{4.1}$$

Let $\mathfrak{F}$ be a developable ruled surface given by equation (4.1) in $\mathfrak{Sol}^3$. Since the tangent plane is constant along rulings of $\mathfrak{F}$, it is clear that the pedal of $\mathfrak{F}$ is a curve. Thus, for the pedal of $\mathfrak{F}$, we can write

$$\bar{\gamma}(s) = \gamma(s) + R(s)\mathbf{T}(s),$$

where $R(s)$ is the distance between the points $\gamma(s)$ and $\bar{\gamma}(s)$.

Theorem 4.1. *Let $\mathfrak{F}$ be tangent developable surface of a unit speed non-geodesic biharmonic curve in $\widetilde{\mathcal{SL}_2(\mathbb{R})}$ and $\bar{\gamma}$ its pedal curve. Then, the parametric equations* *of this pedal curve are given by*

$$\begin{aligned} x_{\mathfrak{F}}(s) &= \frac{1}{\aleph}\sin\varphi\sin[\aleph s+C]+\frac{1}{\aleph}\sin\varphi\cos[\aleph s+C] \\ &+R(s)\sin\varphi\cos[\aleph s+C]-R(s)\sin\varphi\sin[\aleph s+C]+\wp_2, \end{aligned} \tag{4.2}$$

$$\begin{aligned} y_{\mathfrak{F}}(s) &= \frac{1}{\aleph^2+\cos^2\varphi}\sin\varphi\wp_1 e^{\cos\varphi s}(-\aleph\cos[\aleph s+C]+\cos\varphi\sin[\aleph s+C]) \\ &+R(s)\wp_1 e^{\cos\varphi s}\sin\varphi\sin[\aleph s+C], \end{aligned}$$

$$z_{\mathfrak{F}}(s)=\wp_1 e^{\cos\varphi s}+R(s)\wp_1 e^{\cos\varphi s}\cos\varphi,$$

where $\aleph$, C, $\wp_1$, $\wp_2$ *are constants of integration and* $R(s)$ *is the distance between the points* $\gamma(s)$ *and* $\bar{\gamma}(s)$.

Proof. Using (2.1) in (3.5), we obtain

$$\begin{aligned} \mathbf{T} &= (\sin\varphi\cos[\aleph s+C]-\sin\varphi\sin[\aleph s+C], \\ &\wp_1 e^{\cos\varphi s}\sin\varphi\sin[\aleph s+C], \wp_1 e^{\cos\varphi s}\cos\varphi). \end{aligned} \tag{4.3}$$

If we substitute (4.3) into (4.6), we have (4.2). This concludes the proof of Theorem.

We can prove the following interesting results.

Lemma 4.2. *Let* $\gamma: I \longrightarrow \widetilde{\mathcal{SL}_2(\mathbb{R})}$ *be a unit speed non-geodesic biharmonic curve. Then the position vector of* γ *is*

$$\begin{aligned} \gamma(s) &= [\frac{1}{\aleph}\sin\varphi\sin[\aleph s+C]+\frac{1}{\aleph}\sin\varphi\cos[\aleph s+C]+\wp_2 \\ &+[\frac{1}{(\aleph^2+\cos^2\varphi)}\sin\varphi(-\aleph\cos[\aleph s+C]+\cos\varphi\sin[\aleph s+C])]]\mathbf{e}_1 \\ &[\frac{1}{(\aleph^2+\cos^2\varphi)}\sin\varphi(-\aleph\cos[\aleph s+C]+\cos\varphi\sin[\aleph s+C])]\mathbf{e}_2+\mathbf{e}_3, \end{aligned} \tag{4.4}$$

where $\aleph, C, \wp_2$ *are constants of integration.*

Theorem 4.3. *Let* $\mathfrak{F}$ *be tangent developable surface of a unit speed non-geodesic biharmonic curve in* $\widetilde{\mathcal{SL}_2(\mathbb{R})}$ *and* $\bar{\gamma}$ *its pedal curve. Then, the vector equations of this pedal curve is given by*

$$\begin{aligned} \mathfrak{F}(s) &= [\frac{1}{\aleph}\sin\varphi\sin[\aleph s+C]+\frac{1}{\aleph}\sin\varphi\cos[\aleph s+C]+\wp_2 \\ &+[\frac{1}{(\aleph^2+\cos^2\varphi)}\sin\varphi(-\aleph\cos[\aleph s+C]+\cos\varphi\sin[\aleph s+C])] \\ &+R(s)\sin\varphi\cos[\aleph s+C]]\mathbf{e}_1 \\ &[\frac{1}{(\aleph^2+\cos^2\varphi)}\sin\varphi(-\aleph\cos[\aleph s+C]+\cos\varphi\sin[\aleph s+C]) \\ &+R(s)\sin\varphi\sin[\aleph s+C]]\mathbf{e}_2+(1+R(s)\cos\varphi)\mathbf{e}_3, \end{aligned} \tag{4.5}$$

where $\aleph, C, \wp_2$ *are constants of integration.*

Proof. We assume that $\gamma : I \longrightarrow \widetilde{\mathcal{SL}_2}(\mathbb{R})$ be a unit speed biharmonic curve. From 4.3, then one can show that

$$\mathbf{T} = \sin\varphi\cos\left[\aleph s + C\right]\mathbf{e}_1 + \sin\varphi\sin\left[\aleph s + C\right]\mathbf{e}_2 + \cos\varphi\mathbf{e}_3. \tag{4.6}$$

Substituting (4.6) to (4.1), we have (4.5). Thus, the proof is completed.

5. Applications

The obtained parametric equations for Eq. (3.4) is illustrated in Fig. 1:

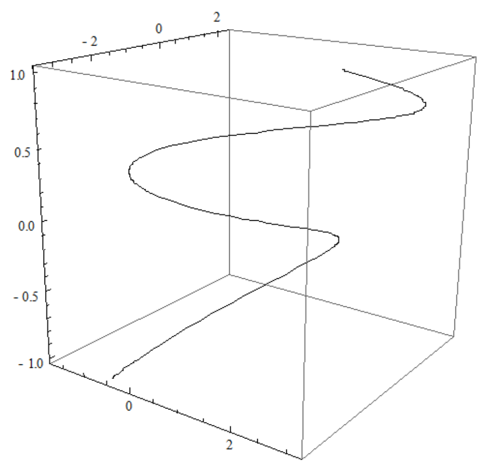

Fig.1

Similarly, the obtained parametric equations for Eq. (4.2) is illustrated for $R(s)$ constant in Fig. 2:

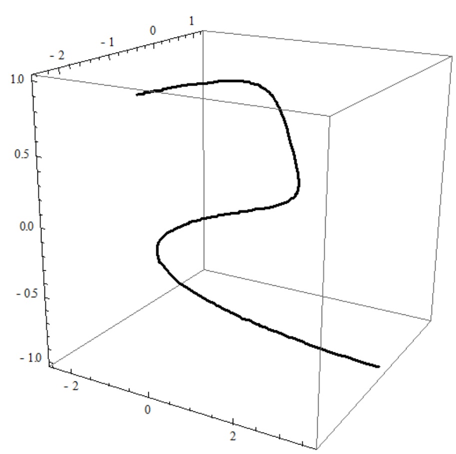

Fig.2

If we use Mathematica both unit speed non-geodesic biharmonic curve and its pedal curve, we have

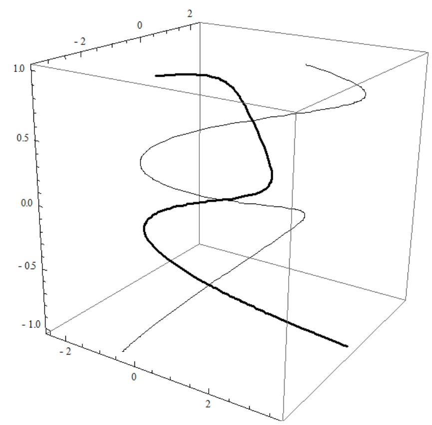

Fig.3

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